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ON THE ERROR TERM IN AN ASYMPTOTIC FORMULA FOR THE SYMMETRIC SQUARE L-FUNCTION

YUK-KAM LAU

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Abstract. Recently Wu proved that for all primes q,

$$\sum_f L(1, \mathrm{sym}^2 f) = \frac{\pi^4}{432} q + O(q^{27/28} \log^B q)$$

where f runs over all normalized newforms of weight 2 and level q. Here we show that 27/28 can be replaced by 9/10.

1. Introduction

Let q be a prime and

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : q|c \right\}.$$

We denote by $S_2(q)$ the space of all holomorphic cusp forms for $\Gamma_0(q)$ of weight 2. With respect to the inner product

$$\langle f, g \rangle \quad = \quad \int_{\Gamma_0(q)\backslash\mathbf{H}} f(z)\overline{g(z)}\, dxdy,$$

$S_2(q)$ is a finite-dimensional Hilbert space, and there is an orthogonal basis $\mathcal{B}_2(q)$ (which is the set of all normalized newforms in $S_2(q)$) such that

(i) each $f \in \mathcal{B}_2(q)$ is a common eigenvector of all Hecke operators T_n with $(n, q) = 1$, i.e. when $f \in \mathcal{B}_2(q)$ and $(n, q) = 1$,

$$T_n f = \lambda_f(n) f;$$

(ii) the Fourier expansion of $f \in \mathcal{B}_2(q)$ is

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) \sqrt{n} e(nz)$$

where $e(\alpha) = e^{2\pi i\alpha}$, $\lambda_f(n)$ is the eigenvalue in (i) if $(n, q) = 1$ and $\lambda_f(n)^2 = l^{-1}\lambda_f(m)^2$ if $n = lm$ where l is a power of q and $(m, q) = 1$ (see [3, (2.19) and (2.24)]).

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For the properties of $\lambda_f(n)$, it is known that they are all real and satisfy the Deligne bound $|\lambda_f(n)| \le \tau(n)$. (Here and in the sequel $\tau(n) = \sum_{d|n} 1$ is the divisor function.) Moreover we have

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \epsilon_q(d)\lambda_f(\frac{mn}{d^2}) \tag{1}$$

where ϵ_q is the principal character mod q. In particular, we see that $\lambda_f(1) = 1$. Associated to each $f \in \mathcal{B}_2(q)$, we define the symmetric square L-function by

$$L(s, \mathrm{sym}^2 f) = \zeta_q(2s)\sum_{n=1}^{\infty} \lambda_f(n^2)n^{-s} \qquad \text{for } \Re e\, s > 1, \tag{2}$$

where $\zeta_q(s) = \prod_{p\nmid q}(1-p^{-s})^{-1}$. This L-function extends to an entire function over $\mathbb{C}$ and it satisfies a functional equation; more precisely, let us write

$$\Lambda(s, \mathrm{sym}^2 f) = \left(\frac{q}{\pi^{3/2}}\right)^s \Gamma(\frac{s+1}{2})^2\Gamma(\frac{s+2}{2})L(s, \mathrm{sym}^2 f). \tag{3}$$

Then we have $\Lambda(s, \mathrm{sym}^2 f) = \Lambda(1-s, \mathrm{sym}^2 f)$. Analogous to the Riemann zeta function, the values attained by $L(s, \mathrm{sym}^2 f)$ in the critical strip are interesting. Particularly for $s = 1$ and all large prime q, we have the asymptotic formula

$$\sum_{f\in\mathcal{B}_2(q)} L(1, \mathrm{sym}^2 f) = \frac{\pi^4}{432}q + O(q^\alpha \log^\beta q)$$

for some constants $0 < \alpha < 1$ and $\beta > 0$. Here, we are concerned with the size of the error term. In [1], Akbary proved that $\alpha = 45/46$ is admissible and recently Wu gave an improvement to $\alpha = 27/28$ (see [5]). Our purpose is to show the refinement below.

Theorem. *Let q be a prime. There is an absolute constant $c > 0$ such that*

$$\sum_{f\in\mathcal{B}_2(q)} L(1, sym^2 f) = \frac{\zeta(2)^3}{2\pi^2}q + O(q^{9/10}\log^c q).$$

(Note that $\zeta(2)^3/(2\pi^2) = \pi^4/432$.)

