A Simplified Aperiodic Cross-Correlation Model for Direct-Sequence Spread-Spectrum Multiple-Access Communication Systems

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ABSTRACT In this paper, the discrete probability density function (pdf) of the average aperiodic cross-correlation function (ACF) is modeled by a triangular function to simplify the evaluation of the bit-error-rate (BER) performance. Both the pdf of ACF for random binary sequence and the pdf of ACF for Gold sequence were employed to examine the simplified linear model. The pdf of multiple-access inteference (MAI) component obtained from probablistic approach was employed to investigate the discrepancy among the approximations. It has been shown that the sum of square error between the average pdf of the MAI component for Gold sequence and that of the simplified linear model was 0.0683% which was negligable, and thus the simplified linear model can be used to simplify the evaluation of the BER performance of an asynchronous direct-sequence spread-spectrum multiple-access (A-DS/SSMA) communication system.

I. INTRODUCTION

In asynchronous direct-sequence spread-spectrum multiple-access (A-DS/SSMA) communication systems, the prime factor that governs the bit-error-rate (BER) performance of the systems is the multiple-access interference (MAI) which depends on the magnitudes of aperiodic cross-correlation function (ACF) between any two pseudo-noise (PN) user sequences in a set. The computation of the MAI involves ${}_K C_2$ cross-correlation functions, where ${}_n C_r = n!/[r!(n-r)!]$, and K is the number of users in the system. Such a computation becomes significant when the spreading period N is large. Thus, a simplified linear model for the discrete probability density function (pdf) of the average ACF is proposed to alleviate the computational complexity. The pdf of the MAI component of a binary phase-shift-keying (BPSK) A-DS/SSMA communications system is employed to verify the model.

The pdf of the average ACF deploying six different sets of well-known PN sequences were examined in Section II, respectively. In Section III, the simplified linear model for the ACF was introduced, including the pdf of the ACF for random binary sequence. The BPSK A-DS/SSMA communication system was employed to examine the pdf of the MAI component in Section IV. In Section V, the simplified linear model was analyzed and compared with the pdf's using Gold sequences and random binary sequence, respectively.

II. THE ACF OF PN SEQUENCES

For a set X of K binary PN sequences, the ACF between any two sequences $\mathbf{x}$, and $\mathbf{y}$ in the set, where $\mathbf{x} = (x_0, x_1, x_2, \cdots, x_{N-1})$ and $\mathbf{y} = (y_0, y_1, y_2, \cdots, y_{N-1})$ are two sequence vectors of period N, is given by [6]

$$C_{x,y}(l) = \begin{cases} \sum_{i=0}^{N-1-l} x_i y_{i+l}, & 0 \le l \le N-1, \\ \sum_{i=0}^{N-1-l} x_{i-l} y_i, & 1-N \le l < 0, \\ 0, & |l| \ge N. \end{cases} \tag{1}$$

The pdf's of the six different sets of PN sequences are obtained by taking the frequency of the magnitude of ACF for all possible combinations of sequence-pairs in a set, as shown in Figure 1 where $N = 63$. Observe that all the pdf 's resemble a triangular function.

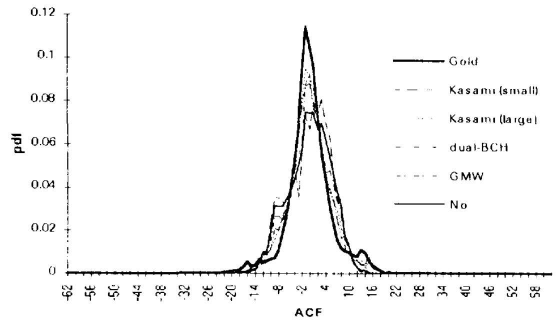

Figure 1. The pdf of ACF for different sets of PN sequence of $N = 63$.

