

Optimal Surrender Strategies for Equity-Indexed Annuity Investors with Partial Information

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Abstract: In this paper we consider an equity-indexed annuity (EIA) investor who wants to determine when he should surrender the EIA in order to maximize his logarithmic utility of the wealth at surrender time. We model the the dynamic of the index using a geometric Brownian motion with regime switching. To be more realistic, we consider a finite time horizon and assume that the Markov chain is unobservable. This leads to the optimal stopping problem with partial information. We give a representation of the value function and an integral equation satisfied by the boundary. In the Bayesian case which is a special case of our model, we obtain analytical results for the value function and the boundary.

Keywords: Equity-indexed annuity; Regime switching; Optimal stopping; Partial information; Logarithmic utility

1 Introduction

An equity-indexed annuity (EIA) is an equity-linked deferred annuity whose returns are based on the performance of a stock index. The sales of EIAs have grown dramatically in recent years. Because of their popularity, EIAs have received considerable attention in the actuarial literature.

There are two main topics in the study of EIAs. One is the valuation of EIAs. There is a vast of literature on this topic, such as Tiong [19], Moore and Young [14], Lin and Tan [11] and Yuen and Yang [20], among others. Since the EIA investors are entitled the right to surrender the product prior to maturity. So it is important for the investors to determine the optimal surrender time. This leads to the another topic, i.e., the optimal surrender strategies for the EIA investors. Considering a geometric Brownian motion model and infinite horizon, Moore and Young [15] discusses the problem for an investor who may surrender the product and invest directly in the index or banking account, and seeks to maximize the expected utility of wealth at time of death. By including time-dependent contract features, such as the minimum annual growth and the hazard rate, Moore [13] extends the work of Moore and Young [15]. Cheung and Yang [1] studies this problem in a discrete-time model with regime switching. For more discussions of guarantees embedded in insurance products and modeling and pricing issues, we refer to the monograph by Hardy [6].

In this paper, we consider a finite horizon model. As the maturity of EIAs is very long in general, it is more realistic to model the appreciation rate of the equity-linked index by a stochastic process. Similar to Yuen and Yang [20] and Cheung and Yang [1], we capture this feature by considering a regime switching model. In contrast to those papers, we consider a continuous-time model and the Markov chain is not observable. This is more reasonable, since usually the investor can only observable the price of the index directly but not the economic states.

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The problem considered in this paper is an optimal stopping problem with partial information. There are many papers on this topic in mathematical finance. Under the geometric Brownian motion model with regime switching, Rishel [18] considers a stock holder who can only observe the price of the stock and wants to determine a selling time to maximize logarithmic utility in an infinite time horizon. The optimal stopping time given in Rishel [18] is that the first time the conditional probability process (see Section 2) falls below a threshold whenever such threshold exists; or never stop it if the threshold does not exist. In terms of mathematics, we are considering the same problem as that in Rishel [18], but with a finite time horizon. As we all know, when the horizon is finite, the optimal stopping problem is analytically more difficult, and the optimal stopping time is the first time when the conditional probability process falls below the value of a time-dependent boundary.

In this paper we give a representation of the value function and a integral equation satisfied by the boundary. In the Bayesian case which is a special case of our model, we obtain analytical results for the value function and the boundary. For more details on the optimal stopping under Bayesian model, we refer the reader to Décamps et al. [2], Klein [10] and Ekström and Lu [3].

This paper is organized as follows. In Section 2, we present our model and a preliminary analysis of the optimal stopping problem. In Section 3, we study the optimal stopping problem. We show that the value function is the solution of a free boundary problem and the boundary satisfies a integral equation. In Section 4, we consider the Bayesian case and obtain analytical results.

2 The Model

Let us consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite time horizon $T > 0$ which is the maturity time of the EIA. Let J_t be a Markov chain with two states $\{\mathbf{e}_1, \mathbf{e}_2\}$, where \mathbf{e}_i ($i = 1, 2$) are column vectors with unity in the i -th position and zero elsewhere. The state of the chain represent hidden states of an economy. The intensity matrix of J_t is given by

$$\mathbf{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix},$$

where $q_1, q_2 > 0$. We assume that J_t is unobservable.