Remark. In decimal form we have $\frac{45}{46} \approx 0.978$, $\frac{27}{28} \approx 0.964$ and $\frac{9}{10} = 0.9$.

2. Some preparation

Lemma 1. *Let $A > 1$ be any fixed constant and $q \ll y \ll q^A$ but $y \notin \mathbb{Z}$. We have*

$$L(1, sym^2 f) = \zeta_q(2)\sum_{n\le y}\frac{\lambda_f(n^2)}{n} + O(q^\epsilon(y^{-1} + (\frac{q}{y})^{2/7}))$$

where $\epsilon > 0$ is an arbitrarily small constant and the implied constant in the O-term depends on ϵ.

Proof. This follows from the truncated Perron's formula. Using the estimate

$$\Gamma(\frac{s+1}{2})^2\Gamma(\frac{s+2}{2}) \asymp |t|^{(3\sigma+1)/2}e^{-3\pi|t|/4}$$

for $s = \sigma + it$ where $\sigma \ll 1$ and $|t| \gg 1$, we can derive from the functional equation the convexity bound: for $0 \le \sigma \le 1$,

$$L(\sigma + it, \mathrm{sym}^2 f) \ll (q|t|^{3/2})^{1-\sigma+\epsilon}. \tag{4}$$

By [2, Lemma 12.1], we see that for any $T \gg 1$,

$$\begin{aligned} &\zeta_q(2)\sum_{n\le y}\frac{\lambda_f(n^2)}{n} \\ (5)\quad = \;& \frac{\zeta_q(2)}{2\pi i}\int_{\epsilon-iT}^{\epsilon+iT}\sum_{n=1}^{\infty}\frac{\lambda_f(n^2)}{n^{1+s}}\frac{y^s}{s}\,ds + O\Big(y^\epsilon\sum_{n=1}^{\infty}\frac{\tau(n)^2}{n^{1+\epsilon}}\min(1,(T|\log\frac{y}{n}|)^{-1})\Big).\end{aligned}$$

To evaluate the O-term, we split the summation over n into three pieces: $n \le y/2$, $n \ge 3y/2$ and $y/2 < n < 3y/2$. As $|\log(y/n)| \gg 1$ in the first two pieces, these two sums are $O(T^{-1}y^\epsilon)$. The third one is

$$\ll y^\epsilon T^{-1}\sum_{\substack{y/2<n<3y/2\\ |n-y|\ge 1}}|y-n|^{-1} + y^{-1+\epsilon} \ll y^\epsilon(T^{-1}+y^{-1}).$$

Thus the overall contribution is absorbed in the O-term in our lemma.

From (2), we can replace $\sum_{n=1}^{\infty}\lambda_f(n^2)n^{-(1+s)}$ in (5) by

$$\zeta_q(2+2s)^{-1}L(1+s,\mathrm{sym}^2 f).$$

Then we apply the residue theorem to the rectangular contour with vertices at $\epsilon \pm iT$ and $-1/2+\epsilon\pm iT$. The integral in (5) equals a sum of two terms: the main term $L(1,\mathrm{sym}^2 f)$ from the pole at $s=0$, and the remainder term which is

$$\begin{aligned}&\ll \int_{-1/2+\epsilon}^{\epsilon}\left|\frac{L(1+\alpha+iT,\mathrm{sym}^2 f)}{\zeta_q(2+2\alpha+i2T)}\right|\frac{y^\alpha}{T}\,d\alpha + y^{-1/2+\epsilon}\\ &\quad\cdot\int_{-T}^{T}\left|\frac{L(1/2+\epsilon+it,\mathrm{sym}^2 f)}{\zeta_q(1+2\epsilon+i2t)}\right|\frac{dt}{1+|t|}.\end{aligned}$$

Using the bound $\zeta(\sigma+it)^{-1} \ll \log(1+|t|)$ for $\sigma \ge 1$ and $|t| \gg 1$, the two O-terms are $\ll (qT)^\epsilon(y^{-1/2}q^{1/2}T^{3/4}+T^{-1})$. The proof is complete after setting $T=(y/q)^{2/7}$.

