

\mathcal{H}_2 and Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Stabilization and Disturbance Attenuation for Differential Linear Repetitive Processes

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Abstract—Repetitive processes are a distinct class of two-dimensional systems (i.e., information propagation in two independent directions) of both systems theoretic and applications interest. A systems theory for them cannot be obtained by direct extension of existing techniques from standard (termed 1-D here) or, in many cases, two-dimensional (2-D) systems theory. Here, we give new results towards the development of such a theory in \mathcal{H}_2 and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ settings. These results are for the sub-class of so-called differential linear repetitive processes and focus on the fundamental problems of stabilization and disturbance attenuation.

Index Terms—Differential linear repetitive processes, \mathcal{H}_2 and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control, linear matrix inequalities.

I. INTRODUCTION

REPETITIVE processes are a distinct class of 2-D systems of both system theoretic and applications interest. The unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, can lead to oscillations which increase in amplitude in the pass-to-pass direction where this instability cannot be analyzed using 1-D systems theory, i.e., the stability property cannot be characterized by 1-D systems stability tests.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then, in a repetitive process the pass profile $y_k(t)$, $0 \leq t \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, the references cited in [13]). Also, in recent years applications

have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes (see, for example, [7]) and iterative algorithms for solving nonlinear dynamic optimal control/optimization problems based on the maximum principle [12]. In the case of iterative learning control for the linear dynamics case, the stability theory for differential (and discrete) linear repetitive processes is one method which can be used to undertake a stability/convergence analysis of a powerful class of such algorithms and thereby produce vital design information concerning the trade-offs required between convergence and transient performance (see, for example, [8]).

Attempts to stabilize and/or meet performance specifications, such as the level of disturbance attenuation, for these processes using standard (or 1-D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2-D systems structure, i.e., information propagation occurs from pass-to-pass and along a given pass. Also, the initial conditions are reset before the start of each new pass and the structure of these can be somewhat complex. For example, if they are an explicit function of points on the the previous pass profile then this alone can destroy stability. In seeking a rigorous foundation on which to analyze such features, it is natural to attempt to exploit structural links which exist between these processes and other classes of 2-D linear systems.

The case of 2-D discrete linear systems recursive in the positive quadrant $(i, j) : i \geq 0, j \geq 0$, (where i and j denote the directions of information propagation) has been the subject of much research effort over the years using, in the main, the well-known Roesser and Fornasini Marchesini state-space models. More recently, productive research has been reported on \mathcal{H}_∞ and \mathcal{H}_2 approaches to analysis and design—see, for example, [3] and [16].

A critical feature of repetitive processes is that information propagation in one of the independent directions, along the pass, only occurs over a finite duration—the pass length. Also, the boundary conditions are reset before the start of each new pass and, as noted above, the structure of these is crucial in terms of stability. Moreover, in this paper we consider so-called differential linear repetitive processes where information propagation along the pass is governed by a matrix differential equation. The systems theory for 2-D discrete linear systems is therefore not applicable.

Stability is obviously a prerequisite in any application of these processes and if it is not present then it must be established by suitable input action constructed using current and/or previous

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pass information. In which context, it is possible to use feedback action on the current pass and/or feedforward action from the previous pass (or passes). The critical role of the previous pass profile dynamics means that the use of current pass action alone is not enough and it must be augmented by feedforward action (from the previous pass). This approach has been the subject of significant research effort and results are beginning to emerge on how to undertake design in the presence of uncertainty. For example, [9] and [11] give preliminary results in an \mathcal{H}_∞ setting.

In this paper, we develop a substantial body of new results in an \mathcal{H}_2 setting and show how they can be combined with \mathcal{H}_∞ results to produce a potentially very powerful mixed $\mathcal{H}_2/\mathcal{H}_\infty$ approach. We begin in the next section by giving the necessary background results.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I , respectively. Moreover, $M \succ 0$ (respectively, $\succeq 0$) denotes a real symmetric positive definite (respectively, semi-definite) matrix. Similarly, $M \prec 0$ (respectively, $\preceq 0$) denotes a real symmetric negative definite (respectively, semi-definite) matrix. We also require the following signal space definition. Finally, (\star) is used to denote the transpose of block entries in LMIs.

Definition 1: Consider a $q \times 1$ vector sequence $\{w_j(t)\}$ defined over the real interval $0 \leq t \leq \infty$ and the nonnegative integers $0 \leq j \leq \infty$, which is written as $\{[0, \infty), [0, \infty)\}$. Then, the L_2 norm of this vector sequence is given by

$$\|w\|_2 = \sqrt{\sum_{j=0}^{\infty} \int_0^{\infty} w_j^T(t) w_j(t) dt}$$

and this sequence is said to be a member of $L_2^q\{[0, \infty), [0, \infty)\}$, or L_2^q for short, if $\|w\|_2 < \infty$.

II. PRELIMINARIES

The most basic form of the state-space model for a differential linear repetitive process over $0 \leq t \leq \alpha$, $k \geq 0$, is

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t) \end{aligned} \quad (1)$$

Here, on pass k , $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector and $u_k(t)$ is the $l \times 1$ vector of control inputs.

To complete the process description, it is necessary to specify the boundary conditions i.e., the state initial vector on each pass and the initial pass profile (i.e., on pass 0). Except where stated otherwise, these are taken to be zero here.

The stability theory [13] for linear repetitive processes is based on an abstract model in a Banach space setting which includes a large number of such processes as special cases. In this setting, a bounded linear operator mapping a Banach space into itself describes the contribution of the previous pass dynamics to the current one and the stability conditions are described in terms of properties of this operator. Noting again the unique feature of these processes, i.e., oscillations that increase in amplitude from pass-to-pass (the k direction in the notation for variables used here), this theory is based on ensuring that such a response cannot occur by demanding that the output sequence of pass profiles generated $\{y_k\}$ has a bounded input bounded output stability property defined in terms of the norm on the underlying Banach space.

In fact, two distinct forms of stability can be defined in this setting which are termed asymptotic stability and stability along the pass, respectively. The former requires this property with respect to the (finite and fixed) pass length and the latter uniformly, i.e., independent of the pass length. Asymptotic stability guarantees the existence of a so-called limit profile defined as the strong limit as $k \rightarrow \infty$ of the sequence $\{y_k\}$ and for the processes under consideration here this limit profile is described by a 1-D differential linear systems state-space model with state matrix $A_{lp} := A + B_0(I - D_0)^{-1}C$. Hence, it is possible for asymptotic stability to result in a limit profile which is unstable as a 1-D differential linear system, e.g., $A = -1$, $B = 0$, $B_0 = 1 + \beta$, $C = 1$, $D = 0$, $D_0 = 0$, where $\beta > 0$ is a real scalar. Stability along the pass prevents this from happening by demanding that the stability property be independent of the pass length, which can be analyzed mathematically by letting $\alpha \rightarrow \infty$.

It is of interest to relate this theory to a physical example in the form of long-wall coal cutting where the pass profile is the thickness (relative to a fixed datum) of the coal left after the cutting machine has moved along the pass length, i.e., the coal face. The stability problem here is caused by the machine's weight as it rests of the previous pass profile during the cutting of the next pass profile. The undulations caused can be very severe and result in productive work having to stop to enable them to be removed. Asymptotic stability here means that after a sufficient number of passes have elapsed the profile produced on each successive pass is the same, i.e., convergence in the pass to pass (i.e., k) direction and this converged value is the limit profile. However, this limit profile can contain growth along it, i.e., nonconvergence in the t direction.

Several equivalent sets of conditions for stability along the pass are known but here the starting point is the 2-D transfer-function matrix description of the process dynamics, and hence the 2-D characteristic polynomial. Since the state on pass 0 plays no role, it is convenient to relabel the state vector as $x_{k+1}(t) \mapsto x_k(t)$ (keeping of course the same interpretation). Also, define the pass-to-pass shift operator as z_2 applied, e.g., to $y_k(t)$ as follows:

$$y_k(t) := z_2 y_{k+1}(t)$$

and for the along the pass dynamics we use the Laplace transform variable s , where it is routine to argue that finite pass length does not cause a problem provided the variables considered are suitably extended from $[0, \alpha]$ to $[0, \infty)$, and here we assume that this has been done.

Let $Y(s, z_2)$ and $U(s, z_2)$ denote the results of applying these transforms to the sequences $\{y_k\}$ and $\{u_k\}$, respectively. Then, the process dynamics can be written as

$$Y(s, z_2) = G_{yu}(s, z_2)U(s, z_2)$$

where the 2-D transfer-function matrix $G_{yu}(s, z_2)$ is given by

$$G_{yu}(s, z_2) = [0 \ I] \left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ D \end{bmatrix}$$

and the 2-D characteristic polynomial by

$$C(s, z_2) := \det \left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \right).$$

It has been shown elsewhere [13] that stability along the pass holds if, and only if

$$C(s, z_2) \neq 0 \quad (2)$$

in $\mathcal{U}(s, z_2) := \{(s, z_2) : \operatorname{Re}(s) \geq 0, |z_2| \leq 1\}$.

