Valuing contingent exotic options: a discounted density approach

Hailiang Yang

Department of Statistics and Actuarial Science
The University of Hong Kong
Hong Kong

Based on a paper with Hans Gerber and Elias Shiu

Brownian motion (Wiener process)

- $X(t) = \mu t + \sigma W(t)$
- $\{W(t)\}$: standard Wiener process
- notation: $D = \frac{\sigma^2}{2}$
- running minimum: $m(t) = \min_{0 \leq s \leq t} X(s)$
- running maximum: $M(t) = \max_{0 \leq s \leq t} X(s)$

2 Exponential stopping of Brownian motion

- τ: exponential random variable
  independent of $\{X(t)\}$
  $f_\tau(t) = \lambda e^{-\lambda t}, \quad t > 0$

- We are interested in $X(\tau)$, $M(\tau)$, ...

- δ: force of interest used for discounting

Three discounted density functions

- $f^{\delta}_{X(\tau)}(x) = \int_0^{\infty} e^{-\delta t} f_{X(t)}(x) f_{\tau}(t) dt$

- $f^{\delta}_{M(\tau)}(m) = \int_0^{\infty} e^{-\delta t} f_{M(t)}(m) f_{\tau}(t) dt$

- $f^{\delta}_{X(\tau),M(\tau)}(x, m) = \int_0^{\infty} e^{-\delta t} f_{X(t),M(t)}(x, m) f_{\tau}(t) dt$

$\alpha < 0$ and $\beta > 0$ solutions of the quadratic equation $D\xi^2 + \mu\xi - (\lambda + \delta) = 0$

- 1). $f^{\delta}_{X(\tau),M(\tau)}(x,m) = \frac{\lambda}{D} e^{-\alpha x - (\beta - \alpha)m} = \frac{\lambda}{D} e^{\alpha(m-x) - \beta m},$
  $-\infty < x \leq m, \;\; m \geq 0$

- 2). $f^{\delta}_{M(\tau)}(m) = \frac{\lambda}{\lambda+\delta} \beta e^{-\beta m}, \qquad m \geq 0$

- 3). $f^{\delta}_{X(\tau)}(x) = \begin{cases} \frac{\lambda}{D(\beta-\alpha)} e^{-\alpha x}, & \text{if } \;\; x < 0, \\ \frac{\lambda}{D(\beta-\alpha)} e^{-\beta x}, & \text{if } \;\; x > 0. \end{cases}$

$$f^{\delta}_{X(\tau),M(\tau)-X(\tau)}(x,z) = \frac{\lambda}{D} e^{-\beta x-(\beta-\alpha)z}, \qquad z \geq \max(-x,0).$$

$$f^{\delta}_{M(\tau),M(\tau)-X(\tau)}(y,z) = \frac{\lambda}{D} e^{-\beta y+\alpha z}, \qquad y \geq 0,\ z \geq 0.$$

$$f^{\delta}_{M(\tau)-X(\tau)}(z) = \frac{\lambda}{\beta D} e^{\alpha z} = \frac{\lambda}{\lambda+\delta}(-\alpha) e^{\alpha z}, \qquad z \geq 0.$$

$m(t) = -\max\{-X(s); \quad 0 \leq s \leq t\}$

If

$$\mathrm{E}[e^{-\delta\tau} g(X(\tau), M(\tau))] = h(\alpha, \beta),$$

then we can translate it to

$$\mathrm{E}[e^{-\delta\tau} g(-X(\tau), -m(\tau))] = h(-\beta, -\alpha).$$

$$
\begin{aligned}
f^{\delta}_{X(\tau),m(\tau)}(x,y) &= \frac{\lambda}{D} e^{-\beta x+(\beta-\alpha)y}, && y \le \min(x,0),\\
f^{\delta}_{X(\tau),X(\tau)-m(\tau)}(x,z) &= \frac{\lambda}{D} e^{-\alpha x-(\beta-\alpha)z}, && z \ge \max(x,0),\\
f^{\delta}_{m(\tau),X(\tau)-m(\tau)}(y,z) &= \frac{\lambda}{D} e^{-\alpha y-\beta z}, && y \le 0, z \ge 0,\\
f^{\delta}_{m(\tau)}(y) = \frac{\lambda}{\beta D} e^{-\alpha y} &= \frac{\lambda}{\lambda+\delta}(-\alpha) e^{-\alpha y}, && y \le 0,\\
f^{\delta}_{X(\tau)-m(\tau)}(z) = \frac{\lambda}{-\alpha D} e^{-\beta z} &= \frac{\lambda}{\lambda+\delta}\beta e^{-\beta z}, && z \ge 0.
\end{aligned}
$$

