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Polar duals of convex and star bodies

Chang-Jian Zhao[1*], Lian-Ying Chen[1] and Wing-Sum Cheung[2]

* Correspondence: chjzhao@yahoo.com.cn
[1]Department of Mathematics, China Jiliang University, Hangzhou 310018, China
Full list of author information is available at the end of the article

Abstract

In this article, some new inequalities about polar duals of convex and star bodies are established. The new inequalities in special case yield some of the recent results.

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1 Notations and preliminaries

The setting for this article is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denotes the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n.

We use $V(K)$ for the n-dimensional volume of convex body K. $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, denotes the support function of $K \in \mathcal{K}^n$; i.e., for $u \in S^{n-1}$

$$h(K, u) = \mathrm{Max}\{u \cdot x : x \in K\}, \tag{1.1}$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denotes the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n, \delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K, u) = \mathrm{Max}\{\lambda \geq 0 : \lambda u \in K\}. \tag{1.2}$$

If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let S^n denotes the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denotes the radial Hausdorff metric, as follows, if $K, L \in S^n$, then $\tilde{\delta}(K, L) = |\rho_K - \rho_L|_\infty$ (See [1,2]).

1.1 L_p-mixed volume and dual L_p-mixed volume

If $K, L \in \mathcal{K}^n$, the L_p-mixed volume $V_p(K, L)$ was defined by Lutwak (see [3]):

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \mathrm{d}S_p(K, u), \tag{1.3}$$

where $S_p(K, \cdot)$ denotes a positive Borel measure on S^{n-1}.

The L_p analog of the classical Minkowski inequality (see [3]) states that: If K and L are convex bodies, then

$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{1.4}$$

with equality if and only if K and L are homothetic.

If K, $L \in S^n$, $p \geq 1$, the L_p-dual mixed volume $\tilde{V}_{-p}(K,L)$ was defined by Lutwak (see [4]):

$$\tilde{V}_{-p}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n+p} \rho(L,u)^{-p} \mathrm{d}S(u), \tag{1.5}$$

where $dS(u)$ signifies the surface area element on S^{n-1} at u.

The following dual L_p-Minkowski inequality was obtained in [2]: If K and L are star bodies, then

$$\tilde{V}_{-p}(K,L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{1.6}$$

with equality if and only if K and L are dilates.

1.2 Mixed bodies of convex bodies

If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, the notation of mixed body $[K_1,..., K_{n-1}]$ states that (see [5]): corresponding to the convex bodies $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ in $\mathbb{R}^n$, there exists a convex body, unique up to translation, which we denote *by*$[K_1,..., K_{n-1}]$.

The following is a list of the properties of mixed body: It is symmetric, linear with respect to Minkowski linear combinations, positively homogeneous, and for $K_i \in \mathcal{K}^n, i = 1, \ldots, n, L_1 \in \mathcal{K}^n$ and $\lambda_i > 0$,

(1) $V_1([K_1, ..., K_{n-1}], K_n) = V(K_1, ..., K_{n-1}, K_n)$;
(2) $[K_1 + L_1, K_2, ..., K_{n-1}] = [K_1, K_2, ..., K_{n-1}] + [L_1, K_2, ..., K_{n-1}]$;
(3) $[\lambda_1 K_1, ..., \lambda_{n-1} K_{n-1}] = \lambda_1 ... \lambda_{n-1} \cdot [K_1, ..., K_{n-1}]$;
(4) $\underbrace{[K, \ldots, K]}_{n-1} = K$.

The properties of mixed body play an important role in proving our main results.

1.3 Polar of convex body

For $K \in \mathcal{K}^n$, the polar body of K, K^* is defined:

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

It is easy to get that

$$\rho(K,u)^{-1} = h(K^*,u). \tag{1.7}$$

Bourgain and Milman's inequality is stated as follows (see [6]).

If K is a convex symmetric body in $\mathbb{R}^n$, then there exists a universal constant $c>0$ such that

$$V(K)V(K^*) \geq c^n \omega_n^2. \tag{1.8}$$

Different proofs were given by Pisier [7].

2 Main results

In this article, we establish some new inequalities on polar duals of convex and star bodies.

Theorem 2.1 *If K, K_1, ..., K_{n-1} are convex bodies in $\mathbb{R}^n$ and let $L = [K_1, ..., K_{n-1}]$, then the L_p-mixed volumes $V_p(K, L)$, $V_p(K^*, L)$, $V_p(B, L)$ satisfy*

$$V_p(K,L)V_p(K^*,L) \geq V_p(B,L)^2. \tag{2.1}$$

Proof From (1.1) and (1.2), it is easy

$$h(K,u) \geq \rho(K,u), \quad K \in \mathcal{K}^n. \tag{2.2}$$

By definition of L_p-mixed volume, we have

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(K,u)^p \mathrm{d}S_p(L;u), \tag{2.3}$$

and

$$V_p(K^*,L) = \frac{1}{n}\int_{S^{n-1}} h(K^*,u)^p \mathrm{d}S_p(L,u). \tag{2.4}$$

Multiply both sides of (2.3) and (2.4), in view of (1.7) and (2.2) and using the Cauchy-Schwarz inequality (see [8]), we obtain

$$\begin{aligned}
&n^2V_p(K,L)V_p(K^*,L)\\
&= \left(\int_{S^{n-1}} h(K,u)^p \mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)\left(\int_{S^{n-1}} \frac{1}{\rho(K,u)^p}\mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)\\
&\geq \left(\int_{S^{n-1}} h(K,u)^{\frac{p}{2}} \cdot \frac{1}{\rho(K,u)^{\frac{p}{2}}}\mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)^2\\
&\geq \left(\int_{S^{n-1}} \mathrm{d}S_p(K_1,\ldots,K_{n-1};u)\right)^2\\
&= n^2V_p^2(B,L).
\end{aligned}$$

Taking $p = n - 1$ in (2.1) and in view of the property (1) of mixed body, we obtain the following result: *If* $K, K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then

$$V(K,K_1,\ldots,K_{n-1})V(K^*,K_1,\ldots,K_n) \geq V(B,K_1,\ldots,K_{n-1})^2. \tag{2.5}$$

This is just an inequality given by Ghandehari [9].