Let $\boldsymbol{\mu} = (\mu_1, \mu_2)$, where μ_i is the appreciation rate of the index if the economy is in state i , for $i = 1, 2$. Without loss of generality, we assume that $\mu_1 > \mu_2$. Let $\sigma > 0$ denote the constant volatility of the index. The dynamic of the index is given by

$$dS_t = \mu_t S_t dt + \sigma S_t d\tilde{W}_t,$$

where \tilde{W}_t is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mu_t = \langle \boldsymbol{\mu}, J_t \rangle$.

We assume that the investor can only observe S_t . Denote by $\mathbb{F} = \{\mathcal{F}_t^S\}_{0 \leq t \leq T}$ the filtration generated by S_t . Now we can define the conditional probability process

$$X_t := \mathbb{P}(J_t = \mathbf{e}_1 | \mathcal{F}_t^S), \quad t \geq 0$$

with initial value $X_0 = x_0 \in (0, 1)$. Then we have (see, Lipster and Shiryaev [12] and Elliott [4], Chapter 18),

$$dX_t = [-q_1 X_t + q_2(1 - X_t)] dt + \omega(1 - X_t)X_t dW_t, \quad t \geq 0, \quad (2.1)$$

where $\omega = (\mu_1 - \mu_2)/\sigma$, and W_t is the innovation process which is given by

$$dW_t = d\tilde{W}_t + \frac{\mu_t - \mu_1 X_t - \mu_2(1 - X_t)}{\sigma} dt.$$

It can be shown that (W_t, \mathbb{F}) is a standard P-Brownian motion. The dynamics of the index can be rewritten as

$$dS_t = [\mu_2 + (\mu_1 - \mu_2)X_t] S_t dt + \sigma S_t dW_t.$$

Similar to Cheung and Yang [1], for simplicity, the effect of mortality and other product features, like the various embedded guarantees, will be ignored. Without loss of generality, we assume the participation rate is one. Then the evolution of the EIA fund value is governed by

$$dY_t = [\mu_2 + (\mu_1 - \mu_2)X_t] Y_t dt + \sigma Y_t dW_t, \quad (2.2)$$

with $Y_0 = y_0 > 0$ which is the initial deposit.

At the surrender time τ , the investor receives Y_τ . The objective of the investor is to maximize the expected logarithmic utility of the surrender value over all \mathbb{F} -stopping time which is bounded by T . Then the value function is given by

$$V_0 = \sup_{0 \leq \tau \leq T} \mathbb{E}[\ln Y_\tau]. \quad (2.3)$$

Applying Itô's formula to $\ln Y_t$, using (2.2), we have

$$\ln Y_t = \ln Y_0 + \int_0^t \left(\mu_2 + (\mu_1 - \mu_2)X_s - \frac{1}{2}\sigma^2 \right) ds + \sigma W_t.$$

Hence, the original problem (2.3) is equivalent to the problem

$$V = \sup_{0 \leq \tau \leq T} \mathbb{E} \left[\int_0^\tau \left(\mu_2 + (\mu_1 - \mu_2)X_s - \frac{1}{2}\sigma^2 \right) ds \right]. \quad (2.4)$$

Note that for all $t \in [0, T]$, the solution of (2.1) satisfies

$$\mathbb{P}(0 < X_t < 1) = 1,$$

see, e.g. Rishel [18]. If $\sigma^2 \geq 2\mu_1$, then the expectation in (2.4) is nonpositive and it is optimal to surrender immediately. If $\sigma^2 \leq 2\mu_2$, then the expectation in (2.4) is nonnegative and it is optimal to hold the EIA to the maturity. To rule out both of these cases, in the following we assume that $2\mu_2 < \sigma^2 < 2\mu_1$.

In order to solve problem (2.4), we are going to study the optimal stopping problem

$$V(t, x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E} \left[\int_0^\tau \left(\mu_2 + (\mu_1 - \mu_2)X_s^x - \frac{1}{2}\sigma^2 \right) ds \right], \quad (t, x) \in [0, T] \times (0, 1), \quad (2.5)$$

where X^x is a solution of (2.1) with initial value $X_0 = x$. Note that $V = V(0, x_0)$.

3 Analysis of the Optimal Stopping Problem

In this section, we are going to study the optimal stopping problem (2.5). The techniques used in this section are similar to that in Jacka [7], Karatzas and Shreve [9], Section 2.7.