This last condition clearly requires that all eigenvalues of the matrix D_0 have modulus strictly less than unity which is the necessary and sufficient condition for asymptotic stability. Under this condition, the sequence of pass profiles generated will converge to the limit profile which is described by a 1-D discrete linear time invariant state-space model with state matrix $A_{lp} := A + B_0(I - D_0)^{-1}C$. As the simple example given above illustrates, however, it is not guaranteed that this limit profile will have even the most basic property of stability, i.e., all eigenvalues of A_{lp} have strictly negative real parts. This is due to the finite pass length, over which duration even an unstable 1-D linear system can only produce a bounded output in response to a bounded input.

If the possibility of a limit profile with unstable along the pass dynamics must be completely excluded, then asymptotic stability must be strengthened, where an obvious intuitive requirement is that the matrix A (which governs the dynamics produced along a given pass) be stable in the sense that all its eigenvalues have strictly negative real parts. Again, however, the simple example given above shows that in general this is not enough to prevent a limit profile with unstable along the pass dynamics. Missing from a stability analysis is an explicit requirement on the coupling between the previous pass profile and current pass state vectors (and hence the current pass profile) which is supplied by (2). In effect, stability along the pass demands the existence of a limit profile with uniformly bounded (with respect to the pass length α) along the pass dynamics and mathematically this can be treated by letting $\alpha \rightarrow \infty$ in analysis.

Consider now, for simplicity, the single-input single-output case with zero input and pass state initial vector on each pass. Then, $y_k(0) = D_0^k y_0(0)$ and hence asymptotic stability demands that the sequence formed from the initial value of the pass profile on each pass does not become unbounded. Also

$$y_k(s) = (C(sI_n - A)^{-1}B_0 + D_0)^k y_0(s)$$

and therefore, under asymptotic stability and the condition that all eigenvalues of the matrix A have strictly negative real parts, stability along the pass requires that each frequency component of the initial profile (and hence on each subsequent pass) is attenuated from pass-to-pass. Note also that asymptotic stability is a necessary condition for stability along the pass.

Stability along the pass treats the process as evolving over the complete positive quadrant, i.e., both k and t are of unbounded duration whereas in application t lies in the finite interval $0 \leq t \leq \alpha$. Hence, there may be individual cases when stability along the pass is too strong. A similar situation can also arise in the case of 2-D discrete linear systems including those described by the well-known Roesser and Fornasini state-space models. This has led to the concept of so-called strong practical stability [1] for this case that can also be extended to the processes considered here. In this paper, however, we focus on the general case and hence the interest is in the stability along the pass.

In this paper, the interest is in the basic systems theoretic property of how to characterize, at the level of computation, a stable along the pass example, including how to stabilize an unstable example. The condition of (2) is difficult to test and also (despite the fact that it can be reformulated to allow the use of Nyquist diagram based tests) it does not form a basis for the design of compensators to guarantee this fundamental property. This, in turn, has led to the emergence of LMI-based conditions which can then be used in this vital role. Here, we will use this route and start from a Lyapunov function as summarized next. The results obtained are sufficient but not necessary and hence are conservative to some extent. They are, however, computationally feasible and this is their major applications oriented interest.

Consider the following candidate Lyapunov function for processes described by (1):

$$\begin{aligned} V(k, t) &= V_1(k, t) + V_2(k, t) \\ &= x_{k+1}^T(t)P_1x_{k+1}(t) + y_k^T(t)P_2y_k(t) \end{aligned} \quad (3)$$

for some $P_1 \succ 0$ and $P_2 \succ 0$, and associated increment

$$\Delta V(k, t) = \dot{V}_1(k, t) + \Delta V_2(k, t) \quad (4)$$

where

$$\begin{aligned} \dot{V}_1(k, t) &= \dot{x}_{k+1}^T(t)P_1x_{k+1}(t) + x_{k+1}^T(t)P_1\dot{x}_{k+1}(t) \\ \Delta V_2(k, t) &= y_{k+1}^T(t)P_2y_{k+1}(t) - y_k^T(t)P_2y_k(t) \end{aligned}$$

Lemma 1: [4] A differential linear repetitive process described by (1) is stable along the pass if

$$\Delta V(k, t) < 0. \quad (5)$$

Lemma 2: [4] A differential linear repetitive process described by (1) is stable along the pass if there exist matrices $P_1 \succ 0$ and $P_2 \succ 0$ such that the following LMI holds:

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^T P_2 & A^T P_1 + P_1 A & P_1 B_0 \\ D_0^T P_2 & B_0^T P_1 & -P_2 \end{bmatrix} \prec 0. \quad (6)$$

As in 1-D systems theory, the choice of Lyapunov function is not unique, but the one here has intuitive appeal in the sense that it is the sum of quadratic terms in the current pass state and previous pass profile vectors, respectively. In particular, noting again the unique characteristic, i.e., oscillations in the pass to pass direction, we have that the first term is related to the condition on the state-space model matrix A (which clearly governs the dynamics produced along any pass (t direction of information propagation) and the second to the effects of the contribution of the previous pass profile (k direction of information propagation).

Some application areas will clearly require the design of compensators which guarantee stability along the pass and also have the maximum possible disturbance attenuation. The relevance of rejecting the effects of disturbances on measurements (and subsequent computations) of variables is well founded physically by noting the conditions in which such examples have to operate, e.g., in long-wall coal cutting and iterative learning control applications, such as using a gantry robot to synchronously place objects on a chain conveyor [2].

This paper considers this key problem area in an \mathcal{H}_∞ and \mathcal{H}_2 frameworks starting from the following process state-space model over $0 \leq t \leq \alpha$, $k \geq 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0 y_k(t) + B u_{k+1}(t) \\ &\quad + B_{11} w_{k+1}(t) + B_{21} \nu_{k+1}(t) \\ y_{k+1}(t) &= C x_{k+1}(t) + D_0 y_k(t) + D u_{k+1}(t) \\ &\quad + B_{12} w_{k+1}(t) + B_{22} \nu_{k+1}(t) \end{aligned} \quad (7)$$

where the vectors $x_{k+1}(t)$, $y_k(t)$ and $u_{k+1}(t)$ are defined as in (1), $w_{k+1}(t)$ and $\nu_{k+1}(t)$ are $r \times 1$ disturbance vectors. Two cases will be considered here, one is when the disturbance is impulsive and the other when it is of finite energy (taken as belonging to L_2). (The boundary conditions are as per the disturbance free case.) Also, it is easy to conclude that stability along the pass of such a process is again governed by (2). Mathematically, we use the \mathcal{H}_2 norm for impulsive inputs and \mathcal{H}_∞ norm to measure the performance objective which is the attenuation of the effects of w_{k+1} and ν_{k+1} . These are introduced next.

It is important to note that the \mathcal{H}_2 norm of the 2-D transfer-function matrix here can be only defined if there is no direct coupling between the impulsive input signal and the pass profile on any pass. Hence, in common with the 1-D linear system case where \mathcal{H}_2 norm is only defined for strictly proper systems, we set $D = 0$ when computing \mathcal{H}_2 norm of the 2-D transfer-function matrix between u and y and set $B_{12} = 0$ for \mathcal{H}_2 norm of the 2-D transfer-function matrix between w and y . Therefore, the \mathcal{H}_2 norm is always defined and never infinite.

A differential linear repetitive process described by (7) is said to have \mathcal{H}_∞ disturbance attenuation (or \mathcal{H}_∞ norm bound) γ_∞ if it is stable along the pass and

$$\sup_{0 \neq \nu \in L_2^r} \frac{\|y\|_2}{\|\nu\|_2} < \gamma_\infty. \quad (8)$$

In effect, this is a worst case bound as it corresponds to a bound on the maximum peak gain of the 2-D frequency response between ν and y and is given, with $\bar{\sigma}(\cdot)$ denoting the maximum singular value of its matrix argument, by

$$\|G_{y\nu}(s, z_2)\|_\infty = \sup_{\omega_1 \in \mathbb{R}, \omega_2 \in [0, 2\pi]} \bar{\sigma} [G(j\omega_1, e^{j\omega_2})]$$

where

$$G_{y\nu}(s, z_2) = [0 \ I] \left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \quad (9)$$

i.e., the 2-D transfer-function matrix between these two vectors. This result invokes Parseval's theorem for 2-D signals. The proof of this for the 2-D discrete case can be found in, for example, [6] and in the Appendix we give the proof for the continuous-discrete signal case which is needed here. (This proof is for the scalar case with an obvious extension to vectors.)

