ON THE LIFTING OF THE NAGATA AUTOMORPHISM

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Abstract. It is proved that the Nagata automorphism (Nagata coordinates, respectively) of the polynomial algebra $F[x,y,z]$ over a field F cannot be lifted to a z-automorphism (z-coordinate, respectively) of the free associative algebra $F\langle x,y,z\rangle$. The proof is based on the following two new results which have their own interests: degree estimate of $Q *_F F\langle x_1,\dots,x_n\rangle$ and tameness of the automorphism group $\mathrm{Aut}_Q(Q *_F F\langle x,y\rangle)$.

1. **Introduction and main results**

The long-standing famous Nagata conjecture for characteristic 0 was proved by Shestakov and Umirbaev [12, 13], and a strong version of the Nagata conjecture was proved by Umirbaev and Yu [14]. That is, the Nagata automorphism $(x-2y(y^2+xz)-(y^2+xz)^2z, y+(y^2+xz)z, z)$ (Nagata coordinates $x-2y(y^2+xz)-(y^2+xz)^2z$ and $y+(y^2+xz)z$ respectively) is (are) wild. In [11, 14], a stronger question (which implies the Nagata conjecture and the strong Nagata conjecture) was raised: whether the Nagata automorphism (coordinates) of the polynomial algebra $F[x,y,z]$ can be lifted to an automorphism (coordinates) of the free associative $F\langle x,y,z\rangle$ over a field F? We can also formulate

The General Lifting Problem. Let $\phi = (f_1,\dots,f_n)$ be an automorphism of the polynomial algebra $F[x_1,\dots,x_n]$ over a field F. Does there exists an F-automorphism $\phi' = (f_1',\dots,f_n')$ of the free associative algebra $F\langle x_1,\dots,x_n\rangle$ such that each f_i is the abelianization of f_i'?

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For $n = 2$, the answer of the above problem is positive, as due to Jung [5] and van der Kulk [6] every automorphism of $F[x, y]$ is composation of linear and elementary automorphisms which are liftable to automorphisms of $F\langle x, y\rangle$. Moreover, Makar-Limanov [9] and Czerniakiewicz [7] proved independently that $\mathrm{Aut}(F\langle x, y\rangle)$ is actually isomorphic to $\mathrm{Aut}(F[x, y])$, which implies that the lifting is unique.

In this paper we prove the following new result, which partially answers the question raised in [14] negatively. The result can be viewed as the first step to attack the general lifting problem. In a forthcoming paper [1], we will deal the general lifting problem.

Theorem 1.1. *Let $(f,\ g)$ be an wild $F[z]$-automorphism of $F[x, y, z] = F[z][x, y]$. Then (f, g, z), as an F-automorphism of $F[x, y, z]$, cannot be lifted to an automorphism of $F\langle x, y, z\rangle$ fixing z.*

The crucial step to prove Theorem 1.1 is the following

Theorem 1.2. *Let (f, g) be a wild $F[z]$-automorphism of $F[x, y, z] = F[z][x, y]$, which can be effectively obtained as the product of the canonical sequence of uniquely determined alternative operations (elementary $F(z)$-automorphisms), and the sequence contains an elementary $F(z)$-automorphism of the type $(x, y + z^{-k}x^l + \dots)$ or $(x + z^{-k}y^l + \dots, y)$ where $l > 1$. Then (f, g, z), as an F-automorphism of $F[x, y, z]$, cannot be lifted to an automorphism of $F\langle x, y, z\rangle$ fixing z.*

Corollary 1.3. *The Nagata automorphism cannot be lifted to an automorphism of $F\langle x, y, z\rangle$ fixing z.*

Corollary 1.4. *Let (f, g) be a wild $F[z]$-automorphism of $F[x, y, z] = F[z][x, y]$. Then neither f nor g can be lifted to a z-coordinate of $F\langle x, y, z\rangle$. In particular, the Nagata coordinates $x - 2y(y^2 + xz) - (y^2 + xz)^2 z$ and $y + (y^2 + xz)z$ cannot be lifted to any z-coordinate of $F\langle x, y, z\rangle$.*