Let $L = B$, we have the following interesting result:

Let K be a convex body and K^* its polar dual, then

$$V_p(K,B)V_p(K^*,B) \geq \omega_n^2. \tag{2.6}$$

Taking $p = n$-1 in (2.6), we have the following result which was given in [9]:

$$W_{n-1}(K)W_{n-1}(K^*) \geq \omega_n^2,$$

with equality if and only if K is an n-ball.

Corollary 2.2 *The L_p-mixed volume of K and K^*, $V_p(K, K^*)$ satisfies*

$$V_p(K^*,K)^n \geq \omega_n^{2(n-p)} V(K)^{2p-n}. \tag{2.7}$$

Proof In view of the property (4) of the mixed body, we have

$$V_p(K,[K,\ \ldots,K]) = V_p(K,K) = V(K).$$

Form (1.4) and taking for $K_1 = K_2 = \ldots = K_{n-1} = K$ in (2.1), we have

$$\begin{aligned} V(K)V_p(K^*,K) &\geq V_p^2(B,K) \\ &\geq V(B)^{\frac{2(n-p)}{n}} V(K)^{\frac{2p}{n}} \\ &= \omega_n^{\frac{2(n-p)}{n}} V(K)^{\frac{2p}{n}}. \end{aligned}$$

Taking $p = n$-1 in (2.7), we have the following result:

$$V(K^*,\underbrace{K,\ldots,K}_{n-1})^n \geq \omega_n^2 V(K)^{n-2}.$$

This is just an inequality given by Ghandehari [9]. The cases $p = 1$ and $n = 2$ give Steinhardt's and Firey's result (see [7]).

A reverse inequality about $\tilde{V}(K^*,\underbrace{K,\ldots,K}_{n-1})$ was given by Ghandehari [9].

$$\tilde{V}(K^*,\underbrace{K,\ldots,K}_{n-1})^n \leq \omega_n^2 V(K)^{n-2}.$$

Theorem 2.3 *Let K be a star body in $\mathbb{R}^n$, K^* be the polar dual of K, then there exist a universal constant c>0 such that*

$$V(K)^{n+2p}\tilde{V}_{-p}(K^*,K)^n \geq (c^n\omega_n^2)^{n+p}, \tag{2.8}$$

where c is the constant of Bourgain and Milman's inequality.

Proof From (1.6) and (1.8), we have

$$\begin{aligned} \tilde{V}_{-p}(K^*,K) &\geq V(K^*)^{\frac{n+p}{n}} V(K)^{-\frac{p}{n}} \\ &= (V(K^*)V(K))^{\frac{n+p}{n}} V(K)^{-\frac{n+2p}{n}} \\ &\geq (c^n\omega_n^2)^{\frac{n+p}{n}} V(K)^{-\frac{n+2p}{n}}. \end{aligned}$$

The following theorem concerning L_p-dual mixed volumes will generalize Santaló inequality.

Theorem 2.4 *Let K_1 and K_2 be two star bodies, $K_1^*$and K_2^*be the polar dual of K_1 and K_2, then there exists a constant c, L_p-dual mixed volumes $\tilde{V}_{-p}(K_1,K_2)$and $\tilde{V}_{-p}(K_1,K_2)\tilde{V}_{-p}(K_1^*,K_2^*) \geq c^n\omega_n^2$ satisfy*

$$\tilde{V}_{-p}(K_1,K_2)\tilde{V}_{-p}(K_1^*,K_2^*) \geq c^n\omega_n^2. \tag{2.9}$$

Proof From (1.6), we have

$$\tilde{V}_{-p}(K_1, K_2) \geq (K_1)^{\frac{n+p}{n}} V(K_2)^{-\frac{p}{n}}. \tag{2.10}$$

For K_1^* and K_2^*, we also have

$$\tilde{V}_{-p}(K_1^*, K_2^*) \geq V(K_1^*)^{\frac{n+p}{n}} V(K_2^*)^{-\frac{p}{n}}. \tag{2.11}$$

Multiply both sides of (2.10) and (2.11) and using Bourgain and Milman's inequality, we obtain

$$\begin{aligned}\tilde{V}_{-p}(K_1, K_2)\tilde{V}_{-p}(K_1^*, K_2^*) &\geq (V(K_1)V(K_1^*))^{-\frac{p}{n}}(V(K_2)V(K_2^*))^{-\frac{p}{n}} \\ &\geq (c^n\omega_n^2)^{\frac{n+p}{n}}(c^n\omega_n^2)^{-\frac{p}{n}} \\ &= c^n\omega_n^2.\end{aligned}$$

Taking for $K_1 = K_2 = K$ in (2.9) and in view of $\tilde{V}_{-p}(K_1, K_2) = \tilde{V}_{-p}(K, K) = V(K)$, (2.9) changes to the Bourgain and Milman's inequality (1.8).

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Author details
[1]Department of Mathematics, China Jiliang University, Hangzhou 310018, China [2]Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

Authors' contributions
C-JZ, L-YC and W-SC jointly contributed to the main results Theorems 2.1, 2.3, and 2.4. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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