Factorization formula

If τ is exponential with mean $1/\lambda$, then the following factorization formula holds,

$$E[e^{-\delta\tau}g_\tau(X)] = E[e^{-\delta\tau}] \times E[g_{\tau^*}(X)],$$

where τ^* is an exponential random variable with mean $1/(\lambda+\delta)$ and independent of X.
Remarks (i) $E[e^{-\delta\tau}] = \frac{\lambda}{\lambda+\delta}$.
(ii) The condition $\delta > 0$ can be replaced by the condition $\delta > -\lambda$.

3. Financial applications

- $S(t)$: stock price
- $S(t) = S(0)e^{X(t)} = S(0)e^{\mu t + \sigma W(t)}, \quad t \geq 0$
- a contingent option provides a payoff at time τ
- Example: τ: time of death
  GMDB (Guaranteed Minimum Death Benefits)

3. Lookback options

- Many equity-indexed annuities credit interest using a high water mark method or a low water mark method

Out-of-the-money fixed strike lookback call option

- Payoff:

$$[S(0)e^{M(\tau)} - K]_+$$

- Time-0 value

$$\begin{aligned}\int_k^\infty [S(0)e^y - K] f^{\delta}_{M(\tau)}(y) \mathrm{d}y &= \frac{\lambda}{\lambda+\delta}[S(0)\frac{\beta e^{-(\beta-1)k}}{\beta-1} - Ke^{-\beta k}] \\ &= \frac{\lambda}{\lambda+\delta}\frac{K}{\beta-1}\left[\frac{S(0)}{K}\right]^{\beta}.\end{aligned}$$

- Another expression for the option value

$$\frac{\lambda}{D}\frac{K}{-\alpha\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}.$$

In-the-money fixed strike lookback call option

- Payoff

$$\max(H, S(0)e^{M(\tau)}) - K.$$

- Rewriting as

$$H - K + [S(0)e^{M(\tau)} - H]_+$$

- Time-0 value

$$\frac{\lambda}{\lambda+\delta}\left\{H - K + \frac{H}{\beta-1}\left[\frac{S(0)}{H}\right]^{\beta}\right\}.$$

Floating strike lookback put option

- Payoff

$$\max(H, \max_{0 \le t \le \tau} S(t)) - S(\tau), \tag{1}$$

  where $H \ge S(0)$.

- Time-0 value

$$\frac{\lambda}{\lambda+\delta}\left\{H + \frac{H}{\beta-1}\left[\frac{S(0)}{H}\right]^{\beta}\right\} - \mathrm{E}[e^{-\delta\tau}S(\tau)].$$

Floating strike lookback put option

- Special case: $H = S(0)$, the time-0 value

$$\begin{aligned} & \frac{\lambda}{\lambda+\delta}\frac{\beta}{\beta-1}S(0) - \mathrm{E}[e^{-\delta\tau}S(\tau)] \\ = \; & \frac{1-\alpha}{-\alpha}\mathrm{E}[e^{-\delta\tau}S(\tau)] - \mathrm{E}[e^{-\delta\tau}S(\tau)] \\ = \; & \frac{1}{-\alpha}\mathrm{E}[e^{-\delta\tau}S(\tau)]. \end{aligned} \tag{2}$$

- This result can be reformulated as

$$\mathrm{E}[e^{-\delta\tau}\max_{0\le t\le\tau} S(t)] = \left(\frac{1}{-\alpha}+1\right)\mathrm{E}[e^{-\delta\tau}S(\tau)].$$

Floating strike lookback put option

Milevsky and Posner (2001) have evaluated (1) with a risk-neutral stock price process and $H = S(0)$. They also assume that the stock pays dividends continuously at a rate proportional to its price. With l denoting the dividend yield rate, $\delta = r$, and $\mu = r - D - l$, the RHS of (2) is

$$\frac{2D}{(r - D - l) + \sqrt{(r - D - l)^2 + 4D(\lambda + r)}} \times S(0)\frac{\lambda}{\lambda + l}.$$

Although it seems rather different from formula (38) on page 117 of Milevsky and Posner (2001), they are the same.