Proposition 3.1. *The function $V(\cdot, \cdot) : [0, T] \times (0, 1) \rightarrow (0, \infty)$ is continuous.*

Proof. It follows from (2.1) that

$$\mathbb{E}[X_t] = X_0 + \mathbb{E} \left[\int_0^t (q_2 - (q_1 + q_2)X_s) ds \right]. \quad (3.1)$$

Let $0 < x_1 \leq x_2 < 1$. By the comparison result (see Karatzas and Shreve [8], Chapter 5), we know that

$$\mathbb{P}(X_s^{x_1} \leq X_s^{x_2}, \forall s \in [0, T-t]) = 1.$$

From the general theory of optimal stopping (see, e.g., Karatzas and Shreve [9], Appendix D and Peskir and Shiryaev [17]), there is an optimal stopping time τ for $V(t, x_2)$ in the sense that

$$V(t, x_2) = \mathbb{E} \left[\int_0^{\tau} \left(\mu_2 + (\mu_1 - \mu_2)X_s^{x_2} - \frac{1}{2}\sigma^2 \right) ds \right].$$

Then we have

$$\begin{aligned} 0 &\leq V(t, x_2) - V(t, x_1) \\ &\leq \mathbb{E} \left[\int_0^{\tau} \left(\mu_2 + (\mu_1 - \mu_2)X_s^{x_2} - \frac{1}{2}\sigma^2 \right) ds \right] - \mathbb{E} \left[\int_0^{\tau} \left(\mu_2 + (\mu_1 - \mu_2)X_s^{x_1} - \frac{1}{2}\sigma^2 \right) ds \right] \\ &= \mathbb{E} \left[\int_0^{\tau} (\mu_1 - \mu_2)(X_s^{x_2} - X_s^{x_1}) ds \right]. \end{aligned} \quad (3.2)$$

Thus, using (3.1), we have

$$\begin{aligned} 0 &\leq V(t, x_2) - V(t, x_1) \\ &\leq \frac{\mu_1 - \mu_2}{q_1 + q_2} (\mathbb{E}[X_{\tau}^{x_2} - X_{\tau}^{x_1}] - (x_2 - x_1)) \\ &\leq \frac{\mu_1 - \mu_2}{q_1 + q_2} (x_2 - x_1), \end{aligned} \quad (3.3)$$

which implies that V is Lipschitz continuous (uniformly) in x .

Let $0 \leq t_1 \leq t_2 \leq T$ and $x \in (0, 1)$. Let τ_1 be an optimal stopping time for $V(t_1, x)$, and define $\tau_2 = \tau_1 \wedge (T - t_2)$. It is easy to see that $0 \leq \tau_1 - \tau_2 \leq t_2 - t_1$. We then have

$$\begin{aligned} 0 &\leq V(t_1, x) - V(t_2, x) \\ &\leq \mathbb{E} \left[\int_0^{\tau_1} \left(\mu_2 + (\mu_1 - \mu_2)X_s^x - \frac{1}{2}\sigma^2 \right) ds \right] - \mathbb{E} \left[\int_0^{\tau_2} \left(\mu_2 + (\mu_1 - \mu_2)X_s^x - \frac{1}{2}\sigma^2 \right) ds \right] \\ &= \mathbb{E} \left[\int_{\tau_2}^{\tau_1} (\mu_1 - \mu_2)X_s^x ds + (\mu_2 - \frac{1}{2}\sigma^2)(\tau_1 - \tau_2) \right] \\ &\leq (\mu_1 - \mu_2)(t_2 - t_1) + (\mu_2 - \frac{1}{2}\sigma^2)(t_2 - t_1) \\ &= (\mu_1 - \frac{1}{2}\sigma^2)(t_2 - t_1). \end{aligned}$$

Recall that we are considering the case with $2\mu_1 > \sigma^2$. So, we have proved that V is Lipschitz continuous (uniformly) in t . This finishes the proof. \square

Now, define the continuation region \mathcal{C} to be

$$\mathcal{C} = \{(t, x) \in [0, T) \times (0, 1) : V(t, x) > 0\},$$

and the stopping region \mathcal{D} to be

$$\mathcal{D} = \{(t, x) \in [0, T) \times (0, 1) : V(t, x) = 0\}.$$

According to the general theory for optimal stopping problems, the stopping time

$$\tau_{\mathcal{D}} = \inf \{0 \leq s \leq T - t : (t + s, X_s^x) \in \mathcal{D}\}$$

is an optimal stopping time in (2.5). Therefore, in the following, we are going to determine the stopping region \mathcal{D} .