Our next task is to extend the admissible range in [5, Lemma 2]. To this end, we modify the mean square estimate result in [4, Corollary 1]. Suppose $M \le q^9$ and $\{a_n\}_{1\le n\le M}$ is a sequence of complex numbers. Then by taking $a_n = 0$ for $M < n \le q^9$, [4, Proposition 1] with $N = q^9$ gives

$$(6)\qquad \sum_{f\in\mathcal{B}_2(q)}\left|\sum_{n\le M}a_n\rho_f(n)\right|^2 \ll q^9(\log q)^{15}\sum_{n\le M}|a_n|^2$$

where $\rho_f(n)=\sum_{lm^2=n}\epsilon_q(m)\lambda_f(l^2)$. (Note that $\mathcal{B}_2(q) = S_2(q)^*$ in [4] for prime q.)

Lemma 2. *Let $M \gg 1$ and suppose that $\{a(n)\}_{M<n\le 2M}$ satisfies*

$$a(n) \ll \frac{(\tau(n)\log n)^A}{n}$$

for some constant $A > 0$. There exists a constant $B = B(A) \ge 0$ such that

$$\sum_{f\in\mathcal{B}_2(q)}\left|\sum_{M<n\le 2M}a(n)\lambda_f(n^2)\right|^2 \ll \max(1,q^9M^{-1})\log^B(qM).$$

The implied constant depends on A.

Proof. When $M \geq q^9$, it follows immediately from [4, Corollary 1] (by taking $N = M$). Consider the case $M < q^9$. From [4, (16)], we have

$$S := \sum_{f \in \mathcal{B}_2(q)} \left| \sum_{M < n \leq 2M} a(n)\lambda_f(n^2) \right|^2 = \sum_{f \in \mathcal{B}_2(q)} \left| \sum_{l < 2M} a_l \rho_f(l) \right|^2$$

where

$$\begin{aligned} a_l &= \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \mu(m)\epsilon_q(m) a(lm^2) \\ &\ll \frac{(\tau(l)\log 2l)^A}{l} \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \frac{(\tau(m)\log 2m)^{2A}}{m^2} \\ &\ll (Ml)^{-1/2}(\log Ml)^B \end{aligned}$$

(see the proof of [4, Corollary 1] as well). B denotes an unspecified positive constant depending on A and its value may differ at each occurrence in the proof. By (6),

$$S \ll q^9 (\log q)^{15} \sum_{l < 2M} (Ml)^{-1} (\log Ml)^B \ll q^9 M^{-1} \log^B (qM).$$

□

Define for $1 \leq x < y$,

$$\omega_f(x, y) = \sum_{x \leq n < y} \frac{\lambda_f(n^2)}{n}.$$

Lemma 3. *Let $x > 0$ and $x < y \ll q^A$ for some constant $A > 0$. Suppose $r \geq 1$ is a fixed integer satisfying $x^r \geq q^9$. Then there exists a constant $D = D(r) > 0$ such that*

$$\sum_{f \in \mathcal{B}_2(q)} \omega_f(x, y)^{2r} \ll (\log q)^D$$

where the implied constant depends on A and r.

Proof. Following the argument in the proof of [4, Lemma 4], one can show that

$$\omega_f(x, y)^r = \sum_{x^r \leq mn < y^r} \lambda_f(m^2) \frac{c(m, n)}{mn}$$

where $c(m, n)$ is independent of f and $c(m, n) = 0$ if n is not of the form $n = dn_1$ where $d|m$ and n_1 is squarefull. Moreover, $|c(m, n)| \leq \tau(mn)^\gamma$ for some integer $\gamma = \gamma(r) > 0$ depending on r. Then we write

$$\omega_f(x, y)^r = \sum_{H = 2^k} \sum_{\substack{x^r \leq mn < y^r \\ H \leq n < 2H}} \lambda_f(m^2) \frac{c(m, n)}{mn}$$

where the first summation runs over all nonnegative integers k. Define

$$c_H(m) = \sum_{\substack{H \leq n < 2H \\ x^r m^{-1} \leq n < y^r m^{-1}}} \frac{c(m, n)}{n}.$$