Fractional floating strike lookback put option

- Payoff

$$[\gamma \max_{0 \le t \le \tau} S(t) - S(\tau)]_+ = S(0)[\gamma e^{M(\tau)} - e^{X(\tau)}]_+.$$

- Notice

$$[\gamma e^{M(\tau)} - e^{X(\tau)}]_+ = e^{M(\tau)}[\gamma - e^{X(\tau)-M(\tau)}]_+$$

Fractional floating strike lookback put option

- Hence

$$
\begin{aligned}
& \mathrm{E}[e^{-\delta\tau} e^{M(\tau)} [\gamma - e^{X(\tau)-M(\tau)}]_+] \\
= \; & \int_0^\infty \int_0^\infty e^y [\gamma - e^{-z}]_+ f^{\delta}_{M(\tau), M(\tau)-X(\tau)}(y, z) \mathrm{d}y \mathrm{d}z \\
= \; & \frac{\lambda}{D} \left[\int_0^\infty e^y e^{-\beta y} \mathrm{d}y \right] \left[\int_0^\infty [\gamma - e^{-z}]_+ e^{\alpha z} \mathrm{d}z \right] \\
= \; & \frac{\lambda}{D} \frac{1}{\beta - 1} \frac{\gamma^{1-\alpha}}{-\alpha(1-\alpha)} \\
= \; & \gamma^{1-\alpha} \frac{\lambda}{\lambda + \delta} \frac{\beta}{(1-\alpha)(\beta - 1)} \\
= \; & \gamma^{1-\alpha} \frac{1}{-\alpha} \mathrm{E}[e^{-\delta\tau} e^{X(\tau)}].
\end{aligned}
$$

Fractional floating strike lookback put option

- This can be rewritten as

$$\mathsf{E}[e^{-\delta\tau}[\gamma e^{M(\tau)} - e^{X(\tau)}]_+] = \gamma^{1-\alpha}\mathsf{E}[e^{-\delta\tau}(e^{M(\tau)} - e^{X(\tau)})].$$

- Time-0 value

$$\mathsf{E}[e^{-\delta\tau}[\gamma \max_{0\le t\le\tau} S(t) - S(\tau)]_+] = \frac{\gamma^{1-\alpha}}{-\alpha}\mathsf{E}[e^{-\delta\tau}S(\tau)],$$

Out-of-the-money fixed strike lookback put option

- Payoff

$$[K - S(0)e^{m(\tau)}]_+,$$

- Time-0 value

$$\int_{-\infty}^{k} [K - S(0)e^{y}] f^{\delta}_{m(\tau)}(y)\mathrm{d}y = \frac{\lambda}{\lambda+\delta} \frac{K}{1-\alpha} \left[\frac{K}{S(0)}\right]^{-\alpha}.$$

In-the-money fixed strike lookback put option

- Payoff

$$K - \min(H, S(0)e^{m(\tau)}) = K - H + [H - S(0)e^{m(\tau)}]_+,$$

- Time-0 value

$$\frac{\lambda}{\lambda+\delta}\left\{K - H + \frac{H}{1-\alpha}\left[\frac{H}{S(0)}\right]^{-\alpha}\right\}.$$

Floating strike lookback call option

- Payoff

$$S(\tau) - \min(H, \min_{0 \le t \le \tau} S(t)),$$

where $0 < H \le S(0)$.

- Time-0 value

$$\mathrm{E}[e^{-\delta\tau} S(\tau)] + \frac{\lambda}{\lambda + \delta} \left\{ -H + \frac{H}{1-\alpha} \left[\frac{H}{S(0)} \right]^{-\alpha} \right\}.$$

- In the special case where $H = S(0)$, the time-0 value

$$\begin{aligned} &\mathrm{E}[e^{-\delta\tau} S(\tau)] - \frac{\lambda}{\lambda+\delta} \frac{-\alpha}{1-\alpha} S(0) \\ &= \mathrm{E}[e^{-\delta\tau} S(\tau)] - \frac{\beta - 1}{\beta} \mathrm{E}[e^{-\delta\tau} S(\tau)] \\ &= \frac{1}{\beta} \mathrm{E}[e^{-\delta\tau} S(\tau)]. \end{aligned}$$