Remark 3.1. It is easy to see that

$$\begin{aligned} V(t, x) &\geq \mathbb{E} \left[\int_0^{T-t} \left(\mu_2 + (\mu_1 - \mu_2) X_s^x - \frac{1}{2} \sigma^2 \right) ds \right] \\ &= \left(\mu_2 - \frac{1}{2} \sigma^2 \right) (T-t) + (\mu_1 - \mu_2) (T-t) \mathbb{E} [X_s^x], \end{aligned}$$

where $0 \leq s \leq T-t$. Then, from (3.1), we have

$$V(t, x) \geq \left(\mu_2 - \frac{1}{2} \sigma^2 \right) (T-t) + (\mu_1 - \mu_2) (T-t) [x - q_1(T-t)].$$

Therefore, if $x > b^* + q_1(T-t)$ where $b^* = \frac{\sigma^2 - 2\mu_2}{2(\mu_1 - \mu_2)}$, we have $V(t, x) > 0$.

Proposition 3.2. *There exists a non-decreasing and right continuous function $b(\cdot) : [0, T] \rightarrow [0, b^*]$, such that*

$$C = \{(t, x) \in [0, T] \times (0, 1) : x > b(t)\}.$$

Proof. For some fixed $t \in [0, T]$, and $0 < x_1 \leq x_2 < 1$, suppose that $(t, x_1) \in C$. To prove the existence of $b(t)$, it is sufficient to show that $(t, x_2) \in C$. Indeed, since $V(t, x)$ is increasing in x , it is easy to show that $(t, x_2) \in C$.

Moreover, for $0 \leq t_1 \leq t_2 < T$ and $x \in (0, 1)$, we have $V(t_1, x) \geq V(t_2, x)$. If $x > b(t_2)$ which implies $V(t_2, x) > 0$, we have $V(t_1, x) \geq V(t_2, x) > 0$ and $x > b(t_1)$. Thus, we have proved that $b(t)$ is increasing.

Since $V(t, x)$ is continuous, C is open. Thus \mathcal{D} is closed. If we take a sequence $\{t_n\} \searrow t$, we see that $\forall n \geq 1, (t_n, b(t_n)) \in \mathcal{D}$. Thus $b(t_+) \leq b(t)$. By the increasing nature of b , we conclude that b is right continuous.

From Remark 3.1, we know that $b(t) \leq b^* + q_1(T-t)$. Since $b(t)$ is increasing, for all $t \in [0, T]$, we have $b(t) \leq b^*$. \square

Theorem 3.1. *The value function $V(t, x)$ is the unique solution of the initial-boundary value problem*

$$\begin{cases} \mathcal{L}v(t, x) = 0, & \text{if } x > b(t), \\ v(t, x) = 0, & \text{if } x \leq b(t), \text{ or } t = T, \end{cases} \quad (3.4)$$

where

$$\mathcal{L}v = \mu_1 x + \mu_2(1-x) - \frac{1}{2} \sigma^2 + v_t + [-q_1 x + q_2(1-x)] v_x + \frac{1}{2} \omega^2(1-x)^2 x^2 v_{xx}.$$

In particular, the partial derivatives V_t, V_x and V_{xx} exist and are continuous in C .

Proof. Obviously, V satisfies the second relation of (3.4). In order to verify the first relation for V , we follow the proof of Theorem 2.7.7 of Karatzas and Shreve [9]. Let us take a point $(t, x) \in C$ and a rectangle $\mathcal{R} = (t_1, t_2) \times (x_1, x_2)$ with $(t, x) \in \mathcal{R} \subset C$. Denoted by $\partial_0 \mathcal{R} = \partial \mathcal{R} \setminus [\{t_1\} \times (x_1, x_2)]$ the parabolic boundary of \mathcal{R} . Now consider the initial value problem

$$\begin{cases} \mathcal{L}v(t, x) = 0, & \text{if } (t, x) \in \mathcal{R}, \\ v(t, x) = V(t, x), & \text{if } (t, x) \in \partial_0 \mathcal{R}. \end{cases}$$

By Theorem 3.6 of Friedman [5], Chapter 6, there exists a unique solution v to this problem, with v_t, v_x and v_{xx} continuous. If we show that $V(t, x) = v(t, x)$ in \mathcal{R} , then the first relation of (3.4) follows.