Table 1 depicts the statistical properties of the pdf's. Note that the highest frequencies of the pdf's are located at the vicinity of the zero-value of ACF. The magnitudes of the minimum and maximum values of ACF are marginally the same except for the GMW sequence. Based on the observation above, the pdf of average ACF can be approximated by a symmetrical triangular function about the origin.

$N = 63$	Gold	small set Kasami	large set Kasami	dual-BCH	GMW	No
total number of sequences	65	8	520	64	12	8
$\max_{\lvert l\rvert<N} C_{x,y}(l)$	20	15	23	18	26	15
$\min_{\lvert l\rvert<N} C_{x,y}(l)$	-22	-17	-24	-18	-33	-17
max. value of pdf $= h$	0.114	0.0811	0.0948	0.0880	0.0924	0.0749
at $C_{x,y}(l)$	-1	3	-1	0	0	-1

Table 1. The statistical properties of the pdf of ACF for different sets of PN sequences.

III. Discrete PDF of ACF

A. Simplified linear model

Suppose that the pdf of ACF can be approximated by a discrete symmetrical triangular function given by

$$\mathrm{P}^{0}_{\text{simp}}\left\{C_{x,y}(l)=z\right\}=\begin{cases}\dfrac{1}{\left(\hat{C}+1\right)^{2}}\left[\hat{C}+1-|z|\right], & \forall|z|\le\hat{C}\\ 0, & \text{otherwise}\end{cases} \qquad (2)$$

where $\hat{C}=\max\limits_{|l|<N}C_{x,y}(l)$ is a positive integer less than $N-2$ and its frequency is greater than 0. The value of $\hat{C}$ can be found from Table 1. For instance, the pdf of ACF is symmetrical about z = -1 for the Gold sequence of N = 63, and $\hat{C}$ = 20. When $\hat{C}=20$, the largest value of discrete pdf, $h=1/\left(\hat{C}+1\right)=0.0476$, as the total cumulative probability is equal to 1. This value of h is much smaller than h = 0.114 found in Table 1. Thus, another approach to find a more appropriate value of $\hat{C}$ is shown as follows. Assume that the simplified linear model is symmetrical about the origin (i.e., $z=0$), the first moment of the random variable (rv) z from equation (2) can be easily derived as,

$$\mathrm{E}\left\{C_{x,y}(l)\right\}=0, \qquad (3)$$

where $\mathrm{E}\{\cdot\}$ denotes the expectation. The second moment of z is given by

$$\mathrm{E}\left\{C^{2}_{x,y}(l)\right\}=\hat{C}\left(\hat{C}+2\right)/6, \qquad (4)$$

and the covariance is

$$\mathrm{E}\left\{z\cdot z'\right\}=-\frac{\hat{C}\left(\hat{C}+2\right)\left(\hat{C}^{2}+2\hat{C}+2\right)}{15\left(\hat{C}+1\right)^{3}}, \quad \text{for } z\ne z'. \qquad (5)$$

Higher order of moments of z about the origin can be easily derived into close-form expressions as the moments consists of only the terms in the form of $\sum_{z=1}^{\hat{C}}z^{r}$. According to the theory of algebra, a close-form solution can be obtained when both z and r are integers.

B. Random Binary Sequences

For a random binary sequence $\mathbf{x}$ of period N, the probability of a 1 or a -1 of each bit x_i, $\forall i=0,1,\cdots,N-1$ in the sequence is equally probable. The N - l terms of $x_i y_{i+l}\in\{-1,+1\}$ are included in the $C_{x,y}(l)$, where $\mathbf{y}$ is another random binary sequence. By assuming N_{+1} 1's and N_{-1} -1's in the N - l terms, the value of N_{+1} can be evaluated in terms of $C_{x,y}(l)$ as $N_{-1}=\left[N-l+C_{x,y}(l)\right]/2$. The pdf of ACF for random binary sequences becomes

$$\mathbf{P}_{\text{rand}}\left\{C_{x,y}(l)=z\right\}=\frac{1}{2N-1}\cdot\sum_{l=-N+z}^{N-z}\binom{N-l}{\frac{1}{2}(N-l+z)}\cdot\frac{1}{2^{N-l}}, \qquad (6)$$

where $\binom{n}{r}=\dfrac{n!}{r!(n-r)!}$.