Proof. Suppose (f, h) is an $F[z]$-automorphism, then obviously (f, h) is the product of (f, g) and an elementary $F[z]$-automorphism of the type (x, h_1). Therefore (f, h) is liftable if and only if (f, g) is liftable. Hence any $F[z]$-automorphism of the type (f, h) is not liftable. Therefore f cannot be lifted to a z-coordinate of $F\langle x, y, z\rangle$. Same for g. $\square$

Crucial to the proof of Theorem 1.2 is the following new result, that implies that the automorphism group $\mathrm{Aut}_Q(Q *_F F\langle x, y\rangle)$ is tame, which has its own interests.

Theorem 1.5 (on degree increasing process)**.** *Let Q be an extension field over a field F. A Q-automorphism of $Q*_F F\langle x,y\rangle$ can be effectively obtained as the product of a sequence of uniquely determined alternating operations (elementary automorphisms) of the following types:*

- $x \to x,\ y \to ryr' + \sum r_0xr_1x\cdots r_kxr_{k+1},$
- $x \to qxq' + q_0\sum yq_1y\cdots q_kyq_{k+1}, \quad y \to y$

where $r, q, r_j, q_j \in Q$.

The following new result of degree estimate is also essential to the proof of Theorem 1.2.

Theorem 1.6 (Degree estimate)**.** *Let Q be an extension field of a field F. Let $A = Q *_F F\langle x_1, \ldots, x_n\rangle$ be a co-product of Q and the free associative algebra $F\langle x_1, \ldots, x_n\rangle$ over F. Suppose $f, g \in A$ are algebraically independent over Q, f^+ and g^+ are algebraically independent over Q; or f^+ and g^+ are algebraically dependent, and neither f^+ is Q-proportional to a power g^+, nor g^+ is Q-proportional to a power f^+. Let $P \in Q *_F F\langle x, y\rangle\backslash Q$. Then*

$$\deg(P(f,g)) \geq \frac{\deg([f,\ g])}{\deg(fg)} w_{\deg(f),\deg(g)}(P),$$

where the degree is the usual homogeneous degree with respect to $x_1, \ldots, x_n$ and $w_{r,s}$ is the weight degree with respect to r, s.

Note that u is **proportional** to v for $u,\ v \in Q *_F F\langle x_1, \ldots, x_n\rangle$ means that there exist $p_1, \ldots, p_m; q_1, \ldots, q_m \in Q$ such that $u = \Sigma_{i=1}^m p_i v q_i$ (it is important that 'proportional' is not reflexive, i.e. u is proportional to v does not imply v is proportional to u), and that f^+ is the highest homogeneous form of f.

Remark 1.7. Theorem 1.6 is still valid for an arbitrary division ring Q over a field F. The proof is almost the same. When $Q = F(z)$ then the result can be directly deduced from the degree estimate in [10, 8] via substitution $\psi:\ x \to P_1(z)xP_2(z); y \to R_1(z)yR_2(z)$ for appropriate $P_i, R_i \in F[z]$. For any element τ in $Q *_F F\langle x, y\rangle$ there exist $P_i, R_i \in F[z]$ such that $\psi(\tau) \in F\langle x, y, z\rangle$. In the sequel we only use this special case regarding the lifting problem.

2. Proofs

Proof of Theorem 1.6 Similar to the proof of the main result of Li and Yu [8], where Bergman's Lemma [3, 4] on centralizers is used. See also Makar-Limanov and Yu [10] for the special case of characteristic 0, where Bergman's Lemma on radical [3, 4] is used. □

Proof of Theorem 1.5. Let $\phi = (f, g)$ be a Q-automorphism in $\mathrm{Aut}_Q(Q *_F F\langle x, y\rangle)$ which is not linear, namely,

$$\deg_{x,y}(f) + \deg_{x,y}(g) \geq 3.$$

By Theorem 1.5, we obtain that either a power of f^+ is proportional to g^+, or a power of g^+ is proportional to f^+. Now the proof is done by induction. □

To prove the main result, we need a few more lemmas.