Let $(t_0, x_0) \in \mathcal{R}$ be given, define the stochastic process

$$N_t^{t_0} := v(t_0 + t, X_t^{x_0}) + \int_0^t \left(\mu_2 + (\mu_1 - \mu_2) X_s^{x_0} - \frac{1}{2} \sigma^2 \right) ds, \quad t \in [0, T - t_0],$$

and define the stopping time

$$\tau := \inf \{s \in [0, T - t_0] : (t_0 + s, X_s^{x_0}) \in \partial_0 \mathcal{R}\}.$$

Applying the Itô's formula yields that $N_{t \wedge \tau}$ is a bounded martingale. Thus

$$v(t_0, x_0) = N_0^{t_0} = \mathbb{E} \left[N_\tau^{t_0} \right] = \mathbb{E} \left[M_\tau^{t_0} \right],$$

where

$$M_t^{t_0} := V(t_0 + t, X_t^{x_0}) + \int_0^t \left(\mu_2 + (\mu_1 - \mu_2) X_s^{x_0} - \frac{1}{2} \sigma^2 \right) ds.$$

Define the stopping time

$$\tau_{t_0} := \inf \{s \in [0, T - t_0] : X_s^{x_0} \leq b(t_0 + s)\}.$$

From the general theory of optimal stopping, we know that $M_t^{t_0}$ is martingale in $[0, \tau_{t_0}]$. Since $(t_0, x_0) \in \mathcal{C}$, we have $\tau \leq \tau_{t_0}$. The optimal sampling theorem yields that

$$v(t_0, x_0) = M_0^{t_0} = V(t_0, x_0).$$

Therefore, v and V agree on \mathcal{R} , and hence V_t, V_x and V_{xx} are defined, continuous, and satisfies the first relation of (3.4) at the arbitrary point $(t, x) \in \mathcal{C}$.

For uniqueness, let v be a solution of (3.4). Clearly, v is bounded on $\bar{\mathcal{C}}$. For any $t_0 \in [0, T]$ and $x_0 > b(t_0)$, applying Itô formula yields that $N_{t \wedge \tau_{t_0}}^{t_0}$ is a martingale. Thus,

$$\begin{aligned} v(t_0, x_0) = N_0^{t_0} &= \mathbb{E} \left[N_{\tau_{t_0}}^{t_0} \right] \\ &= \mathbb{E} \left[\int_0^{\tau_{t_0}} \left(\mu_2 + (\mu_1 - \mu_2) X_s^{x_0} - \frac{1}{2} \sigma^2 \right) ds \right] \\ &= V(t_0, x_0), \end{aligned}$$

where the last equality follows from that τ_{t_0} is the optimal stopping time for $V(t_0, x_0)$. \square

Proposition 3.3. *The function V is C^1 in x .*

Proof. It is sufficient to show that $V_x(t, x)$ is continuous at $b(t)$. Since $V_x(t, b(t)-) = 0$, we only need to show $V_x(t, b(t)+) = 0$. For any $\varepsilon > 0$, let

$$\tau_\varepsilon := \inf \{0 \leq s \leq T - t : X_s^{b(t)+\varepsilon} \leq b(t + s)\}.$$

From (3.2), we have

$$\begin{aligned} 0 &\leq V(t, b(t) + \varepsilon) - V(t, b(t)) \\ &\leq \mathbb{E} \left[\int_0^{\tau_\varepsilon} (\mu_1 - \mu_2) (X_s^{b(t)+\varepsilon} - X_s^{b(t)}) ds \right] \\ &= \mathbb{E} \left[\tau_\varepsilon (\mu_1 - \mu_2) (X_{s_\varepsilon}^{b(t)+\varepsilon} - X_{s_\varepsilon}^{b(t)}) \right], \end{aligned}$$

where $s_\varepsilon \in [0, \tau_\varepsilon]$. It follows from (3.3) that

$$0 \leq V(t, b(t) + \varepsilon) - V(t, b(t)) \leq \varepsilon \mathbb{E} [\tau_\varepsilon (\mu_1 - \mu_2)]. \quad (3.5)$$