The summation term in (6) only takes on values of l when $(N-l+z)$ is even. The computation of the moments of the ACF about the origin for random binary sequence is not as straight forward as those using the linear model. Figure 2 shows the pdf of ACF of random binary sequences for N = 63. The sum of square error between the pdf of ACF for Gold sequence and that of random binary sequence is 0.260%; while the sum of square error between that of Gold sequence and the simplified linear model is 0.754% which is larger than the former value.

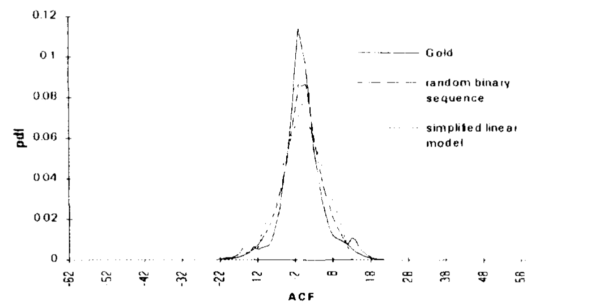

Figure 2. The pdf of ACF for Gold sequence, random binary sequence and the simplified linear model for N = 63.

From [5], the first and second moments of ACF are given by

$$\begin{aligned}&\mathrm{E}\left\{C_{x,y}(l)\right\}=0,\\&\mathrm{E}\left\{C^{2}_{x,y}(l)\right\}=N-|l|,\\&\mathrm{E}\left\{C_{x,y}(l)C_{x,y}(l')\right\}=0,\ \forall\, l\ne l',\end{aligned} \qquad (7)$$

However, higher order of moments of ACF using random binary sequences cannot be derived into close-form expressions. The advantage of using the simplified linear model is that the statistics of the ACF can be obtained in a relative simple manner. In order to find the suitable value of $\hat{C}$ for equation (2), equate $\mathrm{E}\{r_{x,y}\}$ of random binary sequence to that of the simplified ACF model as follows. The cross-correlation parameter defined in [4], $r_{x,y}$, for a BPSK A-DS/SSMA system is given by

$$r_{x,y}=\sum_{l=1-N}^{N-1}\left[2C^{2}_{x,y}(l)+C_{x,y}(l)C_{x,y}(l+1)\right]. \qquad (8)$$

It can be easily shown that

$$\mathrm{E}\left\{r^{\text{random}}_{x,y}\right\}=2N^{2} \qquad (9)$$

for random binary sequences, and

$$\mathrm{E}\{r_{x,y}^{\mathrm{sim}}\} = \hat{C}(\hat{C}+2)(2N-1)/3 \tag{10}$$

for the simplified linear model. An appropriate value of $\hat{C}$ used to approximate random sequence can be obtained by equating equation (9) to (10), viz;

$$\hat{C} = \left\lfloor \sqrt{3N+1} - 1 \right\rfloor \approx \left\lfloor \sqrt{3N} - 1 \right\rfloor, \tag{11}$$

where $\lfloor \alpha \rfloor$ denotes the integer part of real number α. The maximum value of pdf becomes $h \approx 1/\left\lfloor \sqrt{3N} \right\rfloor$. Also by applying (11) with $N = 63$, $\hat{C} = 12$ and $h = 0.077$ as shown in Table 2. Observe that as the integer $\hat{C}$ is the truncated real number obtained from (11), $\mathrm{E}\{r_{x,y}^{\mathrm{sim}}\}$ is smaller than the cross-correlation parameter for Gold sequence, $\mathrm{E}\{r_{x,y}^{\mathrm{Gold}}\}$, and also smaller than $\mathrm{E}\{r_{x,y}^{\mathrm{random}}\}$. Figure 2 shows the pdf's of ACF employing Gold sequences, random binary sequences and the simplified model for $N = 63$, respectively. The maximum value of pdf, h, using the simplified linear model of (2) is the smallest. Since the simplified model is a linear function, the square error between the pdf of Gold sequence and the pdf of simplified linear model is larger than that of between Gold sequence and random binary sequence. The random binary sequence is thus superior to the simplified linear model in approximating the pdf of the ACF for Gold sequences.