Definition. Let D be a domain containing a field K, E the field of fractions of D. A monomial $\in E *_K K\langle x, y\rangle$ of the following form,

$$......ptq......$$

where $t \in E\backslash D$, $p, q \in \{x, y\}$, is called a *sandwich monomial*, or just a *sandwich* for short.

Lemma 2.1 (on sandwich preserving)**.** *In the constructive decomposation in Theorem 1.5, suppose a sandwich ...ptq... (where $p, q \in \{x, y\}$, $t \in F(z)\backslash F[z]$, appears on some step during the process of the effective decomposation, then there will be some sandwich in any future step.*

Proof. Let f be the polynomial obtained in the $(n-1)^{-th}$ step of the effective operation in Theorem 1.5, $k = \deg(f)$. Take all sandwiches s_α of the maximum total degree with respect to x and y. Let $S = \sum s_\alpha$ be their sum. Let $T = \sum t_\beta$ be the sum of components (monomials) t_β of f maximum total degree respect to x and y. It is possible that $s_\alpha = t_\beta$ for some α, β, then $\deg(s_\alpha) = \deg(t_\beta)$ for all α, β. In this case $T = S + D$.

Suppose the n^{-th} step has the following form $x \to x, y \to y + G(x)$. Let $\bar{G}$ be the sum of monomials in G with the maximum degree. Then $\bar{G}(x) = \widetilde{G}(x, \ldots, x)$ where

$$\widetilde{G}(x_1, \ldots, x_n) = \sum_i q_{i,1} x_1 q_{i,2} x_2 \cdots x_m q_{i,m+1}, \quad q_{ij} \in F(z),$$

m be the degree of the n^{-th} step operation (elementary automorphism).

Let $\deg(S) < \deg(T)$. Consider elements of the form $\widetilde{G}(S, T, \ldots, T)$. It is a linear combination of sandwiches. All of them have the following form

$$q_0 s_i q_1 t_2 \cdots t_{m-1} q_m, \quad q_i \in Q.$$

Their sum is not zero, because for any polynomial of the form $H = \sum_i q_{i_1} x_1 q_{i_2} x_2 \cdots x_m q_{i_{m+1}}$ such that $H(x, \ldots, x) \neq 0$ and for any $S, T \notin Q$, $H(S, T, \ldots, T) \neq 0$.

If $\deg(S) = \deg(T)$, we consider elements of the form $\widetilde{G}(S, S, \ldots, S)$. It is a linear combination of sandwiches. All of them have the following form

$$q_0 s_1 q_1 \cdots s_m q_m, \quad q_i \in Q.$$

s_i are monomials from S. Their sum is not zero, because for any polynomial of the form $H = \sum_i q_{i_1} x_1 q_{i_2} x_2 \cdots x_m q_{i_{m+1}}$ such that $H(x, \ldots, x) \neq 0$ and for any $S \in Q$, $H(S, \ldots, S) \neq 0$.

Now we are going to prove (via degree estimate) that they cannot cancel out by other monomials (which must be sandwiches). That is, there are no other sandwiches which are in this form. They cannot be produced by $H(R_1, \ldots, R_m)$ where R_i are monomials either from S or T. This can be easily seen from the following argument:

Suppose $\deg(S) = \deg(T)$ and $D \neq 0$. Then if we substitute monomials forming S and D in different 'words', the outcomes would be different. Similarly suppose $\deg(S) < \deg(T)$ and $D \neq 0$, then substituting monomials forming S and T in different 'words', the outcomes would be different. Suppose $D = 0$, i.e. $S = T$. Then $H(T, \ldots, T) = H(S, \ldots, S)$. Obviously in this case we need to do nothing.

Suppose we get such a sandwich because an element from $F(z)\backslash F[z]$ appears between summands of polynomials, obtained by the the (previous) $(n-1)^{-th}$ step. It means that 'fractional coefficient' in $F(z)\backslash F[z]$ appears in the position in some term, between two monomials obtained on the $(n-1)$-th step. Let us describe this situation in more details.