Then, using $\sum_{\substack{n\le z\\ \text{squarefull}}} \tau(n)^\gamma \ll z^{1/2}(\log z)^{2^\gamma}$, we have

$$\begin{aligned} c_H(m) &\ll \tau(m)^\gamma \sum_{d|m} \frac{1}{d} \sum_{\substack{H\le dn<2H\\ n\,\text{squarefull}}} \frac{\tau(n)^\gamma}{n} \\ &\ll \tau(m)^\gamma \Big(\sum_{\substack{d|m\\ d>\sqrt{H}}} d^{-1} + \sum_{\substack{d|m\\ d<\sqrt{H}}} d^{-1} \sum_{\substack{H/d\le n<2H/d\\ n\,\text{squarefull}}} \frac{\tau(n)^\gamma}{n} \Big) \\ &\ll H^{-1/2}(\tau(m)(\log m)(\log H))^D. \end{aligned} \tag{7}$$

Here we use D to denote a positive constant (depending on r) which may assume different values at other places. Making use of (7) for $H \ge q$,

$$\omega_f(x,y)^r = \sum_{H=2^k<q} \sum_{x^r/(2H)<m\le y^r/H} \lambda_f(m^2)\frac{c_H(m)}{m} + O(q^{-1/2}\log^D q).$$

Squaring both sides and averaging over all $f \in \mathcal{B}_2(q)$ yields

$$\begin{aligned} &\sum_{f\in\mathcal{B}_2(q)} \omega_f(x,y)^{2r} \\ \ll &\left(\sum_{H=2^k<q} H^{-1} \sum_f \left| \sum_{x^r/(2H)<m\le y^r/H} \lambda_f(m^2) \frac{c_H(m)\sqrt{H}}{m} \right|^2 + 1 \right) \log^D q \end{aligned} \tag{8}$$

as $(\sum_{i\in I} a_i)^2 \ll |I|\sum_{i\in I} a_i^2$ and $|\mathcal{B}_2(q)| \ll q$. For each H, we split the range of the summation over m into dyadic intervals $M < m \le 2M$ where $M \ge x^r/(2H)$. It follows from Lemma 2 and (7) that

$$\sum_f \left| \sum_{x^r/(2H)<m\le y^r/H} \lambda_f(m^2)\frac{c_H(m)\sqrt{H}}{m} \right|^2 \ll \max(1, q^9 x^{-r} H)\log^D q.$$

Inserting it into (8), we conclude that

$$\sum_{f\in\mathcal{B}_2(q)} \omega_f(x,y)^{2r} \ll \log^D q \sum_{H=2^k<q} \max(H^{-1}, q^9 x^{-r}),$$

and our result follows in view of the condition $x^r \ge q^9$. □

3. Proof of the Theorem

Define for $f \in \mathcal{B}_2(q)$, $w_f = 4\pi\langle f, f\rangle$, which is a positive real number. We have from [3, Lemma 2.5] that $w_f = (2\pi^2)^{-1} q L(1, \mathrm{sym}^2 f)$ and from [3, Corollary 2.2] (with $\tau_3((m,n)) \le \tau((m,n))^2 \le \tau(m)\tau(n)$) that

$$\sum_{f\in\mathcal{B}_2(q)} w_f^{-1}\lambda_f(m^2)\lambda_f(n^2) = \delta(m,n) + O(q^{-1}(mn)^{1/2}(\tau(m)\tau(n))^2\log 2mn) \tag{9}$$

for $\min(m,n) < q$, where $\delta(\cdot,\cdot)$ is the Kronecker delta. (Note that $w_f^{-1} = \omega_f$ in [4].) In particular, $\sum_f w_f^{-1} \ll 1$ as $\lambda_f(1) = 1$.

We split the sum over n in Lemma 1 into two subsums $\sum_{n\le x} + \sum_{x<n\le y}$ where $1 < x < q < y$. (Our choice will be $x = q^{9/10}$ and $y = q^{173/110}$.) Squaring the formula in Lemma 1 together with the bound $L(1,\mathrm{sym}^2 f)\ll \log^3 q$ (from [4, (18)]), we deduce that

$$\begin{aligned}\sum_{f\in\mathcal{B}_2(q)} L(1,\mathrm{sym}^2 f) &= \frac{q}{2\pi^2}\sum_{f\in\mathcal{B}_2(q)} w_f^{-1}L(1,\mathrm{sym}^2 f)^2\\ &= \frac{q}{2\pi^2}\zeta_q(2)(S_1+2S_2+S_3)+O(q^\epsilon(y^{-1}+(\frac{q}{y})^{2/7}))\end{aligned}\tag{10}$$