N	$\hat{C}$	$\hat{C}/N$	$\mathrm{E}\{r_{x,y}^{\mathrm{sim}}\}$	h	$\mathrm{E}\{r_{x,y}^{\mathrm{random}}\}$	$\mathrm{E}\{r_{x,y}^{\mathrm{Gold}}\}$
63	12	0.190	7000	0.077	7938	7818
255	26	0.102	123517	0.037	130050	130088

Table 2. The $\mathrm{E}\{r_{x,y}\}$ of Gold sequences using different approaches.

IV. System Model

The system model of a BPSK A-DS/SSMA K-user system over an AWGN channel is deployed to examine the proposed simplified linear model. The MAI of the A-DS/SSMA system is given by [4] as

$$\mathrm{MAI} = T\sqrt{P/2}\sum_{\substack{k=1\\k\neq i}}^{K} I_{k,i}, \tag{12}$$

where P is the common signal power and T is the data bit duration. The MAI component, $I_{k,i}$, is the interference between k-th and i-th users which is given by

$$I_{k,i} = N^{-1}A_{k,i}\cos\phi_k, \tag{13}$$

where N is the number of chips per user data bit and the phase angle rv ϕ_k is assumed uniformly distributed over $[0, 2\pi)$. The parameter, $A_{k,i}$, is the cross-correlation parameter between k-th and i-th users given by

$$A_{k,i} = T_c^{-1}b_0^{(k)}\left[\theta_{k,i}(\gamma_k)\hat{R}_\psi(s_k) + \theta_{k,i}(\gamma_k+1)R_\psi(s_k)\right];\ b_0^{(k)} = b_{-1}^{(k)}. \tag{14a}$$

$$A_{k,i} = T_c^{-1}b_0^{(k)}\left[\hat{\theta}_{k,i}(\gamma_k)\hat{R}_\psi(s_k) + \hat{\theta}_{k,i}(\gamma_k+1)R_\psi(s_k)\right];\ b_0^{(k)} \neq b_{-1}^{(k)}. \tag{14b}$$

where $s_k = \tau_k - \gamma_k T_c$ and the chip duration $T_c = T/N$. The relative time delay between the k-th user and intended i-th user is given by $\gamma_k = \lfloor \tau_k / T_c \rfloor$. The vector $\left(b_{-1}^{(k)}, b_0^{(k)}\right)$ represents a pair of consecutive data bits of the k-th user. The functions $R_\psi(s)$ and $\hat{R}_\psi(s)$ are the partial auto-correlation functions of the chip waveform as defined in [1]. The parameters [6], $\theta_{k,i}(\cdot)$ and $\hat{\theta}_{k,i}(\cdot)$ denote the even and odd periodic cross-correlation functions (PCF) which are defined by