Another commonly used performance measure for analysis and synthesis is the \mathcal{H}_2 norm. It is widely recognized that \mathcal{H}_2 norm is a useful tool to optimize the transient behavior of a system. In this paper, we will minimize \mathcal{H}_2 norm of the 2-D

transfer-function matrix between w and y , i.e.,

$$G_{yw}(s, z_2) = [0 \ I] \left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \quad (10)$$

to reduce the pass profile energy in response to impulse disturbances or the variance against white noise disturbances.

Consider the state-space model (7). Then, the main goal in this paper is to guarantee stability along the pass process and achieve the required performance specifications, as defined in terms of the \mathcal{H}_2 and \mathcal{H}_∞ norms as appropriate. We will seek to do this by assuming that all entries in the current pass state vector are available for measurement. Then, (noting again the critical role of the previous pass profile vector and hence the weakness of current pass action alone) the following control law can be defined which is termed static since it has no internal dynamics:

$$u_{k+1}(t) = [K_1 \ K_2] \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix} \quad (11)$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. If the current pass state vectors are not available for measurement we still can use this method accompanied by a state observer.

In general, the model available for design will only be an approximation to the process dynamics. Hence, we also deal here with robustness, where as in the 1-D linear systems case we assume that the unmodeled dynamics lie within well-defined model classes (or assumptions). Here, we consider the following two.

- 1) *Norm-bounded uncertainty model:* Here, we assume that the uncertainty present can be modeled as additive perturbations to the nominal process state-space model matrices. In particular, noting from (2) that stability along the pass is defined in terms of the so-called augmented nominal plant matrix

$$M_0 = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}$$

we say that a process is subject to so-called norm-bounded uncertainty if its augmented plant matrix, denoted by M here, can be written in the form

$$M = M_0 + \Delta M = M_0 + HFE. \quad (12)$$

Here, H and E are some known constant matrices, and F is an unknown constant matrix which satisfies

$$F^T F \preceq I \quad (13)$$

- 2) *Polytopic uncertainty model:* Here, we assume that M is only known to lie in a given fixed polytope of matrices described by

$$M \in \text{Co}(M_1, M_2, \dots, M_h) \quad (14)$$

where Co denotes the convex hull. In particular, for $i = 1, 2, \dots, h$, M can be written as

$$M := \left\{ X : X = \sum_{i=1}^h \alpha_i M_i, \alpha_i \geq 0, \sum_{i=1}^h \alpha_i = 1 \right\}.$$

Uncertainties satisfying either of these models are termed admissible.

Now we are in a position to state the problems solved in this paper.

The \mathcal{H}_2 problem.

Given a disturbance attenuation level γ_2 , find (11) such that stability along the pass holds for the controlled process and also

$$\|G_{yw}^{cl}(s, z_2)\|_2 < \gamma_2$$

where $G_{yw}^{cl}(s, z_2)$ denotes the 2-D transfer-function matrix between y and w .

The robust \mathcal{H}_2 problem.

Given an uncertain process and disturbance attenuation level γ_2 , find (11) such that stability along the pass holds for the controlled process and

$$\|G_{yw}^{cl}(s, z_2)\|_2 < \gamma_2$$

for admissible uncertainties.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

Given the disturbance attenuation level γ_∞ , find (11) such that stability along the pass holds for the controlled process and also

$$\|G_{y\nu}^{cl}(s, z_2)\|_\infty < \gamma_\infty$$

and the quantity

$$\|G_{yw}^{cl}(s, z_2)\|_2$$

is minimized.

III. \mathcal{H}_∞ AND \mathcal{H}_2 NORMS AND STABILITY ALONG THE PASS

A. \mathcal{H}_∞ Norm

We will first state the following known result on computing an upper bound on the \mathcal{H}_∞ norm as an LMI constraint.

Lemma 3: [10] A differential linear repetitive process described by (7) is stable along the pass and has \mathcal{H}_∞ disturbance attenuation $\gamma_\infty > 0$ if there exist matrices $R_1 \succ 0$, $R_2 \succ 0$ and $R_3 \succ 0$ such that

$$\begin{bmatrix} -S & S\hat{A}_2 & S\hat{D}_1 & 0 \\ \hat{A}_2^T S & \hat{A}_1^T P + P\hat{A}_1 - R & P\hat{B}_1 & L^T \\ \hat{D}_1^T S & \hat{B}_1^T P & -\gamma_\infty^2 I & 0 \\ 0 & L & 0 & -I \end{bmatrix} < 0 \quad (15)$$

where $P = \text{diag}(R_1, 0)$, $S = \text{diag}(R_3, R_2)$, $R = \text{diag}(0, R_2)$ and

$$\hat{B}_1 = \begin{bmatrix} B_{21} \\ 0 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad L = [0 \quad I]. \quad (16)$$

This result is the so-called bounded real lemma for differential linear repetitive processes. Note also that the matrix R_3 has no influence on the result (but is needed in its proof). Hence, it can be deleted to give the following result.

Lemma 4: [10] For some prescribed $\gamma_\infty > 0$, suppose that there exist matrices $R_1 \succ 0$, and $R_2 \succ 0$ such that the following LMI holds for a differential linear repetitive process described by (7)

$$\begin{bmatrix} -R_2 & R_2 C & R_2 D_0 & R_2 B_{22} \\ C^T R_2 & A^T R_1 + R_1 A & R_1 B_0 & R_1 B_{21} \\ D_0^T R_2 & B_0^T R_1 & -R_2 + I & 0 \\ B_{22}^T R_2 & B_{21}^T R_1 & 0 & -\gamma_\infty^2 I \end{bmatrix} < 0. \quad (17)$$

Then, this process is stable along the pass and also $\|G_{y\nu}(s, z_2)\|_\infty < \gamma_\infty$.

Motivated by 1-D system theory, where the \mathcal{H}_∞ norm is used as a measure of system robustness, the above result has the following interpretation. Keeping the \mathcal{H}_∞ norm of the controlled process 2-D transfer-function matrix from ν to y below the level γ_∞ guarantees that the process under consideration is robust to unstructured perturbations of the form

$$\nu = \Delta y, \quad \|\Delta\|_\infty \leq \gamma_\infty.$$

This means that choosing a lower value of γ_∞ reduces the robustness to unmodeled dynamics (as measured in this way) and vice-versa.

B. \mathcal{H}_2 Norm

In the 1-D linear systems case, the \mathcal{H}_2 norm coincides with the total output energy in the impulse response. Moreover, this observation leads immediately to algorithms for computing this norm from the state-space model. Next we develop an LMI (and hence state-space-based) method for computing the \mathcal{H}_2 norm for the differential linear repetitive processes considered here.

Consider first a single input stable along the pass process (note again that this property can be analyzed mathematically by letting the pass length $\alpha \rightarrow \infty$) described by (1) with zero boundary conditions. Also, let the $m \times 1$ vector $g_k^h(t)$ denote the response of the system at pass k due to an impulse applied to the h th ($h, 1 \leq h \leq l$) input channel at $k = 0$, i.e.,

$$u_{k+1}(t) = \begin{cases} \delta(t)e^h, & \text{for } k = 0 \\ 0, & \text{for } k = 1, 2, \dots \end{cases} \quad (18)$$

where $\delta(t)$ is the Dirac delta function and e^h is the h th standard basis vector in \mathbb{R}^l . Then, following the definition of the \mathcal{H}_2 norm for 1-D systems, where it is a measure of the energy (L_2 norm) in the impulse response, the \mathcal{H}_2 norm of a process of the form considered here is given by

$$\|G\|_2 = \sqrt{\sum_{h=1}^l \sum_{k=0}^{\infty} \int_0^{\infty} (g_{k+1}^h)^T(t) g_{k+1}^h(t) dt}. \quad (19)$$

Next, introduce the following matrices

$$\begin{aligned}\widehat{A}_1 &= \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}, \\ \Omega &= \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \Psi = [C \quad D_0]\end{aligned}\quad (20)$$

and vectors

$$\begin{aligned}\zeta_{k+1}(t) &= \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix}, \quad \zeta_{k+1}^h(t) = \begin{bmatrix} x_{k+1}^h(t) \\ y_k^h(t) \end{bmatrix} \\ \xi_{k+1}^h(t) &= \begin{bmatrix} \dot{x}_{k+1}^h(t) \\ y_{k+1}^h(t) \end{bmatrix}.\end{aligned}\quad (21)$$

Then

$$\begin{aligned}\xi_{k+1}^h(t) &= (\widehat{A}_1 + \widehat{A}_2)\zeta_{k+1}^h(t) + \Omega\delta(t)\epsilon^h \\ &= \begin{cases} \widehat{\Omega}_h\delta(t), & \text{for } k=0 \\ (\widehat{A}_1 + \widehat{A}_2)\zeta_{k+1}^h(t), & \text{otherwise} \end{cases}\end{aligned}\quad (22)$$

and the process impulse response $g_{k+1}^h(t)$, $1 \leq h \leq l$, is

$$g_{k+1}^h(t) = \begin{cases} 0, & \text{for } k=0 \\ \Psi\zeta_{k+1}^h(t), & \text{otherwise} \end{cases}\quad (23)$$

where $\widehat{\Omega}_h$ denotes h th column of the matrix Ω .