Fractional floating strike lookback call option

- Payoff

$$\begin{aligned}[S(\tau) - \gamma \min_{0 \le t \le \tau} S(t)]_+ &= S(0)[e^{X(\tau)} - \gamma e^{m(\tau)}]_+. \\ &= S(0)e^{m(\tau)}[e^{X(\tau)-m(\tau)} - \gamma]_+\end{aligned}$$

Fractional floating strike lookback call option

- Its expected discounted value is $S(0)$ times the following expectation

$$
\begin{aligned}
& \mathsf{E}[e^{-\delta\tau} e^{m(\tau)} [e^{X(\tau)-m(\tau)} - \gamma]_+] \\
= \quad & \frac{\lambda}{D} \left[\int_{-\infty}^{0} e^{y} e^{-\alpha y} \mathrm{d}y \right] \left[\int_{0}^{\infty} [e^{z} - \gamma]_+ e^{-\beta z} \mathrm{d}z \right] \\
= \quad & \frac{\lambda}{D} \frac{1}{1-\alpha} \frac{\gamma^{1-\beta}}{\beta(\beta-1)} \\
= \quad & \frac{1}{\gamma^{\beta-1}} \frac{\lambda}{\lambda+\delta} \frac{-\alpha}{(1-\alpha)(\beta-1)} \\
= \quad & \frac{1}{\gamma^{\beta-1}} \frac{1}{\beta} \mathsf{E}[e^{-\delta\tau} e^{X(\tau)}].
\end{aligned}
$$

Fractional floating strike lookback call option

- This can be rewritten as

$$\mathrm{E}[e^{-\delta\tau}[e^{X(\tau)} - \gamma e^{m(\tau)}]_+] = \gamma^{-(\beta-1)}\mathrm{E}[e^{-\delta\tau}(e^{X(\tau)} - e^{m(\tau)})].$$

- We have

$$\mathrm{E}[e^{-\delta\tau}[S(\tau) - \gamma \min_{0\le t\le\tau} S(t)]_+] = \frac{1}{\beta\gamma^{\beta-1}}\mathrm{E}[e^{-\delta\tau}S(\tau)].$$

High-low option

- Payoff

$$\max(\overline{H}, \max_{0\le t\le \tau} S(t)) - \min(\underline{H}, \min_{0\le t\le \tau} S(t)),$$

where $0 < \underline{H} \le S(0) \le \overline{H}$.

- Time-0 value

$$\frac{\lambda}{\lambda+\delta}\left\{\overline{H} + \frac{\overline{H}}{\beta-1}\left[\frac{S(0)}{\overline{H}}\right]^{\beta} - \underline{H} + \frac{\underline{H}}{1-\alpha}\left[\frac{\underline{H}}{S(0)}\right]^{-\alpha}\right\}.$$

- In the special case where $\underline{H} = S(0) = \overline{H}$, time-0 value

$$S(0)\frac{\lambda}{\lambda+\delta}\frac{\beta-\alpha}{(\beta-1)(1-\alpha)} = \frac{\beta-\alpha}{-\alpha\beta}\mathrm{E}[e^{-\delta\tau}S(\tau)].$$

- This can be rewritten as

$$\left(\frac{1}{-\alpha}+\frac{1}{\beta}\right)\mathrm{E}[e^{-\delta\tau}S(\tau)],$$

Barrier options

- A barrier option is an option whose payoff depends on whether or not the price of the underlying asset has reached a predetermined level or *barrier*.
- Knock-out options are those which go out of existence if the asset price reaches the barrier, and *knock-in options* are those which come into existence if the barrier is reached.