$$\theta_{x,y}(l) = C_{x,y}(l) + C_{x,y}(l-N),\ \text{and} \tag{15}$$

$$\hat{\theta}_{x,y}(l) = C_{x,y}(l) - C_{x,y}(l-N) \tag{16}$$

The random variables, τ_k and s_k, are independent uniformly distributed over $[0, T)$ and $[0, T_c)$, respectively. The user data is assumed equally probable, i.e., $\mathrm{P}\{b_j^{(k)} = -1\} = \mathrm{P}\{b_j^{(k)} = 1\} = 1/2$. For PN sequence of rectangular waveform, $\Psi(t) = 1$ for $0 \le t < T_c$ and $\Psi(t) = 0$ otherwise. Consequently, the partial auto-correlation functions $R_\psi(s) = s$ and $\hat{R}_\psi(s) = T_c - s$. The pdf of $y = \cos\phi$ is given by $p_Y(y) = \left(\pi\sqrt{1-y^2}\right)^{-1}$ when $|y| < 1$ and $p_Y(y) = 0$ otherwise. Thus the MAI component, $I_{k,i}$, is a rv distributed over $[-1,+1]$. As all the pdf's of each individual rv in Equation (13) and (14) are known and independent, the resultant pdf of $I_{k,i}$ was obtained by means of convolution and multiplication of the pdf's of each rv accordingly.

V. Analysis on the PDF of MAI Component

A. *The PDF of MAI component for* $b_0^{(k)} = b_{-1}^{(k)}$

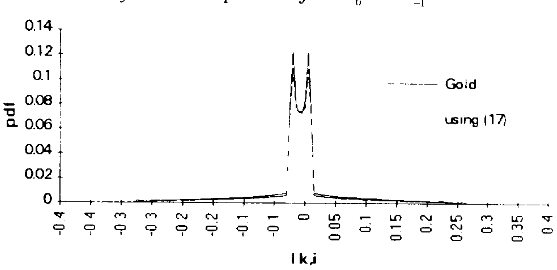

Figure 3. The pdf of MAI component, $I_{k,i}$, assuming even PCF for Gold sequence of $N = 63$.

Figure 3 shows the pdf of $I_{k,j}$ of $N = 63$ for Gold sequence when $b_0^{(k)} = b_{-1}^{(k)}$. Four sequences are arbitrarily selected from the set of Gold sequences as shown in Table 3. Observe that the pdf are concentrated and gradually dispersed about the origin. Figure 5 shows that the pdf of MAI component is symmetric about -0.075 which is approximately equal to zero. Table 4 depicts the mean and variance of $I_{k,j}$. It was found that the mean and variance of $I_{k,j}$ for sequence-pairs (1,2), (1,3), and (1,4) are the same. This implies that the pdf of $I_{k,j}$ for any sequence-pair in the set of Gold sequences are exactly the same. It is due to the even PCF, $\theta_{x,y}(\cdot)$, of Gold code is a 3-value function [7] given by

Seq. No.	4 Gold sequences of $N = 63$
1	111111010101100110111011010010011100010111100101000110000010000
2	110100010000101100101010010011110000011011100110001110101111111
3	001011000101001010010001000001101100001100000011001000101011111
4	100101011101110000101110011011100100011010010110000001010011111

Table 3. Four sequences in the set of Gold sequences of $N = 63$.

Gold $N = 63$	sequences 1 & 2		sequences 1 & 3	
	$b_0^{(k)} = b_{-1}^{(k)}$	$b_0^{(k)} \neq b_{-1}^{(k)}$	$b_0^{(k)} = b_{-1}^{(k)}$	$b_0^{(k)} \neq b_{-1}^{(k)}$
$E\{I_{k,j}\}$	-9.70e-5	-6.20e-5	-1.14e-4	-8.50e-5
$Var\{I_{k,j}\}$	5.390e-3	8.811e-3	5.219e-3	1.206e-2
$E\{I_{k,j}\}$ over data bit	-7.948e-5		-9.098e-5	
$Var\{I_{k,j}\}$ over data bit	7.101e-3		8.726e-3	

(a)