Let

$$x \to \sum v_i, \quad y \to \sum u_i$$

be an automorphism, obtained on the $(n-1)$-th step. Consider n-th step:

$$x \to x,\ y \to y + \sum_i q_0^i x q_1^i \cdots x q_{n_i}^i.$$

Let $v_i = a_i \bar{v}_i b_i$ where $a_i, b_i \in Q, \bar{v}_i$ begins with either x or y and also ends with either x or y.

Suppose the leftmost factor $q \in F(z)\backslash F[z]$ corresponding to the leftmost factor q in monomial s_i in s appears in the corresponding sandwich w. Then it has the form

$$w = q_0^i v_{\alpha_1} q_1^i \cdots a_{\alpha_k} \bar{v}_{\alpha_k} b_{\alpha_k} q_k^i a_{\alpha_{k+1}} \bar{v}_{\alpha_{k+1}} b_{\alpha_{k+1}} \cdots a_{\alpha_{n_i}} \bar{v}_{\alpha_{n_i}} b_{\alpha_{n_i}} q_{n_i}^i$$

and the position $\bar{v}_{\alpha_k} b_{\alpha_k} q_k^i a_{\alpha_{k+1}} \bar{v}_{\alpha_{k+1}}$ corresponds to the position of fractional coefficient in the sandwich $s \cdot \prod_{i=1}^{n-1} v_i$ living inside $s = s_1 q s_2$, s_1 ends with x or y, s_2 begins with x or y. Then

$$q = b_{\alpha_k} q_k^i a_{\alpha_{k+1}}, s_1 = q_0^i v_{\alpha_1} q_1^i \cdots a_{\alpha_k} \bar{v}_{\alpha_k}, s_2 T_{i_2} \cdots T_{i_n} =$$

$$= \bar{v}_{\alpha_{k+1}} b_{\alpha_{k+1}} \cdots a_{\alpha_{n_i}} \bar{v}_{\alpha_{n_i}} b_{\alpha_{n_i}} q_{n_i}^i.$$

Only in that case cancellation is possible. Here T_i are monomial summands of T.

Now let us compare the degrees. $\deg(s_1) < \deg(s)$, $\sum_{i=k}^n \deg(v_i) \leq (n_i - k + 1)\deg(T) \leq (m-1)\deg(T)$. Hence

$$\deg(W) < \deg(s) + (m-1)\deg(T) = \deg(\widetilde{G}(s, T, \ldots, T))$$

so any cancellation is impossible. □

Lemma 2.2 (on coefficient improving). *a) Let $x' = pxq$; $p, q \in F(z)$, $M_{\vec{q}}(x) = xq_1xq_2 \cdots x$, $q_i' = q^{-1} q_i p^{-1}$. Then $M_{\vec{q}}(x') = pM_{\vec{q}'}(x)q$.*

b) Take the process in Theorem 1.5 without sandwiches. Then after each step (except the last step), the outcome (f, g) has the following properties:

- *Both f and g are sandwich-free.*
- *The left coefficients of f and g belong to $F[z]$. Moreover, the two coefficients are relatively prime.*
- *The right coefficients of f and g belong to $F[z]$. Moreover, the two coefficients are relatively prime.*

Now we can clearly see that the outcome of the last step also has the above property.

Proof. a) is obvious; b) is a consequence of a). □

Lemma 2.3. *Let $f \in F(z) *_F F\langle x, y\rangle$, $P(u) \in F(z) * F[u]$ such that each monomial has degree ≥ 2 respect to x and y. Suppose that one of the coefficients of P has zero right z-degree and one of the coefficients of f has zero right z-degree, and there is no coefficients of P and f with negative right z-degree. Then $P(f)$ has one of the coefficients with zero right z-degree and the degree (respect to x and y) of corresponding term is strictly more then* $\deg(f)$.

Proof. Consider the highest degree monomials of P and f with zero right z-degree, let $\widetilde{P}, \widetilde{f}$ will be their sums. Let g be sum of terms of f with zero right z-degree, h be the sum of terms of f of maximal degree.

Now consider again the highest degree monomials in $P(u)$ with zero right z-degree and substitute $\widetilde{f}$ on the rightmost position instead of u and h on other positions of u. We shall get some terms with non-zero sum T (same argument as in the proof of sandwich lemma). All such terms have zero right z-degree.