Parity relation

- $$\text{Knock-out option} + \text{Knock-in option} = \text{Ordinary option}.$$
- Notation: L denotes the barrier and $\ell = \ln[L/S(0)]$

Up-and-out and up-and-in options ($L > S(0)$ ($\ell > 0$))

- Payoffs

$$I_{([\max_{0\le t\le \tau} S(t)]<L)} b(S(\tau)) = I_{(M(\tau)<\ell)} b(S(0)e^{X(\tau)})$$

-

$$I_{([\max_{0\le t\le \tau} S(t)]\ge L)} b(S(\tau)) = I_{(M(\tau)\ge\ell)} b(S(0)e^{X(\tau)})$$

The expected discounted values

- Up-and-out

$$\int_0^\infty \left[\int_{-\infty}^{y} I_{(y<\ell)} b(S(0)e^x) f^{\delta}_{X(\tau),M(\tau)}(x,y)\mathrm{d}x\right]\mathrm{d}y$$
$$= \frac{\lambda}{D}\int_0^\ell \left[\int_{-\infty}^{y} b(S(0)e^x)e^{-\alpha x}\mathrm{d}x\right] e^{-(\beta-\alpha)y}\mathrm{d}y$$

- Up-and-in

$$\frac{\lambda}{D}\int_\ell^\infty \left[\int_{-\infty}^{y} b(S(0)e^x)e^{-\alpha x}\mathrm{d}x\right] e^{-(\beta-\alpha)y}\mathrm{d}y;$$

Down-and-out and down-and-in options $(0 < L < S(0)\ (\ell < 0))$

- Payoffs

$$I_{([\min_{0\le t\le\tau} S(t)]>L)} b(S(\tau)) = I_{(m(\tau)>\ell)} b(S(0)e^{X(\tau)})$$

-

$$I_{([\min_{0\le t\le\tau} S(t)]\le L)} b(S(\tau)) = I_{(m(\tau)\le\ell)} b(S(0)e^{X(\tau)})$$

The expected discounted values

- $$\frac{\lambda}{D}\int_{\ell}^{0}\left[\int_{y}^{\infty} b(S(0)e^{x})e^{-\beta x}\mathrm{d}x\right]e^{(\beta-\alpha)y}\mathrm{d}y$$
- $$\frac{\lambda}{D}\int_{-\infty}^{\ell}\left[\int_{y}^{\infty} b(S(0)e^{x})e^{-\beta x}\mathrm{d}x\right]e^{(\beta-\alpha)y}\mathrm{d}y,$$

Notation

$$
\begin{aligned}
A_1(n) &= \frac{\lambda}{D}\frac{S(0)^n}{(n-\alpha)(\beta-n)}, \\
A_2(n) &= \frac{\lambda}{D}\frac{L^n}{(n-\alpha)(\beta-n)}\left[\frac{S(0)}{L}\right]^{\beta}, \\
A_3(n) &= \frac{\lambda}{D}\frac{L^n}{(n-\alpha)(\beta-n)}\left[\frac{L}{S(0)}\right]^{-\alpha}, \\
A_4 &= \frac{\lambda}{D}\frac{K^n}{(n-\alpha)(\beta-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha} = \frac{\kappa K^n}{n-\alpha}\left[\frac{K}{S(0)}\right]^{-\alpha},
\end{aligned}
$$

Notation

$$A_5 = \frac{\lambda}{D}\frac{K^{n-\alpha}L^{\alpha}}{(n-\alpha)(\beta-\alpha)}\left[\frac{S(0)}{L}\right]^{\beta} = \frac{\kappa K^{n-\alpha}L^{\alpha}}{n-\alpha}\left[\frac{S(0)}{L}\right]^{\beta},$$

$$A_6 = \frac{\lambda}{D}\frac{K^{n}}{(\beta-n)(\beta-\alpha)}\left[\frac{S(0)}{K}\right]^{\beta} = \frac{\kappa K^{n}}{\beta-n}\left[\frac{S(0)}{K}\right]^{\beta},$$

$$A_7 = \frac{\lambda}{D}\frac{K^{-(\beta-n)}L^{\beta}}{(\beta-n)(\beta-\alpha)}\left[\frac{L}{S(0)}\right]^{-\alpha} = \frac{\kappa K^{-(\beta-n)}L^{\beta}}{\beta-n}\left[\frac{L}{S(0)}\right]^{-\alpha},$$

$$A_8 = \frac{\lambda}{D}\frac{K}{-\alpha(1-\alpha)(\beta-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha} = \frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha},$$