Gold $N = 63$	sequences 1 & 4		average	
	$b_0^{(k)} = b_{-1}^{(k)}$	$b_0^{(k)} \neq b_{-1}^{(k)}$	$b_0^{(k)} = b_{-1}^{(k)}$	$b_0^{(k)} \neq b_{-1}^{(k)}$
$E\{I_{k,j}\}$	-9.70e-5	-8.29e-5	-1.14e-4	-4.90e-5
$Var\{I_{k,j}\}$	5.390e-3	9.321e-3	5.219e-3	6.769e-3
$E\{I_{k,j}\}$ over data bit	-8.992e-5		-8.156e-5	
$Var\{I_{k,j}\}$ over data bit	7.355e-3		5.994e-3	

(b)

Table 4. The mean and variance of MAI component, $I_{k,j}$, for Gold sequences of $N = 63$.

$$\theta_{k,j}(l) = \begin{cases} -1 & \text{for } 2^n - 2^{n-e} - 1 \text{ values of } l, \\ -1 + 2^{(n+e)/2} & \text{for } 2^{n-e-1} + 2^{(n-e-2)/2} \text{ values of } l, \\ -1 - 2^{(n+e)/2} & \text{for } 2^{n-e-1} - 2^{(n-e-2)/2} \text{ values of } l. \end{cases} \quad (17)$$

where n is the degree of primitive polynomials generating Gold sequences of period $N = 2^n - 1$, and $e = 1$ when n is odd, and $e = 2$ when n is even. Figure 3 also illustrates the pdf of $I_{k,j}$ by deploying the pdf of $\theta_{k,j}(l)$ from (17). Observe that the difference between these two pdf's is marginal as the sum of square error between these two pdf's is equal to 0.0763%.

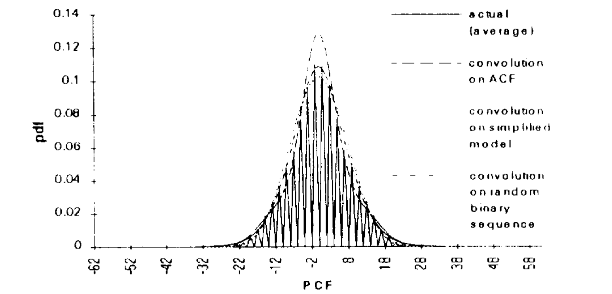

Figure 5. The pdf of odd PCF for Gold sequence of $N = 63$.

As the random variables $C_{k,j}(l)$ and $-C_{k,j}(l-N)$ are assumed independent, the pdf of the sum of two rv's $\{C_{k,j}(l) - C_{k,j}(l-N)\}$, is equal to the convolution of the pdf's of $C_{k,j}(l)$ with that of $-C_{k,j}(l-N)$. The resultant pdf resembles that of odd PCF employing Gold sequence as shown in Figure 5.

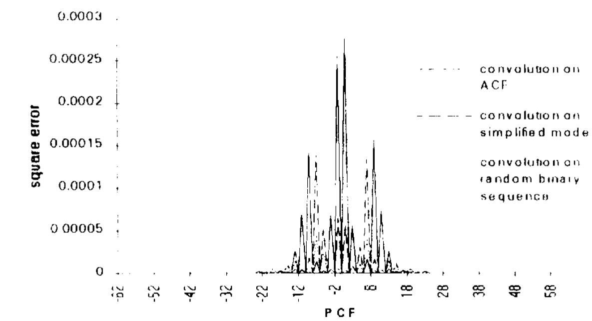

Figure 6. The sum of square error between the pdf of odd PCF for Gold sequence of $N = 63$ and three different schemes.

Figure 6 depicts the sum of square error between the pdf of the odd PCF for Gold sequence and the resultant pdf by means of convolution, which is equal to 1.698 %. Figure 5 also depicts various pdf's obtained from the convolution of ACF's. Both Figures 5 and 6 indicate that the difference between the pdf of the simplified linear model and that of Gold sequence is small as its corresponding sum of square error is equal to 0.835%. The sum of square error between the pdf of Gold sequence and that of random binary sequence obtained by convolution method is equal to 0.266% which marks the lowest difference.