It remains to prove that such terms cannot cancel out from the other terms. First of all, we need to consider only terms of P with zero right z-degree, other terms can not make any influence. Second, we have to consider substitutions only of terms with zero right z-degree on the rightmost positions of u. Let V be their sum.

But the sum of highest terms satisfying this conditions is equal to T and T is the highest homogeneous component of V, hence $V \neq 0$. $\square$

Corollary 2.4. *Let f be a polynomial, $P \in F(z) * F[x]$ such that each monomial has degree ≥ 2. Suppose that one of the coefficients of $P(f)$ has (zero)negative right z-degree. Then $P(f)$ has one of the coefficients with (zero) negative right z-degree and degree of corresponding term is strictly more then* $\deg(f)$.

There is just the 'dual' left version of lemma 2.3 and corollary 2.4.

As a consequence of the above corollary, we get

Lemma 2.5. *In the step $x \to x$, ($z \to z$ because we are working with z-automorphisms) $y \to y + x^k z^{-l}$, $k > 1$ of the degree-strictly-increasing process, applied to the automorphism $x \to x +$ higest terms, $y \to y +$ higest terms causes some negative power(s).*

In order to prove Theorem 1.1 we need a similar statement which is also a consequence of the Corollary 2.4.

Lemma 2.6. *In the step $x \to x$, ($z \to z$ because we are working with z-automorphisms) $y \to y + P(x)$, such that P has negative powers of z as left coefficients of some monomial of degree ≥ 2 in the degree-strictly-increasing process causes some negative power(s) on any succeeding step.*

Lemma 2.3, Lemma 2.6 and Corollary 2.4 says that any further step of non-linear operation either contains terms of negative power with bigger degree, or does not interfere in the process. Hence they imply the following

Lemma 2.7. *a) Consider stage in strictly increasing process of following form.*

$$x \to T_1 + h_1, \quad y \to T_2 + h_2$$

where T_i are sums of the terms with negative powers of z to the right, h_i – are sums of the terms without negative powers of z to the right. If T_i are $F(z)$-linear independent, then the negative powers can not be cancelled in the strictly increasing process.

b) Suppose T_1 is the sum of the terms with negative powers of z to the right, h_1 – is the sum of the terms without negative powers of z to the right, T_2 is the sum of the terms with zero powers of z to the right, h_2 is the sum of the terms with positive powers of z to the right.
If T_i are $F(z)$-linear independent, then the negative powers can not be cancelled in the strictly increasing process.

In order to prove Theorem 1.1 we need slight generalization of the previous lemma, which also follows from the Lemma 2.3 and Corollary 2.4.

Proposition 2.8. *Consider stage in strictly increasing process of following form:*

$$x \to T_1 + h_1 + g_1, \quad y \to T_2 + h_2 + g_2$$

where T_i are sums of the terms with negative powers of z to the right, h_i – are sums of the terms with zero powers of z to the right, g_i are sums of the terms with positive powers of z to the right.

If T_i are $F(z)$-linear independent, or wedge product of vectors

$$(T_1, T_2) \bigwedge_{F[z_l, z_r]} (h_1, h_2) \neq 0,$$

then the negative powers can not be cancelled in the strictly increasing process. Wedge product is taken respect to left and right $F(z)$-actions, i.e. as $F[z_l, z_r]$-modula, monomial (respect to x, y, and inner positions of z) are considered as basis vectors.

Proof. Pbviously, any linear operation cannot cancel the negative powers of z, but Lemma 2.3 and corollary 2.4 allows us to consider only such operations. □

Remark 2.9. Considering the substitutions $z \to z + c$ one can get similar results for negative powers of $z + c$ (or via considering other valuations of $F(z)$).

Lemma 2.10. *Consider the step in the strictly increasing process of following form.*

$$x \to T + h_1', \quad y \to U + h_2'$$

where T is the sum of the terms with negative powers of z to the right, U the sum of the terms with negative powers of z to the right, h_1 is the *sum of the terms without negative powers of z to the right, h_2 is the sum of the terms with positive powers of z to the right.*

If T and U are $F(z)$-linear independent, then the negative powers cannot be cancelled in the strictly increasing process.