Notation

$$
\begin{aligned}
A_9 &= \frac{\lambda}{D}\frac{K^{1-\alpha}L^{\alpha}}{-\alpha(1-\alpha)(\beta-\alpha)}\left[\frac{S(0)}{L}\right]^{\beta} \\
&= \frac{\kappa K^{1-\alpha}L^{\alpha}}{-\alpha(1-\alpha)}\left[\frac{S(0)}{L}\right]^{\beta}, \\
A_{10} &= \frac{\lambda}{D}\frac{K}{\beta(\beta-1)(\beta-\alpha)}\left[\frac{S(0)}{K}\right]^{\beta} = \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}, \\
A_{11} &= \frac{\lambda}{D}\frac{K^{-(\beta-1)}L^{\beta}}{\beta(\beta-1)(\beta-\alpha)}\left[\frac{L}{S(0)}\right]^{-\alpha} \\
&= \frac{\kappa K^{-(\beta-1)}L^{\beta}}{\beta(\beta-1)}\left[\frac{L}{S(0)}\right]^{-\alpha}.
\end{aligned}
$$

Up-and-out all-or-nothing call option

The option value is

$$
\begin{cases}
0, & \text{if } L < K, \\
\frac{\lambda}{D} \int_0^{\ell} [\int_k^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L \geq K \text{ and } S(0) > K, \\
\frac{\lambda}{D} \int_k^{\ell} [\int_k^y S(0)^n e^{nx} e^{-\alpha x} \mathrm{d}x] e^{-(\beta-\alpha)y} \mathrm{d}y, & \text{if } L \geq K \text{ and } S(0) \leq K
\end{cases}
$$

$$
= \begin{cases}
0, & \text{if } L < K, \\
A_1(n) - A_2(n) - A_4 + A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\
A_6 - A_2(n) + A_5, & \text{if } L \geq K \text{ and } S(0) \leq K.
\end{cases}
$$

Up-and-out all-or-nothing put option

The option value is

$$\begin{cases} \frac{\lambda}{D}\int_0^{\ell}[\int_{-\infty}^{y} S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x]e^{-(\beta-\alpha)y}\mathrm{d}y, & \text{if } L < K, \\ \frac{\lambda}{D}\int_0^{\ell}[\int_{-\infty}^{k} S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x]e^{-(\beta-\alpha)y}\mathrm{d}y, & \text{if } L \geq K \& S(0) > K \\ \frac{\lambda}{D}\{\int_0^{k}[\int_{-\infty}^{y} S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x]e^{-(\beta-\alpha)y}\mathrm{d}y & \\ \quad + \int_k^{\ell}[\int_{-\infty}^{k} S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x]e^{-(\beta-\alpha)y}\mathrm{d}y\}, & \text{if } L \geq K \& S(0) \leq K \end{cases}$$

$$= \begin{cases} A_1(n) - A_2(n), & \text{if } L < K, \\ A_4 - A_5, & \text{if } L \geq K \text{ and } S(0) > K, \\ A_1(n) - A_5 - A_6, & \text{if } L \geq K \text{ and } S(0) \leq K. \end{cases}$$

up-and-out option with payoff $S(\tau)^n$

$$\frac{\lambda}{D}\int_0^{\ell}\left[\int_{-\infty}^{y} S(0)^n e^{nx} e^{-\alpha x}\mathrm{d}x\right] e^{-(\beta-\alpha)y}\mathrm{d}y = A_1(n) - A_2(n).$$

This is the sum of the value of the up-and-out all-or-nothing put option and the value of the up-and-out all-or-nothing call option.

Up-and-out call option

The value is

$$\begin{cases} 0, & \text{if } \; L < K, \\ A_1(1) - A_2(1) - A_1(0)K & \\ +A_2(0)K + A_8 - A_9, & \text{if } \; L \geq K \text{ and } S(0) > K, \\ A_2(0)K + A_{10} - A_2(1) - A_9, & \text{if } \; L \geq K \text{ and } S(0) \leq K. \end{cases}$$

Up-and-out put option

The value is

$$\begin{cases} A_1(0)K - A_2(0)K - A_1(1) + A_2(1), & \text{if} \;\; L < K, \\ A_8 - A_9, & \text{if} \;\; L \geq K \text{ and } S(0) > K, \\ A_1(0)K - A_1(1) + A_{10} - A_9, & \text{if} \;\; L \geq K \text{ and } S(0) \leq K. \end{cases}$$