The following is the first major result of this paper and gives a sufficient condition for stability along the pass together with an upper bound on the \mathcal{H}_2 norm of the 2-D transfer-function matrix (between control u and pass profile y).

Theorem 1: A differential linear repetitive process described by (7) is stable along the pass and has \mathcal{H}_2 norm bound $\gamma_2 > 0$, i.e., $\|G_{yw}(s, z_2)\|_2 < \gamma_2$, if there exist matrices $P_1 \succ 0$ and $P_2 \succ 0$ such that the following LMIs hold:

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1 + P_1A + C^TC & P_1B_0 + C^TD_0 \\ D_0^TP_2 & B_0^TP_1 + D_0^TC & -P_2 + D_0^TD_0 \end{bmatrix} \prec 0\quad (24)$$

and

$$\text{trace}(B^TP_1B) - \gamma_2^2 < 0.\quad (25)$$

Proof: Clearly, the LMI (24) can be rewritten as

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1 + P_1A & P_1B_0 \\ D_0^TP_2 & B_0^TP_1 & -P_2 \end{bmatrix} + \begin{bmatrix} 0 \\ C^T \\ D_0^T \end{bmatrix} [0 \quad C \quad D_0] \prec 0.\quad (26)$$

Also, since the second term on the left-hand side of (26) is non-negative definite, it follows immediately that

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1 + P_1A & P_1B_0 \\ D_0^TP_2 & B_0^TP_1 & -P_2 \end{bmatrix} \prec 0$$

and hence, by the LMI of (6), stability along the pass holds. Next, we show that the prescribed \mathcal{H}_2 performance bound γ_2 holds.

To proceed, we again use a Lyapunov approach and introduce

$$\begin{aligned}\Delta V^h(k, t) &= \dot{x}_{k+1}^{hT}(t)P_1x_{k+1}^h(t) + x_{k+1}^{hT}(t)P_1\dot{x}_{k+1}^h(t) \\ &\quad + y_{k+1}^{hT}(t)P_2y_{k+1}^h(t) - y_k^{hT}(t)P_2y_k^h(t)\end{aligned}\quad (27)$$

and hence

$$\begin{aligned}\Delta V^h(k, t) &= [\dot{x}_{k+1}^{hT}(t) \quad y_{k+1}^{hT}(t)] \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1}^h(t) \\ y_k^h(t) \end{bmatrix} \\ &\quad + [x_{k+1}^{hT}(t) \quad y_k^{hT}(t)] \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_{k+1}^h(t) \\ y_{k+1}^h(t) \end{bmatrix} \\ &\quad + [\dot{x}_{k+1}^{hT}(t) \quad y_{k+1}^{hT}(t)] \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x_{k+1}^h(t) \\ y_{k+1}^h(t) \end{bmatrix} \\ &\quad - [x_{k+1}^{hT}(t) \quad y_k^{hT}(t)] \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x_{k+1}^h(t) \\ y_k^h(t) \end{bmatrix}.\end{aligned}$$

Next, using the notation introduced in (21) we can write

$$\begin{aligned}\Delta V^h(k, t) &= \xi_{k+1}^{hT}(t)P\xi_{k+1}^h(t) + \zeta_{k+1}^{hT}(t)P\xi_{k+1}^h(t) \\ &\quad + \xi_{k+1}^{hT}(t)R\xi_{k+1}^h(t) - \zeta_{k+1}^{hT}(t)R\xi_{k+1}^h(t)\end{aligned}$$

where $P = \text{diag}(P_1, 0)$, $R = \text{diag}(0, P_2)$. Also, since the process is stable along the pass then $\lim_{t \rightarrow \infty} x_{k+1}(t) = 0$ and $\lim_{k \rightarrow \infty} y_{k+1}(t) = 0$. Furthermore, we have that

$$\begin{aligned}\sum_{k=0}^{\infty} \int_0^{\infty} \Delta V(k, t) dt &= \sum_{k=0}^{\infty} \int_0^{\infty} (\dot{V}_1(k, t) + \Delta V_2(k, t)) dt \\ &= - \sum_{k=0}^{\infty} x_{k+1}^T(0)P_1x_{k+1}(0) \\ &\quad - \int_0^{\infty} y_0^T(t)P_2y_0(t) dt.\end{aligned}$$

Hence, taking into account the zero boundary conditions the following holds:

$$\sum_{k=0}^{\infty} \int_0^{\infty} \Delta V(k, t) dt = 0.$$

On the other hand, it is easy to see that

$$\begin{aligned}\sum_{k=0}^{\infty} \int_0^{\infty} \dot{V}_1(k, t) dt &= \int_0^{\infty} (\dot{x}_1^T(t)P_1x_1(t) + x_1^T(t)P_1\dot{x}_1(t)) dt \\ &\quad + \int_0^{\infty} \sum_{k=1}^{\infty} (\zeta_{k+1}^T(t)\widehat{A}_1^TP\xi_{k+1}(t) + \zeta_{k+1}^T(t)P\widehat{A}_1\xi_{k+1}(t)) dt\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} \int_0^{\infty} \Delta V_2(k, t) dt &= \int_0^{\infty} (y_1^T(t)P_2y_1(t) - y_0^T(t)P_2y_0(t)) dt \\ &\quad + \int_0^{\infty} \left(\sum_{k=1}^{\infty} \zeta_{k+1}^T(t) (\widehat{A}_2^TR\widehat{A}_2 - R) \zeta_{k+1}(t) \right) dt.\end{aligned}$$

Now, based on (22) it can be verified that

$$\begin{aligned}\int_0^{\infty} (\dot{x}_1^T(t)P_1x_1(t) + x_1^T(t)P_1\dot{x}_1(t)) dt \\ = \Omega^TP\Omega + \int_0^{\infty} \zeta_1^T(t) (\widehat{A}_1^TP + P\widehat{A}_1) \zeta_1(t) dt\end{aligned}$$

and

$$\begin{aligned} & \int_0^{\infty} (y_1^T(t)P_2y_1(t) - y_0^T(t)P_2y_0(t)) dt \\ &= \Omega^T R \Omega + \int_0^{\infty} \zeta_1^T(t) \left(\hat{A}_2^T R \hat{A}_2 - R \right) \zeta_1(t) dt. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \int_0^{\infty} \Delta V(k, t) dt \\ &= \Omega^T (P + R) \Omega \\ &+ \int_0^{\infty} \sum_{k=0}^{\infty} \left(\zeta_{k+1}^T(t) \left(\hat{A}_1^T P + P \hat{A}_1 + \hat{A}_2^T R \hat{A}_2 - R \right) \right. \\ &\quad \left. \times \zeta_{k+1}(t) \right) dt. \end{aligned} \quad (28)$$

Next, based on (19) and (23), we have that

$$\begin{aligned} \|G\|_2^2 &= \sum_{h=1}^l \sum_{k=0}^{\infty} \int_0^{\infty} g_{k+1}^{hT}(t) g_{k+1}^h(t) dt \\ &= \sum_{h=1}^l \left(\sum_{k=0}^{\infty} \int_0^{\infty} \zeta_{k+1}^{hT}(t) \Psi^T \Psi \zeta_{k+1}^h(t) dt \right) \end{aligned} \quad (29)$$

and hence, on using (28)

$$\begin{aligned} \|G\|_2^2 &= \sum_{h=1}^l \left(\sum_{k=0}^{\infty} \int_0^{\infty} \zeta_{k+1}^{hT}(t) \Psi^T \Psi \zeta_{k+1}^h(t) dt \right. \\ &\quad \left. + \hat{\Omega}_h^T W \hat{\Omega}_h + \sum_{k=0}^{\infty} \int_0^{\infty} \Delta V^h(k, t) dt \right). \end{aligned}$$

Routine manipulations now show that the above equation is equivalent to

$$\begin{aligned} \|G\|_2^2 &= \sum_{h=1}^l \left(\hat{\Omega}_h^T W \hat{\Omega}_h + \sum_{k=0}^{\infty} \int_0^{\infty} \zeta_{k+1}^T(t) \Lambda \zeta_{k+1}(t) dt \right) \\ &= \text{trace}(\Omega^T W \Omega) + \sum_{k=0}^{\infty} \int_0^{\infty} \zeta_{k+1}^T(t) \Lambda \zeta_{k+1}(t) dt \\ &= \chi_1 + \chi_2 \end{aligned} \quad (30)$$

where

$$\Lambda = \Psi^T \Psi + \hat{A}_1^T P + P \hat{A}_1 + \hat{A}_2^T S \hat{A}_2 - R$$

and $W = \text{diag}(P_1, P_2)$, $S = \text{diag}(P_3, P_2)$, and $P_3 \succ 0$ are any given matrices of the required dimensions. Also, it follows immediately from (25) that $\chi_1 < \gamma_2^2$ and an obvious application of the Schur's complement formula to (the right-hand side of) χ_2 yields (24) (where we have also made use of the fact that $-P_3 \prec 0$). Also, if (24) holds, then $\chi_2 < 0$ and these facts together imply that $\chi_1 + \chi_2 < \gamma_2^2$. However, $\gamma_2 > 0$ by assumption and hence $\|G\|_2 < \gamma_2$ and the proof is complete. ■

Remark 1: Based on the above derivations, it can be seen that the \mathcal{H}_2 norm can be used as a tool to optimize the transient behavior of the process by suitably choosing the columns of the matrix B .