Since $b(t)$ is increasing, we have

$$\tau_\varepsilon \leq \tau_\varepsilon^* := \inf \{0 \leq s \leq T - t : X_s^{b(t)+\varepsilon} \leq b(t)\}.$$

However, $\tau_\varepsilon^* \rightarrow 0$, as $\varepsilon \rightarrow 0$. Dividing (3.5) by ε and letting $\varepsilon \rightarrow 0$, we obtain $V_x(t, b(t)+) = 0$. \square

Proposition 3.4. *The boundary $b(t)$ is continuous on $[0, T)$, and $b(T-) = b^*$.*

Proof. It follows from Proposition 3.2 that b is right continuous and increasing on $[0, T)$. In order to prove the left continuity, define $b(T) = b^*$. We shall suppose that $b(t_0-) < b(t_0)$ for some $t_0 \in (0, T)$, and obtain a contradiction.

Under the assumption $b(t_0-) < b(t_0)$, there exists $\delta > 0$ such that $b(t_0-) < b_1 < b(t_0)$, where

$$b_1 = b^* - \frac{\delta}{\mu_1 - \mu_2}.$$

Let $t \in [0, t_0)$ and $x \in (b(t), b_1)$ be given. Noting that $V(t, x)$ is non-increasing with respect to t , it follows from Theorem 3.1 that

$$\begin{aligned} \frac{1}{2}\omega^2(1-x^2)x^2V_{xx}(t, x) &\geq -\left[\mu_1x + \mu_2(1-x) - \frac{1}{2}\sigma^2\right] + [q_1x - q_2(1-x)]V_x(t, x) \\ &\geq \delta - q_2V_x(t, x). \end{aligned}$$

Since $0 < b(t) < x < b_1 < 1$, we have

$$\begin{aligned} V_{xx}(t, x) &\geq \frac{2\delta}{\omega^2(1-x^2)x^2} - \frac{2q_2}{\omega^2(1-x^2)x^2}V_x(t, x) \\ &\geq \frac{2\delta}{\omega^2} - \frac{2q_2}{\omega^2(1-b_1^2)b(t)^2}V_x(t, x). \end{aligned}$$

Therefore for $x_0 \in (b(t_0-), b_1)$,

$$\begin{aligned} V(t, x_0) &= \int_{b(t)}^{x_0} \int_{b(t)}^y V_{xx}(t, z) dz dy \\ &\geq \int_{b(t)}^{x_0} \int_{b(t)}^y \left[\frac{2\delta}{\omega^2} - \frac{2q_2}{\omega^2(1-b_1^2)b(t)^2} V_x(t, z) \right] dz dy \\ &= \frac{\delta}{\omega^2} (x_0 - b(t))^2 - \frac{2q_2}{\omega^2(1-b_1^2)b(t)^2} \int_{b(t)}^{x_0} V(t, y) dy. \end{aligned}$$

Letting $t \nearrow t_0$ and using the continuity of V , we obtain

$$\begin{aligned} V(t_0, x_0) &\geq \frac{\delta}{\omega^2} (x_0 - b(t_0-))^2 - \frac{2q_2}{\omega^2(1-b_1^2)b(t_0-)^2} \int_{b(t_0-)}^{x_0} V(t_0, y) dy \\ &= \frac{\delta}{\omega^2} (x_0 - b(t_0-))^2 \\ &> 0, \end{aligned}$$

which implies that $(t_0, x_0) \in C$. But $x_0 < b_1 < b(t_0)$, and we obtain a contradiction. Therefore $b(t)$ is continuous on $[0, T)$, and $b(T-) = b^*$. \square

The following proposition characterizes the value function $V(t, x)$ and the boundary $b(t)$. For simplicity, we assume V is regular enough to apply the Itô's formula (see Peskir [16] and Ekström and Lu [3]). However, without this assumption, a proof can be given by using similar method as that in Karatzas and Shreve [9], Theorem 2.7.9.