where

$$\begin{aligned}S_1 &= \sum_f w_f^{-1}\left(\sum_{n\le x}\frac{\lambda_f(n^2)}{n}\right)^2,\\ S_2 &= \sum_f w_f^{-1}\left(\sum_{n\le x}\frac{\lambda_f(n^2)}{n}\right)\left(\sum_{x<n\le y}\frac{\lambda_f(n^2)}{n}\right),\\ S_3 &= \sum_f w_f^{-1}\left(\sum_{x<n\le y}\frac{\lambda_f(n^2)}{n}\right)^2.\end{aligned}$$

It follows from the bound $w_f^{-1}\ll q^{-1}\log q$ (see [4, (20)]) and Lemma 3 that if $x^r\ge q^9$,

$$\begin{aligned}S_3 &\ll \frac{\log q}{q}\sum_f \omega_f(x,y)^2 \ll (\sum_f \omega_f(x,y)^{2r})^{1/r}|\mathcal{B}_2(q)|^{1-1/r}q^{-1}\log q\\ &\ll q^{-1/r}\log^{c_{11}} q.\end{aligned}\tag{11}$$

Throughout c_i, $i=1,2,\cdots$, denote unspecified positive constants. Using (9), we obtain that for $x<q$,

$$\begin{aligned}S_1 &= \sum_{n\le x} n^{-2}+O(q^{-1}\sum_{m,n\le x}(mn)^{-1/2}\tau(m)^2\tau(n)^2\log 2mn)\\ &= \zeta(2)+O(x^{-1}+q^{-1}x\log^{c_{11}} q).\end{aligned}\tag{12}$$

To treat S_2, we split it into two parts: let $z=qx^{-1}$,

$$\begin{aligned}S_2 &= \sum_f w_f^{-1}\sum_{n\le z}\frac{\lambda_f(n^2)}{n}\sum_{x<n\le y}\frac{\lambda_f(n^2)}{n}+\sum_f w_f^{-1}\sum_{z<n\le x}\frac{\lambda_f(n^2)}{n}\sum_{x<n\le y}\frac{\lambda_f(n^2)}{n}\\ &= S_{21}+S_{22},\ \text{say}.\end{aligned}\tag{13}$$

By (9), we have, provided that $z\le x$ (or equivalently $x\ge q^{1/2}$),

$$S_{21} \ll q^{-1}(\log^{c_{11}} q)\sum_{m\le z}\sum_{n\le y}\tau(m)^2\tau(n)^2(mn)^{-1/2}\ll\sqrt{\frac{y}{qx}}\log^{c_{11}} q.$$

Applying the argument in (12), we get that

$$\sum_f w_f^{-1}\left(\sum_{z<n\le x}\frac{\lambda_f(n^2)}{n}\right)^2 \ll z^{-1}+q^{-1}x\log^{c_{11}} q\ll q^{-1}x\log^{c_{11}} q.$$

By $ab \ll |a|^2 + |b|^2$ and (11), we have $S_{22} \ll (q^{-1/r} + q^{-1}x)\log^{c_{11}} q$. Hence, by (13),

$$S_2 \ll (q^{-1/r} + q^{-1}x + (\frac{y}{qx})^{1/2}) \log^{c_{11}} q.$$

Putting this estimate, (11) and (12) into (10), we infer that as $\zeta_q(2) = \zeta(2) + O(q^{-2})$,

$$\sum_{f \in \mathcal{B}_2(q)} L(1, \mathrm{sym}^2 f) = \frac{\zeta(2)^3}{2\pi^2} q + qO\big((q^{-1/r} + q^{-1}x)\log^{c_{11}} q \\ + q^\epsilon(x^{-1} + (\frac{y}{qx})^{1/2} + (\frac{q}{y})^{2/7})\big).$$

Subject to the condition $x^r \geq q^9$, we take $x = q^{9/r}$ and select $r = 10$, $x = q^{9/10}$ and $y = q^{173/110}$ by equating $q^{-1/r} = q^{-1}x$. This ends the proof.

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Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

E-mail address: yklau@maths.hku.hk