Remark 2: The \mathcal{H}_2 norm bound here can be minimized using the following linear objective minimization algorithm

$$\begin{aligned} & \min_{P_1 \succ 0, P_2 \succ 0, \mu > 0} \mu \\ & \text{subject to (24) and (25) with } \mu = \gamma_2^2 \end{aligned} \quad (31)$$

Remark 3: The analysis of this section assumed zero boundary conditions. If these are in fact nonzero then (22) and (23) become

$$\begin{aligned} \zeta_{k+1}^h(t) &= (\hat{A}_1 + \hat{A}_2) \zeta_{k+1}^h(t) + \Omega \delta(t) \epsilon^h \\ &= \begin{cases} \hat{\Omega}_h + \hat{A}_2 \zeta_{k+1}^h(t), & \text{for } k = 0 \\ (\hat{A}_1 + \hat{A}_2) \zeta_{k+1}^h(t), & \text{otherwise} \end{cases} \end{aligned}$$

and

$$g_{k+1}^h(t) = \begin{cases} D_0 y_k^h(t), & \text{for } k = 0 \\ \Psi \zeta_{k+1}^h(t), & \text{otherwise} \end{cases}$$

respectively. Obviously, nonzero boundary conditions introduce the terms $\hat{A}_2 \zeta_{k+1}^h(t)$ and $D_0 y_k^h(t)$ at $k = 0$. With these adjustments, the analysis follows exactly as for the case of zero boundary conditions.

IV. \mathcal{H}_2 STABILIZATION

In this section, we solve the problem of designing (11) to ensure stability along the pass plus a given level of disturbance attenuation as measured by the \mathcal{H}_2 norm. Hence, we can consider the process state-space model (7) with the disturbance input vector ν deleted.

Application of (11) (with $B_{12} = 0$ in the process state-space model) results in the controlled process state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A + BK_1)x_{k+1}(t) + (B_0 + BK_2)y_k(t) \\ &\quad + B_{11}w_{k+1}(t) \\ y_{k+1}(t) &= (C + DK_1)x_{k+1}(t) + (D_0 + DK_2)y_k(t) \end{aligned} \quad (32)$$

and hence the 2-D transfer-function matrix between the disturbance vector and the current pass profile is given by

$$\begin{aligned} G_{yw}^{cl}(s, z_2) &= [0 \ I] \\ &\quad \times \begin{bmatrix} sI - (A + BK_1) & -(B_0 + BK_2) \\ -z_2(C + DK_1) & I - z_2(D_0 + DK_2) \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}. \end{aligned}$$

The following result gives a solution to the this problem with a design algorithm.

Theorem 2: The process (32) is stable along the pass and has prescribed \mathcal{H}_2 disturbance attenuation bound $\gamma_2 > 0$ if there exist matrices $W_1 \succ 0$, $W_2 \succ 0$, X , N_1 and N_2 such that the LMI (33), shown at the bottom of the next page, holds together with the LMI, and

$$\begin{aligned} & \text{trace}(X) < \gamma_2^2 \\ & \begin{bmatrix} X & B_{11}^T \\ B_{11} & W_1 \end{bmatrix} \succ 0 \end{aligned} \quad (34)$$

where X is an additional symmetric matrix of compatible dimensions. If these conditions hold, the control law matrices K_1 and K_2 are given by

$$K_1 = N_1 W_1^{-1}, \quad K_2 = N_2 W_2^{-1}. \quad (35)$$

Proof: Applying Theorem 1 to (32) yields the following sufficient condition for stability along the pass, as shown by (36) at the bottom of the page, where $\Lambda_1 = A^T P_1 + P_1 A + K_1^T B^T P_1 + P_1 B K_1$. Now set $W_1 = P_1^{-1}$, $W_2 = P_2^{-1}$, and pre- and post-multiply both sides of this last inequality by $\text{diag}(W_2, W_1, W_2, I)$ to obtain (37), as shown at the bottom of the page, where $\Lambda_2 = W_1 A^T + A W_1 + W_1 K_1^T B^T + B K_1 W_1$. Next, make use of the following change of variables $N_1 = K_1 W_1$ and $N_2 = K_2 W_2$ to yield (33). Also, in this case the inequality (25) becomes

$$\text{trace} \left([B_{11}^T \ 0] \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \right) < \gamma_2^2 \quad (38)$$

which is equivalent to (34). To see this, first note that this last expression is equivalent to

$$\begin{bmatrix} I & -B_{11}^T W_1^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} X & B_{11}^T & 0 \\ B_{11} & W_1 & 0 \\ 0 & 0 & W_2 \end{bmatrix} \times \\ \begin{bmatrix} I & 0 & 0 \\ -W_1^{-1} B_{11} & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} X - B_{11}^T W_1^{-1} B_{11} & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & W_2 \end{bmatrix}$$

and also

$$X \succ B_{11}^T W_1^{-1} B_{11}.$$

This guarantees that the constraint (34) is satisfied when (38) holds and the proof is complete. \blacksquare

Remark 4: The \mathcal{H}_2 disturbance rejection bound γ_2 in the LMI of (34) can be minimized by using linear objective minimization algorithm

$$\min_{W_1 \succ 0, W_2 \succ 0, X, N_1, N_2, \mu > 0} \mu \\ \text{subject to (33) and (34) with } \mu = \gamma_2^2. \quad (39)$$

V. \mathcal{H}_2 STABILIZATION IN THE PRESENCE OF UNCERTAINTY

In this section, the aim is to design (11) to ensure stability along the pass of a process with uncertainty in its state-space model description and a guaranteed bound on disturbance rejection as measured by the \mathcal{H}_2 norm. The analysis will make use of the following well known result.

Lemma 5: [5] Let Σ_1, Σ_2 be real matrices of appropriate dimensions. Then, for any matrix \mathcal{F} satisfying $\mathcal{F}^T \mathcal{F} \preceq I$ and a scalar $\epsilon > 0$ the following inequality holds:

$$\Sigma_1 \mathcal{F} \Sigma_2 + \Sigma_2^T \mathcal{F}^T \Sigma_1^T \preceq \epsilon^{-1} \Sigma_1 \Sigma_1^T + \epsilon \Sigma_2^T \Sigma_2.$$

Clearly, it is not necessary to assume that both the state and pass profile vectors have the same type of uncertainty associated with them. Here, we assume that the former is subject to polytopic uncertainty (i.e., the matrices A, B_0 , and B of the state-space model (1) are only known to lie in a given fixed polytope of matrices) as described by (14) i.e.,

$$[A \ B_0 \ B] \in \text{Co}([A^i \ B_0^i \ B^i]), \quad i = 1, 2, \dots, h. \quad (40)$$

$$\begin{bmatrix} -W_2 & CW_1 + DN_1 & D_0 W_2 + DN_2 & 0 \\ N_1^T D^T + W_1 C^T & W_1 A^T + A W_1 + N_1^T B^T + B N_1 & B_0 W_2 + B N_2 & W_1 C^T + N_1^T D^T \\ N_2^T D^T + W_2 D_0^T & W_2 B_0^T + N_2^T B^T & -W_2 & W_2 D_0^T + N_2^T D^T \\ 0 & CW_1 + DN_1 & D_0 W_2 + DN_2 & -I \end{bmatrix} < 0 \quad (33)$$

$$\begin{bmatrix} -P_2 & P_2 C + P_2 D K_1 & P_2 D_0 + P_2 D K_2 & 0 \\ K_1^T D^T P_2 + C^T P_2 & \Lambda_1 & P_1 B_0 + P_1 B K_1 & C^T + K_1^T D^T \\ K_2^T D^T P_2 + D_0^T P_2 & B_0^T P_1 + K_2^T B^T P_1 & -P_2 & D_0^T + K_2^T D^T \\ 0 & C + D K_1 & D_0 + D K_2 & -I \end{bmatrix} < 0 \quad (36)$$

$$\begin{bmatrix} -W_2 & CW_1 + D K_1 W_1 & D_0 W_2 + D K_2 W_2 & 0 \\ W_1 K_1^T D^T + W_1 C^T & \Lambda_2 & B_0 W_2 + B K_2 W_2 & W_1 C^T + W_1 K_1^T D^T \\ W_2 K_2^T D^T + W_2 D_0^T & W_2 B_0^T + W_2 K_2^T B^T & -W_2 & W_2 D_0^T + W_2 K_2^T D^T \\ 0 & CW_1 + D K_1 W_1 & D_0 W_2 + D K_2 W_2 & -I \end{bmatrix} < 0 \quad (37)$$

The current pass profile updating equation in (1) is assumed to be subject to norm-bounded uncertainty as defined by (12) and (13), i.e.,

$$y_{k+1}(t) = (C + \Delta C)x_{k+1}(t) + (D_0 + \Delta D_0)y_k(t) + (D + \Delta D)u_{k+1}(t) \quad (41)$$

where

$$[\Delta C \quad \Delta D_0 \quad \Delta D] = H_2 \mathcal{F} [E_1 \quad E_2 \quad E_3] \quad (42)$$

and the matrix \mathcal{F} satisfies (13).