Proposition 3.5. *The function $V(t, x)$ admits the representation*

$$V(t, x) = \int_0^{T-t} \mathbb{E} \left[\mathbf{1}_{\{X_s^x > b(t+s)\}} \left((\mu_1 - \mu_2) X_s^x + \mu_2 - \frac{1}{2} \sigma^2 \right) \right] ds, \quad (3.6)$$

where the boundary $b(t)$ satisfies

$$0 = \int_0^{T-t} \mathbb{E} \left[\mathbf{1}_{\{X_s^{b(t)} > b(t+s)\}} \left((\mu_1 - \mu_2) X_s^{b(t)} + \mu_2 - \frac{1}{2} \sigma^2 \right) \right] ds. \quad (3.7)$$

Proof. Fix a $t \in [0, T]$ and $X_0 = x \in (0, 1)$. Applying Itô's formula to $V(t + s, X_s^x)$ yields

$$\begin{aligned} & V(T, X_{T-t}^x) - V(t, x) \\ &= \int_0^{T-t} V_t(t + s, X_s^x) ds + \int_0^{T-t} [-q_1 X_s^x + q_2(1 - X_s^x)] V_x(t + s, X_s^x) ds \\ & \quad + \int_0^{T-t} \omega(1 - X_s^x) X_s^x V_x(t + s, X_s^x) dW_s + \frac{1}{2} \int_0^{T-t} \omega^2(1 - X_s^x)^2 (X_s^x)^2 V_{xx}(t + s, X_s^x) ds. \end{aligned}$$

By Theorem 3.1, taking the expected value of the above equation gives

$$V(t, x) = \int_0^{T-t} \mathbb{E} \left[\mathbf{1}_{\{X_s^x > b(t+s)\}} \left((\mu_1 - \mu_2) X_s^x + \mu_2 - \frac{1}{2} \sigma^2 \right) \right] ds.$$

Letting $x = b(t)$ in (3.6) yields (3.7). \square

Remark 3.2. Using similar techniques to Peskir [16], we also can show that $b(t)$ is the unique solution of (3.7). Also, we can prove that the pair (V, b) is the unique solution to (3.4) (see e.g. Karatzas and Shreve [9], Section 2.7).

4 The Bayesian Case

In this section, we study (3.6) and (3.7) in a special case with the intensity matrix $\mathbf{Q} = 0$ in the model of previous section. This is the so-called Bayesian case where the unobserved drift process μ_t is simply a random variable $\mu_t = \mu$, and (2.1) is rewritten as

$$dX_t = \omega(1 - X_t)X_t dW_t, \quad t \geq 0, \quad (4.1)$$

with $X_0 = x_0 \in (0, 1)$. To calculate (3.6) and (3.7) analytically, let $\phi_t = X_t/(1 - X_t)$. Applying the Itô formula yields that

$$d\phi_t = \omega^2 X_t \phi_t dt + \omega \phi_t dW_t.$$

First, by a similar method as that in Klein [10] and Ekström and Lu [3], we introduce a new probability measure \mathbb{P}^* . To do this, define a new process W^* by

$$dW_t^* = \omega X_t dt + dW_t.$$

The new probability measure is defined by

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} \Big|_{\mathcal{F}_T^x} &= \exp \left\{ -\frac{1}{2} \int_0^T \omega^2 X_t^2 dt - \int_0^T \omega X_t dW_t \right\} \\ &= \exp \left\{ \frac{1}{2} \int_0^T \omega^2 X_t^2 dt - \int_0^T \omega X_t dW_t^* \right\}. \end{aligned}$$

By Girsanov's theorem, W^* is a \mathbb{P}^* -Brownian motion, and

$$d\phi_t = \omega \phi_t dW_t^* \quad (4.2)$$

under measure \mathbb{P}^* . Define the likelihood process

$$\eta_t = \exp \left\{ -\frac{1}{2} \int_0^T \omega^2 X_t^2 dt + \int_0^T \omega X_t dW_t^* \right\}$$

which is a \mathbb{P}^* -martingale. It can be shown that

$$\eta_t = \frac{1 + \phi_t}{1 + \phi_0} \quad (4.3)$$

(see Ekström and Lu [3]).