Double barrier option

Payoff:

$$\pi(S(\tau))I\{a < m(\tau), M(\tau) < b\}$$

Double barrier option

Using a formula from B & S (2002), we have

$$
\begin{aligned}
& E[b(S(0)e^{X(\tau)})I\{a < m(\tau), M(\tau) < b\}] \\
= \; & \frac{\kappa}{\Xi - \Xi^{-1}} \Big[\Xi \int_a^0 b(S(0)e^z)e^{-\alpha z} dz + \Xi^{-1} \int_a^0 b(S(0)e^z)e^{-\beta z} dz \\
& + \Xi \int_0^b b(S(0)e^z)e^{-\beta z} dz + \Xi^{-1} \int_0^b b(S(0)e^z)e^{-\alpha z} dz \\
& - \Upsilon^{-1} \int_a^b b(S(0)e^z)e^{-\alpha z} dz - \Upsilon \int_a^b b(S(0)e^z)e^{-\beta z} dz \Big],
\end{aligned}
$$

where $\Xi = e^{\frac{1}{2}(b-a)(\beta-\alpha)}$ and $\Upsilon = e^{\frac{1}{2}(a+b)(\beta-\alpha)}$.

Double barrier option

When $\delta > 0$, we have

$$\begin{aligned} & E[e^{-\delta\tau} b(S(0)e^{X(\tau)}) I\{a < m(\tau), M(\tau) < b\}] \\ = \ & \lambda \int_0^\infty e^{-(\lambda+\delta)t} b(S(0)e^{X(t)}) I\{a < m(t), M(t) < b|\} dt \\ = \ & \frac{\kappa}{\Xi - \Xi^{-1}} \Big[\Xi \int_a^0 b(S(0)e^z) e^{-\alpha z} dz + \Xi^{-1} \int_a^0 b(S(0)e^z) e^{-\beta z} dz \\ & + \Xi \int_0^b b(S(0)e^z) e^{-\beta z} dz + \Xi^{-1} \int_0^b b(S(0)e^z) e^{-\alpha z} dz \\ & - \Upsilon^{-1} \int_a^b b(S(0)e^z) e^{-\alpha z} dz - \Upsilon \int_a^b b(S(0)e^z) e^{-\beta z} dz \Big]. \end{aligned}$$

Several stocks

- $$\mathbf{X}(t) = (X_1(t), X_2(t), \cdots, X_n(t))'$$

  n-dimensional Brownian motion.
- $\boldsymbol{\mu}$ the mean vector
- $\mathbf{C}$ the covariance matrix of $\mathbf{X}(1)$
- $$g_t(\mathbf{X})$$

  a real-valued functional of the process up to time t.
- $\mathbf{h}$ an n-dimensional vector of real numbers

$$\mathsf{E}[e^{-\delta\tau}e^{\mathbf{h}'\mathbf{X}(\tau)}g_\tau(\mathbf{X})] = \mathsf{E}[e^{-\delta(\mathbf{h})\tau}g_\tau(\mathbf{X});\mathbf{h}], \tag{3}$$

where

$$\begin{aligned}\delta(\mathbf{h}) &= \delta - \ln[M_{\mathbf{X}(1)}(\mathbf{h})] \\ &= \delta - \mathbf{h}'\mu - \frac{1}{2}\mathbf{h}'\mathbf{C}\mathbf{h}.\end{aligned}$$

Proof of (3)

- Conditioning on $\tau = t$, the LHS (3) is

$$\int_0^\infty e^{-\delta t} \mathrm{E}[e^{\mathbf{h}'\mathbf{X}(t)} g_t(\mathbf{X})] f_\tau(t) \mathrm{d}t.$$

- By the factorization formula in the method of Esscher transforms, the expectation inside the integrand can be written as the product of two expectations,

$$\begin{aligned} &\mathrm{E}[e^{\mathbf{h}'\mathbf{X}(t)}] \times \mathrm{E}[g_t(\mathbf{X}); \mathbf{h}] \\ &= [M_{\mathbf{X}(1)}(\mathbf{h})]^t \times \mathrm{E}[g_t(\mathbf{X}); \mathbf{h}]. \end{aligned}$$