Now we have the following result which gives an existence condition for a solution to the design problem here.

Theorem 3: Suppose that (11) is applied to a differential linear repetitive process described by (7) (with the disturbance vector ν deleted) in the presence of uncertainty which satisfies (40) and (42). Then, the resulting controlled process is stable along the pass for all admissible uncertainties and has prescribed \mathcal{H}_2 disturbance attenuation $\gamma_2 > 0$ if there exist matrices $W_1 \succ 0$, $W_2 \succ 0$, X , N_1 , and N_2 and a scalar $\epsilon > 0$, such that the LMIs (34) and (43), as shown at the bottom of the page, are feasible and where

$$\Lambda_4 = W_1 A^{iT} + A^i W_1 + N_1^T B^{iT} + B^i N_1.$$

If these conditions hold, the control law matrices K_1 and K_2 are again given by (35).

Proof: An important property of LMIs is that they form a convex constraint on the decision variables vector. This means that any convex combination of solutions taken from a feasible set of LMIs is also a solution. Hence, in case of uncertainty modeled with a polytopic model [see (14)], we only need to find a solution for all vertices of the polytope to obtain this for all elements of the uncertainty set.

Given this fact and the conditions of Theorem 2, it follows immediately that the second equation at the bottom of the page holds, and the remaining difficulty is that the matrices

$\{\Delta C, \Delta D_0, \Delta D\}$ are unknown and hence underlying LMI cannot be solved. To overcome this, i.e., to find an LMI problem formulation, rewrite the second term as

$$\begin{aligned} & \begin{bmatrix} H_2 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{F} & 0 & 0 & 0 \\ 0 & \mathcal{F} & 0 & 0 \\ 0 & 0 & \mathcal{F} & 0 \\ 0 & 0 & 0 & \mathcal{F} \end{bmatrix} \begin{bmatrix} 0 & 0 & \Lambda_5 & \Lambda_6 \\ 0 & 0 & \Lambda_5 & \Lambda_6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Lambda_5^T & \Lambda_5^T & 0 & 0 \\ \Lambda_6^T & \Lambda_6^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{F}^T & 0 & 0 & 0 \\ 0 & \mathcal{F}^T & 0 & 0 \\ 0 & 0 & \mathcal{F}^T & 0 \\ 0 & 0 & 0 & \mathcal{F}^T \end{bmatrix} \\ & \times \begin{bmatrix} H_2^T & 0 & 0 & 0 \\ 0 & H_2^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0 \end{aligned}$$

where $\Lambda_5 = E_1 W_1 + E_3 N_1$, $\Lambda_6 = E_2 W_2 + E_3 N_2$.

Use (in an obvious manner and hence the details are omitted) of the result of Lemma 5 followed by application of the Schur's complement formula now gives (43) and the proof is complete. ■

Remark 5: This last result is based on choosing a single Lyapunov function for both the \mathcal{H}_2 and \mathcal{H}_∞ criteria. In the 1-D systems case, this is a well-known procedure termed the ‘‘Lyapunov shaping paradigm’’ in the literature [14].

VI. MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ ANALYSIS

In this section, we address the question of when can (11) be designed for processes described by (7) to minimize the \mathcal{H}_2 norm from w to y , denoted here by $\|G_{yw}^{cl}(s, z_2)\|_2$, and keep the \mathcal{H}_∞ norm from ν to y , denoted here by $\|G_{y\nu}^{cl}(s, z_2)\|_\infty$, below some prescribed level. Note also that if only w is present then this problem reduces to the \mathcal{H}_2 problem already solved in this paper. Similarly, if only ν is present, then we obtain the \mathcal{H}_∞ problem solved in [10]. Hence, the problem here has a solution if the inequalities (36), (25), and the first equation shown at the

$$\begin{bmatrix} -I + \epsilon H_2 H_2^T & (*) & (*) & (*) & (*) & (*) \\ 0 & -W_2 + \epsilon H_2 H_2^T & (*) & (*) & (*) & (*) \\ W_1 C^T + N_1^T D^T & N_1^T D^T + W_1 C^T & \Lambda_4 & (*) & (*) & (*) \\ W_2 D_0^T + N_2^T D^T & N_2^T D^T + W_2 D_0^T & W_2 B_0^{iT} + N_2^T B^{iT} & -W_2 & (*) & (*) \\ 0 & 0 & E_1 W_1 + E_3 N_1 & E_2 W_2 + E_3 N_2 & -\epsilon I & (*) \\ 0 & 0 & E_1 W_1 + E_3 N_1 & E_2 W_2 + E_3 N_2 & 0 & -\epsilon I \end{bmatrix} < 0 \quad (43)$$

$$\begin{aligned} & \begin{bmatrix} -I & (*) & (*) & (*) \\ 0 & -W_2 & (*) & (*) \\ W_1 C^T + N_1^T D^T & N_1^T D^T + W_1 C^T & \Lambda_4 & (*) \\ W_2 D_0^T + N_2^T D^T & N_2^T D^T + W_2 D_0^T & W_2 B_0^{iT} + N_2^T B^{iT} & -W_2 \end{bmatrix} \\ & + \begin{bmatrix} 0 & (*) & (*) & (*) \\ 0 & 0 & (*) & (*) \\ W_1 \Delta C^T + N_1^T \Delta D^T & W_1 \Delta C^T + N_1^T \Delta D^T & 0 & (*) \\ W_2 \Delta D_0^T + N_2^T \Delta D^T & W_2 \Delta D_0^T + N_2^T \Delta D^T & 0 & 0 \end{bmatrix} < 0 \end{aligned}$$

bottom of the page, hold for some $P_1 \succ 0$, $P_2 \succ 0$, $R_1 \succ 0$, and $R_2 \succ 0$.

The main problem now is that we cannot linearize simultaneously the terms $K_1 R_1$, $K_1 P_1$ and $K_2 R_2$, $K_2 P_2$. This can be overcome by enforcing $P_1 = R_1$ and $P_2 = R_2$ (at the possible expense of increased conservativeness). Under these assumptions, the following result provides the LMI condition for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design.

Theorem 4: Suppose that (11) is applied to a differential linear repetitive process described by (7). Then, the resulting controlled process is stable along the pass and has prescribed \mathcal{H}_2 and \mathcal{H}_∞ norms bounds $\gamma_2 > 0$ and $\gamma_\infty > 0$, respectively, if there exist matrices $W_1 \succ 0$, $W_2 \succ 0$, X , N_1 , and N_2 such that the LMIs (33), (34), and (44) shown at the bottom of the page hold. Also, if this is the case the control law matrices K_1 and K_2 are given by (35).

Proof: This follows immediately from the results given or referenced above and hence the details are omitted. ■

Remark 6: The special case of (11) which minimizes the \mathcal{H}_2 norm of the resulting 2-D transfer-function matrix from w to y and keeps the \mathcal{H}_∞ of that from ν to y under some prescribed level γ_∞ , can be found using the following convex optimization procedure:

$$\begin{aligned} & \min_{W_1 \succ 0, W_2 \succ 0, X, N_1, N_2, \mu > 0} \mu \\ & \text{subject to (33), (34) and (44) with } \mu = \gamma_2^2 \end{aligned} \quad (45)$$

Remark 7: It is important to note that by adjusting γ_∞ we can trade off between \mathcal{H}_∞ and \mathcal{H}_2 performance. Hence, a tradeoff curve allows the designer to choose (11) as an appropriate compromise between robustness (measured with \mathcal{H}_∞ norm) and performance (measured with \mathcal{H}_2 norm).