Let E^* be the expectation with respect to P^* . Noting that $X_t = \phi_t/(1 + \phi_t)$, and using (4.3), we have

$$\begin{aligned}
& E \left[\mathbf{1}_{\{X_s^x > b(t+s)\}} \left((\mu_1 - \mu_2)X_s^x + \mu_2 - \frac{1}{2}\sigma^2 \right) \right] \\
&= \frac{1}{1 + \phi_0} E \left[(1 + \phi_0) \mathbf{1}_{\{X_s^x > b(t+s)\}} \left((\mu_1 - \mu_2)X_s^x + \mu_2 - \frac{1}{2}\sigma^2 \right) \right] \\
&= \frac{1}{1 + \phi_0} E^* \left[\eta_T (1 + \phi_0) \mathbf{1}_{\{\frac{\phi_s}{1+\phi_s} > b(t+s)\}} \left((\mu_1 - \mu_2) \frac{\phi_s}{1 + \phi_s} + \mu_2 - \frac{1}{2}\sigma^2 \right) \right] \\
&= \frac{1}{1 + \phi_0} E^* \left[\eta_s (1 + \phi_0) \mathbf{1}_{\{\frac{\phi_s}{1+\phi_s} > b(t+s)\}} \left((\mu_1 - \mu_2) \frac{\phi_s}{1 + \phi_s} + \mu_2 - \frac{1}{2}\sigma^2 \right) \right] \\
&= \frac{1}{1 + \phi_0} E^* \left[(1 + \phi_s) \mathbf{1}_{\{\frac{\phi_s}{1+\phi_s} > b(t+s)\}} \left((\mu_1 - \mu_2) \frac{\phi_s}{1 + \phi_s} + \mu_2 - \frac{1}{2}\sigma^2 \right) \right] \\
&= \frac{\mu_1 - \frac{1}{2}\sigma^2}{1 + \phi_0} E^* \left[\mathbf{1}_{\{\phi_s > \frac{b(t+s)}{1-b(t+s)}\}} \phi_s \right] + \frac{\mu_2 - \frac{1}{2}\sigma^2}{1 + \phi_0} E^* \left[\mathbf{1}_{\{\phi_s > \frac{b(t+s)}{1-b(t+s)}\}} \right],
\end{aligned}$$

where $\phi_0 = x/(1 - x)$. From (4.2), we know that

$$\phi_t = \phi_0 e^{-\frac{1}{2}\omega^2 t + \omega W_t^*}.$$

Thus we have

$$E^* \left[\mathbf{1}_{\{\phi_s > \frac{b(t+s)}{1-b(t+s)}\}} \phi_s \right] = \phi_0 N(d_1(t, s)), \quad \text{and} \quad E^* \left[\mathbf{1}_{\{\phi_s > \frac{b(t+s)}{1-b(t+s)}\}} \right] = N(d_2(t, s)),$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, and

$$\begin{aligned}
d_1(t, s) &= \frac{\ln[\phi_0(1 - b(t+s))] - \ln b(t+s)}{\omega \sqrt{s}} + \frac{\omega \sqrt{s}}{2}, \\
d_2(t, s) &= d_1(t, s) - \omega \sqrt{s}.
\end{aligned}$$

Consequently, we have

$$V(t, x) = \int_0^{T-t} \left[\frac{\phi_0}{1 + \phi_0} (\mu_1 - \frac{1}{2}\sigma^2) N(d_1(t, s)) + \frac{1}{1 + \phi_0} (\mu_2 - \frac{1}{2}\sigma^2) N(d_2(t, s)) \right] ds, \quad (4.4)$$

where $\phi_0 = x/(1 - x)$. Setting $x = b(t)$ gives

$$0 = \int_0^{T-t} \left[b(t) (\mu_1 - \frac{1}{2}\sigma^2) N(d_1^*(t, s)) + (1 - b(t)) (\mu_2 - \frac{1}{2}\sigma^2) N(d_2^*(t, s)) \right] ds, \quad (4.5)$$

where

$$\begin{aligned}
d_1^*(t, s) &= \frac{\ln[b(t)(1 - b(t+s))] - \ln[(1 - b(t))b(t+s)]}{\omega \sqrt{s}} + \frac{\omega \sqrt{s}}{2}, \\
d_2^*(t, s) &= d_1^*(t, s) - \omega \sqrt{s}.
\end{aligned}$$

Thus in the Bayesian case we can rewrite the results of Proposition 3.5 in an analytical form giving by (4.4) and (4.5).

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