- Hence

$$\int_0^\infty e^{-\delta t} \mathrm{E}[e^{\mathbf{h}'\mathbf{X}(t)} g_t(\mathbf{X})] f_\tau(t) \mathrm{d}t = \int_0^\infty e^{-\delta(\mathbf{h})t} \mathrm{E}[g_t(\mathbf{X}); \mathbf{h}] f_\tau(t) \mathrm{d}t.$$

Application of (3)

- $\mathbf{k}$ n-dimensional vector of real numbers
- $q_t(\mathbf{k}'\mathbf{X})$ real-valued functional of the process up to time t
- $$\mathrm{E}[e^{-\delta\tau}e^{\mathbf{h}'\mathbf{X}(\tau)}q_\tau(\mathbf{k}'\mathbf{X})] = \mathrm{E}[e^{-\delta(\mathbf{h})\tau}q_\tau(\mathbf{k}'\mathbf{X}); \mathbf{h}].$$
- The quadratic equation becomes

$$\begin{aligned}&\tfrac{1}{2}\mathrm{Var}[\mathbf{k}'\mathbf{X}(1); \mathbf{h}]\xi^2 + \mathrm{E}[\mathbf{k}'\mathbf{X}(1); \mathbf{h}]\xi - [\lambda + \delta(\mathbf{h})] \\ &= \tfrac{1}{2}\mathbf{k}'\mathbf{C}\mathbf{k}\xi^2 + \mathbf{k}'(\mu + \mathbf{C}\mathbf{h})\xi - (\lambda + \delta - \mathbf{h}'\mu - \tfrac{1}{2}\mathbf{h}'\mathbf{C}\mathbf{h})\end{aligned}$$

Special case: $n = 2$

- $S_1(t) = S_1(0)e^{X_1(t)}$ and $S_2(t) = S_2(0)e^{X_2(t)}$
- $\mu = (\mu_1, \mu_2)'$
- $$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

Margrabe option

Payoff:

$$[S_1(\tau) - S_2(\tau)]_+. \tag{4}$$

If we rewrite (4) as

$$e^{X_2(\tau)}[S_1(0)e^{X_1(\tau)-X_2(\tau)} - S_2(0)]_+,$$

$$\mathrm{E}[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ | S_1(0) < S_2(0)] = \frac{\kappa^* S_2(0)}{\beta^*(\beta^* - 1)} \left[\frac{S_1(0)}{S_2(0)}\right]^{\beta^*}.$$

Here,

$$\begin{aligned} \kappa^* &= \frac{\lambda}{D^*(\beta^* - \alpha^*)}, \\ D^* &= \tfrac{1}{2}\mathrm{Var}[X_1(1) - X_2(1)] = \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2), \end{aligned}$$

and $\alpha^* < 0$ and $\beta^* > 0$ are the zeros of

$$\begin{aligned} & D^*\xi^2 + (\mu_1 - \mu_2 + \rho\sigma_1\sigma_2 - \sigma_2^2)\xi - (\lambda + \delta - \mu_2 - \tfrac{1}{2}\sigma_2^2) \\ & = \ln[M_{\mathbf{X}(1)}((\xi, 1 - \xi)')] - (\lambda + \delta). \end{aligned}$$

- If we write (4) as

$$e^{X_1(\tau)}[S_1(0) - S_2(0)e^{X_2(\tau)-X_1(\tau)}]_+,$$

-

$$\begin{aligned} & \mathrm{E}[e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ | S_1(0) < S_2(0)] \\ &= \frac{\kappa^{**} S_1(0)}{-\alpha^{**}(1-\alpha^{**})} \left[\frac{S_1(0)}{S_2(0)}\right]^{-\alpha^{**}}. \end{aligned}$$

Here,

$$\begin{aligned} \kappa^{**} &= \frac{\lambda}{D^{**}(\beta^{**} - \alpha^{**})}, \\ D^{**} &= \tfrac{1}{2}\mathrm{Var}[X_2(1) - X_1(1)] = D^*, \end{aligned}$$

and $\alpha^{**} < 0$ and $\beta^{**} > 0$ are the zeros of

$$\ln[M_{\mathbf{X}(1)}((1-\xi, \xi)')] - (\lambda + \delta).$$

Hence

$$\alpha^* = 1 - \beta^{**}$$

and

$$\beta^* = 1 - \alpha^{**}.$$

Thus, $\kappa^* = \kappa^{**}$