VII. ALTERNATIVE \mathcal{H}_2 SOLUTION

Here, we solve the \mathcal{H}_2 problem in the presence of uncertainty based on the solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. This allows us to cope with uncertainty under an \mathcal{H}_2 performance constraint, and hence we can immediately proceed to solve the \mathcal{H}_2 problem for differential linear repetitive processes

in the case when the matrices A , B_0 , C , and D_0 are subject to norm-bounded uncertainty [see (12) and (13)].

Consider therefore a differential linear repetitive process over $0 \leq t \leq \alpha$, $k \geq 0$, with state-space model

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A + \Delta A)x_{k+1}(t) + Bu_{k+1}(t) \\ &\quad + (B_0 + \Delta B_0)y_k(t) + B_{11}w_{k+1}(t) \\ y_{k+1}(t) &= (C + \Delta C)x_{k+1}(t) + Du_{k+1}(t) \\ &\quad + (D_0 + \Delta D_0)y_k(t) \end{aligned} \quad (46)$$

where again ΔA , ΔB_0 , ΔC , and ΔD_0 represent admissible uncertainties, and it is assumed that they can be written in the form

$$\begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \sigma^{-1} \mathcal{F} \begin{bmatrix} E_1 & E_2 \end{bmatrix} \quad (47)$$

where H_1 , H_2 , E_1 , and E_2 are known constant matrices and \mathcal{F} is an unknown matrix with constant entries which satisfies (13). The design parameter σ here can be considered as a term available for use to attenuate the effects of the uncertainty.

It is a well-known fact in the 1-D linear systems case that linear fractional transformations (LFT) provide a general framework design in the presence of (certain types of) uncertainty. The basic use of an LFT (see, for example, [17]) is to isolate the uncertainty and represent it as interconnection of the nominal process with uncertainty block—see Fig. 1 (where LRP denotes an uncertainty free differential linear repetitive process of the form considered here).

Application of this technique to the process model of (46) (with the current pass input terms deleted as they do not have uncertainty associated with them) gives

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0 y_k(t) + H_1 \nu_{k+1}(t) + B_{11} w_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0 y_k(t) + H_2 \nu_{k+1}(t) \\ z_{k+1}(t) &= E_1 x_{k+1}(t) + E_2 y_k(t) + H_3 \nu_{k+1}(t) \\ \nu_{k+1}(t) &= \sigma^{-1} \mathcal{F} z_{k+1}(t). \end{aligned}$$

$$\begin{bmatrix} -R_2 & CR_1 + DK_1 R_1 & D_0 R_2 + DK_2 R_2 & B_{22} & 0 \\ R_1 C^T + R_1 K_1^T D^T & R_1 A^T + R_1 K_1^T B^T + AR_1 + BK_1 R_1 & B_0 R_2 + BK_2 R_2 & B_{21} & 0 \\ R_2 D_0^T + R_2 K_2^T D^T & R_2 B_0^T + R_2 K_2^T B^T & -R_2 & 0 & R_2 \\ B_{22}^T & B_{21}^T & 0 & -\gamma_\infty^2 I & 0 \\ 0 & 0 & R_2 & 0 & -I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} -W_2 & CW_1 + DN_1 & D_0 W_2 + DN_2 & B_{22} & 0 \\ W_1 C^T + N_1^T D^T & W_1 A^T + N_1^T B^T + AW_1 + BN_1 & B_0 W_2 + BN_2 & B_{21} & 0 \\ W_2 D_0^T + N_2^T D^T & W_2 B_0^T + N_2^T B^T & -W_2 & 0 & W_2 \\ B_{22}^T & B_{21}^T & 0 & -\gamma_\infty^2 I & 0 \\ 0 & 0 & W_2 & 0 & -I \end{bmatrix} \prec 0 \quad (44)$$

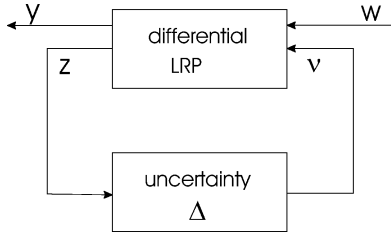


Fig. 1. LFT of the process with uncertainty.

Now, substituting the third equation into the last one in the above set of four and solving for $\nu_{k+1}(t)$ gives

$$\nu_{k+1}(t) = (I - \sigma^{-1}\mathcal{F}H_3)^{-1}\sigma^{-1}\mathcal{F}(E_1x_{k+1}(t) + E_2y_k(t)).$$

The process model (46) can be rewritten in the form of (7) as

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A + H_1\Upsilon E_1)x_{k+1}(t) + (B_0 + H_1\Upsilon E_2)y_k(t) \\ &\quad + Bu_{k+1}(t) + B_{11}w_{k+1}(t) \\ y_{k+1}(t) &= (C + H_2\Upsilon E_1)x_{k+1}(t) + (D_0 + H_2\Upsilon E_2)y_k(t) \\ &\quad + Du_{k+1}(t) \end{aligned} \quad (48)$$

where

$$\Upsilon = (I - \sigma^{-1}\mathcal{F}H_3)^{-1}\sigma^{-1}\mathcal{F}$$

and clearly $\det(I - \sigma^{-1}\mathcal{F}H_3) \neq 0$ is required for well-posedness of the feedback interconnection. In particular, setting $H_3 = 0$ in (48) gives this property and hence

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A + \sigma^{-1}H_1\mathcal{F}E_1)x_{k+1}(t) \\ &\quad + (B_0 + \sigma^{-1}H_1\mathcal{F}E_2)y_k(t) + Bu_{k+1}(t) \\ &\quad + B_{11}w_{k+1}(t) \\ y_{k+1}(t) &= (C + \sigma^{-1}H_2\mathcal{F}E_1)x_{k+1}(t) \\ &\quad + (D_0 + \sigma^{-1}H_2\mathcal{F}E_2)y_k(t) + Du_{k+1}(t) \end{aligned}$$

which is of the form (46). Hence, on applying (11) we have

$$\begin{aligned} \begin{bmatrix} \dot{x}_{k+1}(t) \\ y_{k+1}(t) \end{bmatrix} &= \left(\begin{bmatrix} A + BK_1 & B_0 + BK_2 \\ C + DK_1 & D_0 + DK_2 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} \right) \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} w_{k+1}(t) \end{aligned} \quad (49)$$

and if the matrices describing the uncertainty can be written in the form (47) we have the following result.

Theorem 5: Suppose that (11) is applied to a differential linear repetitive process described by (46) with uncertainty structure satisfying (47) and (13). Then, the resulting controlled

process is stable along the pass and has prescribed \mathcal{H}_2 norm bound $\gamma_2 > 0$ if there exist matrices $W_1 \succ 0$, $W_2 \succ 0$, X , N_1 , and N_2 such that the LMIs (33) and (34) of Theorem 2 hold together with

$$\begin{bmatrix} -W_2 & (*) & (*) & (*) & (*) \\ W_1C^T + N_1^TD^T & \Lambda_7 & (*) & (*) & (*) \\ W_2D_0^T + N_2^TD^T & W_2B_0^T + N_2^TB^T & -W_2 & (*) & (*) \\ H_2^T & H_1^T & 0 & -\sigma^2I & (*) \\ 0 & E_1W_1 & E_2W_2 & 0 & -I \end{bmatrix} \prec 0 \quad (50)$$

where $\Lambda_7 = W_1A^T + AW_1 + BN_1 + N_1^TB^T$.

Also, if these conditions hold, control law matrices K_1 and K_2 are given by (35).

Proof: Introduce the matrices

$$\begin{aligned} \bar{A}_1 &= \begin{bmatrix} A + BK_1 & B_0 + BK_2 \\ 0 & 0 \end{bmatrix} \\ \bar{A}_2 &= \begin{bmatrix} 0 & 0 \\ C + DK_1 & D_0 + DK_2 \end{bmatrix} \\ \bar{H}_1 &= \begin{bmatrix} H_1 \\ 0 \end{bmatrix}, \quad \bar{H}_2 = \begin{bmatrix} 0 \\ H_2 \end{bmatrix}, \quad E = [E_1 \quad E_2]. \end{aligned}$$

Then, from the associated Lyapunov function (interpret (3), (4) and Lemma 1), it follows immediately that a process described by (48) is stable along the pass if there exist matrices $P_1 \succ 0$, $P_2 \succ 0$, and $P_3 \succ 0$ such that the following LMI holds:

$$\begin{aligned} (\bar{A}_1 + \bar{H}_1\sigma^{-1}\mathcal{F}E)^T P + P(\bar{A}_1 + \bar{H}_1\sigma^{-1}\mathcal{F}E) - R + \\ (\bar{A}_2 + \bar{H}_2\sigma^{-1}\mathcal{F}E)^T S(\bar{A}_2 + \bar{H}_2\sigma^{-1}\mathcal{F}E) \prec 0 \end{aligned}$$

where $P = \text{diag}(P_1, 0)$, $S = \text{diag}(P_3, P_2)$, $R = \text{diag}(0, P_2)$. An obvious application of the Schur's complement formula to this last expression now yields the equation shown at the bottom of the page.