B. The PDF of MAI component for $b_0^{(k)} \neq b_{-1}^{(k)}$

Figure 7 shows the pdf of the average $I_{k,j}$ for $b_0^{(k)} \neq b_{-1}^{(k)}$. Observe that the pdf's are of different shapes and symmetric about the origin. There are peaks at each average $I_{k,j}$ interval of 0.025 as it has been shown by Lehnert [1]. The pdf of $I_{k,i}$ using the simplified linear model has small spread (i.e., small variance of $I_{k,i}$) about the origin which yields the highest peak of pdf. The sum of square error between the average pdf and the pdf of the simplified linear model is equal to 0.256%.

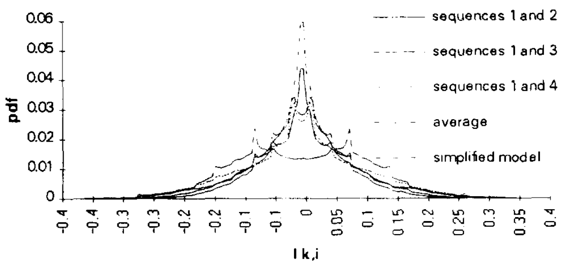

Figure 7. The pdf of MAI component, $I_{k,i}$, assuming odd PCF for Gold sequence of $N = 63$.

C. The PDF of MAI component averaged over data bit

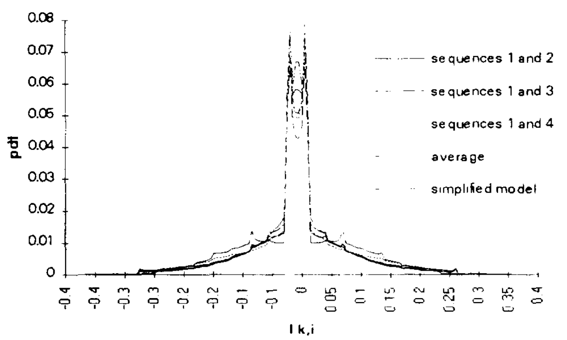

Figure 8. The pdf of MAI component averaged over data bit for Gold sequence of $N = 63$.

Figure 8 gives the pdf of $I_{k,i}$ averaged over the data bit by assuming $P\{b_j^{(k)} = -1\} = P\{b_j^{(k)} = 1\} = 1/2$. Observe that the pdf's are of similar shape and symmetrically distributed about slightly different values. It can be easily shown that these small differences have negligible effect on the BER performance as the sum of square error between the average pdf of the MAI component for Gold sequence and that of the simplified linear model is 0.0683%. Thus the simplified linear model can be deployed to simplify the evaluation of the BER performance of an A-DS/SSMA communication system.

VI. Conclusions

Based on the pdf patterns observed from six sets of well-known PN sequences, a simplified linear model for the pdf of ACF was introduced to simplify the evaluation of the BER performance of an A-DS/SSMA communication system. Although the comparison shows that the pdf of ACF using random binary sequence is superior to that of the simplified linear model in modeling the pdf of the MAI component, $I_{k,i}$, but its higher moments and other statistics are relatively more difficult to obtain when compared with the simplified model. Furthermore, the difference between the pdf of $I_{k,i}$ using the simplified model and that of Gold sequence is negligible. This implies that the simplified model can be used to evaluate the approximate BER performance. In addition, it was found that the pdf of odd PCF can be obtained by the convoluting the rv's, $C_{x,y}(l)$ with $-C_{x,y}(l-N)$. It means that $C_{x,y}(l)$ and $-C_{x,y}(l-N)$ can be regarded as independent rv's. The benefits of using the probabilistic approach to evaluate the BER performance is to minimize the computation time. Arbitrarily tight BER bounds [1,2] can be also obtained by employing this method. With the supplement of the simplified linear model of the pdf of ACF, the evaluation of BER performance of the A-DS/SSMA communication systems employing different sets of PN sequences can be further simplified.

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