Now make an obvious application of the result of Lemma 5, and pre- and post-multiply the outcome by $\text{diag}(\epsilon^{-(1/2)}I, \epsilon^{-(1/2)}I, \epsilon^{-(1/2)}I)$. Also, introduce the notation $\bar{P}_1 = \epsilon^{-1}P_1$, $\bar{P}_2 = \epsilon^{-1}P_2$, and then an obvious application of the Schur's complement formula gives the equation shown at the bottom of the next page, where $\Lambda_8 = A^T\bar{P}_1 + \bar{P}_1A + \bar{P}_1BK_1 + K_1^TB^T\bar{P}_1$.

Note that this last condition is not linear in $\bar{P}_1, \bar{P}_2, \bar{P}_3, K_1$, and K_2 . However, this difficulty can be avoided by employing the following transformations. First, pre- and post-multiply the last expression by $\text{diag}(\bar{P}_2^{-1}, \bar{P}_1^{-1}, \bar{P}_2^{-1}, I, I)$ and finally set $W_1 = \bar{P}_1^{-1}$, $W_2 = \bar{P}_2^{-1}$, $N_1 = K_1\bar{P}_1^{-1}$, $N_2 = K_2\bar{P}_2^{-1}$ to obtain (50). ■

$$\begin{aligned} \begin{bmatrix} -P_2 & (*) & (*) & (*) & (*) \\ C^TP_2 + K_1^TD^TP_2 & A^TP_1 + P_1A + P_1BK_1 + K_1^TB^TP_1 & (*) & (*) & (*) \\ D_0^TP_2 + K_2^TD^TP_2 & B_0^TP_1 + K_2^TB^TP_1 & (*) & (*) & (*) \\ (*) & (*) & (*) & (*) & (*) \\ (*) & (*) & (*) & (*) & (*) \end{bmatrix} + \begin{bmatrix} 0 \\ E_1^T \\ E_2^T \end{bmatrix} \mathcal{F}^T [\sigma^{-1}H_2^TP_2 \quad \sigma^{-1}H_1^TP_1 \quad 0] \\ + \begin{bmatrix} \sigma^{-1}P_2H_2 \\ \sigma^{-1}P_1H_1 \\ 0 \end{bmatrix} \mathcal{F}[0 \quad E_1 \quad E_2] \prec 0 \end{aligned}$$

Remark 8: Design to minimize σ and keep the \mathcal{H}_2 norm of the 2-D transfer-function matrix from ν to y below some prescribed level γ_2 can be undertaken using the following linear objective minimization procedure:

$$\min_{W_1 \succ 0, W_2 \succ 0, X, N_1, N_2, \mu > 0} \mu$$

subject to (34), (35) and (50) with $\mu = \sigma^2$

which, due to the presence of the term σ^{-1} in the uncertainty model of (47), provides the essential advantage of allowing us to extend the uncertainty boundaries, i.e., increase the robustness.

The last result (i.e., the LMI (50)) shows that there exists a link between robust \mathcal{H}_2 stabilization of processes described by (48) and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ stabilization of processes described by (7). This means that the same control law solves the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and robust \mathcal{H}_2 problems.

To see this, assume that $\sigma = \gamma_\infty$ and apply the same transformation used to obtain (44) from (15) where the matrices \hat{B}_1 , \hat{D}_1 and L of (16) are now given by

$$\hat{B}_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 0 \\ H_2 \end{bmatrix}, \quad L = [E_1 \quad E_2].$$

This link can also be established using the Lyapunov stability condition of Lemma 1. In particular from (13)

$$\sigma^{-2} \mathcal{F}^T \mathcal{F} \leq \sigma^{-2} I$$

and hence, assuming $\sigma = \gamma_\infty$, we have

$$\begin{aligned} \nu_{k+1}^T(t) \nu_{k+1}(t) &= \gamma_\infty^{-2} z_{k+1}^T(t) \mathcal{F}^T \mathcal{F} z_{k+1}(t) \\ &\leq \gamma_\infty^{-2} z_{k+1}^T(t) z_{k+1}(t) \end{aligned}$$

which can be rewritten as

$$\gamma_\infty^2 \nu_{k+1}^T(t) \nu_{k+1}(t) - z_{k+1}^T(t) z_{k+1}(t) \leq 0.$$

Hence, the inequality

$$\Delta V(k, t) + z_{k+1}^T(t) z_{k+1}(t) - \gamma_\infty^2 \nu_{k+1}^T(t) \nu_{k+1}(t) < 0 \quad (51)$$

can hold only if the term $\Delta V(k, t) < 0$, i.e., stability along the pass holds. Also, this inequality can be regarded as arising from the associated Hamiltonian for differential linear repetitive processes—see [10] for further details (in addition to its use in the \mathcal{H}_∞ analysis in this chapter). Moreover, if (51) holds then the process is stable along the pass and the \mathcal{H}_∞ norm from ν

to y is kept below the prescribed level $\gamma_\infty > 0$. Finally, routine manipulations establish the link between robust \mathcal{H}_2 control and $\mathcal{H}_2/\mathcal{H}_\infty$ control detailed above.

VIII. CONCLUSION

This paper has developed substantial new results on the control of differential linear repetitive processes which are a distinct class of 2-D linear systems of both systems theoretic and applications interest. These results show that it is possible to define an \mathcal{H}_2 norm for these processes which has a well-defined physical basis. It has been shown that stability along the pass can be expressed in terms of the \mathcal{H}_2 norm and that a control law activated by a linear combination of the current pass state vector and the previous pass profile can be designed with a prescribed degree of disturbance attenuation. Moreover, this setting allows us to consider the more realistic case when the disturbance vector varies from pass-to-pass as opposed to previous work [15] where it was necessary to assume that the disturbances considered were constant from pass-to-pass (but full decoupling as opposed to attenuation was possible).

It has also been shown how these \mathcal{H}_2 results can be combined with those (partially) obtained previously to provide a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ setting for analysis and compensator design in the presence of uncertainty and disturbances. Overall, these results represent a very significant step towards a complete systems theory with implementable design algorithms for the processes considered. An obvious next stage is to consider design to meet detailed performance specifications.

APPENDIX

PROOF OF THE PARSEVAL'S THEOREM

Lemma 7:

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |w_k(t)|^2 dt = \frac{1}{2\pi} \frac{1}{\Omega} \int_{-\infty}^{\infty} |W(e^{j\omega_1 T}, j\omega_2)|^2 d\omega_1 d\omega_2 \quad (52)$$

where $\Omega := 2\pi/T$ and

$$W(e^{j\omega_1 T}, j\omega_2) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} w_k(t) e^{-j\omega_1 T k} e^{-j\omega_2 t} dt$$

is the double or hybrid (continuous-discrete) Fourier transform of the signal sequence $w_k(t)$.

Proof: First note that by a double inverse Fourier transform, i.e.,

$$w_k(t) = \frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} W(e^{j\omega_1 T}, j\omega_2) e^{j\omega_2 t} e^{j\omega_1 T k} d\omega_1 d\omega_2$$

$$\begin{bmatrix} -\bar{P}_2 & (*) & (*) & (*) & (*) \\ C^T \bar{P}_2 + K_1^T D^T \bar{P}_2 & \Lambda_8 & (*) & (*) & (*) \\ D_0^T \bar{P}_2 + K_2^T D^T \bar{P}_2 & B_0^T \bar{P}_1 + K_2^T B^T \bar{P}_1 & -\bar{P}_2 & (*) & (*) \\ H_2^T \bar{P}_2 & H_1^T \bar{P}_1 & 0 & -\sigma^2 I & (*) \\ 0 & E_1 & E_2 & 0 & -I \end{bmatrix} < 0$$

we have

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |w_k(t)|^2 dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} w_k(t) \frac{1}{\Omega} \Xi dt$$

where

$$\Xi = \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} W^*(e^{j\omega_1 T}, j\omega_2) e^{-j\omega_2 t} e^{-j\omega_1 T k} d\omega_1 d\omega_2.$$

The right-hand side in this last expression can be rewritten as

$$\frac{1}{\Omega} \frac{1}{2\pi} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \int_{-\infty}^{\infty} W^*(e^{j\omega_1 T}, j\omega_2) X d\omega_1 d\omega_2$$

where

$$X := \left[\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\omega_2 t} e^{-j\omega_1 T k} dt \right] d\omega_1 d\omega_2$$

which is the Fourier transform $W(e^{j\omega_1 T}, j\omega_2)$ and the proof is complete. ■

In this paper, we assume that T is normalized to unity and hence $\Omega = 2\pi$.

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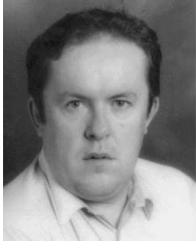
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