Proof. By induction. Input of composition with polynomials with x-degree ≥ 2 cannot be cancelled (otherwise some negative power appears in the highest terms, and the $F(z)$-independence preserves). But the x-linear term action only produces the $F(z)$-linear combinations. $\square$

Consider, for instance, the elementary automorphism

$$x \to x, \quad y \to y + z^n x^k.$$

It can be lifted to an Q-automorphism

$$x \to x, \quad y \to y + z^{n_0} x^{k_1} z^{n_1} \cdots x^{k_s} z^{n_s}, \quad \sum k_i = k, \ \sum n_i = n.$$

Though $n < 0$, n_0 and n_s can still be non-negative. It is necessary to deal with that kind of situation by the next lemma.

Lemma 2.11. *Consider a elementary mapping*

$$x \to x; \quad y \to y + P(x)$$

such that $P(x)$ has a monomial of the following form:

$$z^{k_1} x z^{k_2} x \cdots x z^{k_s}$$

where one of $k_i < 0$ for some i such that $1 < i < s$. Then if such an elementary transformation occurs in the strictly increasing process, it must produce some sandwich.

Proof. First of all, due to the Lemma 2.1, we may assume without loss of generality that there exists no sandwiches before this step.

Consider z^{k_i}, the minimum power of z, lying before the variables for all monomials in P. Next, consider the monomials in P of the minimum degree containing z^{k_i} between x's and among them, i.e. the monomials such that z^{k_i} positioned on the left-most possible position (but then $i > 1$, it should be a sandwich position). Let us denote such terms T_i.

Let

$$\varphi(x) = \sum u_i, \quad \varphi(y) = \sum v_i$$

will be an automorphism, obtained by the previous step. Due to the Lemma 2.1 we may assume that no terms come from u_i, v_i are sandwiches.

Now we consider u_i with minimal right z-degree n_r, and among them – terms with minimal degree (respect to x and y). Let u^r_j will be such terms, $u^r = \sum u^r_j$. Because x is one of the u_i, $n_r \leq 0$. Similarly we consider u_i with minimal left z-degree n_l, terms u^l_j and their sum $u^l = \sum u^l_j$. We also get $n_l \leq 0$.

Now for any monomial T_j, consider the element

$$E_{T_j} = q_0^{(j)} x \cdots u^r z^{n_i} u^l \cdots x q_s^{(j)}$$

obtained by replacement of u^r and u^l into the positions of x surrounding occurrence of z^{n_i} as discussed previously, the resulting power of z would be equal to $n_r + n_l + n_i \leq n_i < 0$.

Now E_{T_j} can be presented as a sum $E_{T_j} = \sum M_{E_{T_j}}$, where $M_{E_{T_j}}$ are monomials. Monomials $M_{E_{T_j}}$ are sandwiches, they may appear only that way which was described previously and hence cannot cancell by other monomials. Hence we must have a sandwich. $\square$

3. Proofs of the main theorems

Proof of Theorem 1.2.
Suppose the automorphism (f,g) can be lifted to a z-automorphism of $F\langle x,y,z\rangle$. Then it induces an automorphism of $F(z) *_F F\langle x,y\rangle$ and can be obtained by the process described in the Lemma on coefficient improving.

Then at some steps some negative powers of z appear either between variables or on the right or on the left and it will be preserved to the end, due to Lemma 2.1 and Lemma 2.11, or Lemmas 2.10, 2.7, 2.5.

Hence in the lifted automorphism, there exists some negative power of z. A contradiction. $\square$

Proof of Theorem 1.1.
Let (f,g) be a wild $F[z]$-automorphism of $F[x,y,z]$ such that it is not of the type in Theorem 1.2. Consider corresponding strictly increasing process. We shall need few more statements.
The following lemma is a consequence of Proposition 2.8.

Lemma 3.1. *In the strictly increasing process. Consider the steps with negative powers of z appearing to the right.*

$$\varphi : \ x \to x + P(y), y \to y$$

Let

$$\psi : \ x \to x, y \to y + Q_1(x) + Q_2(x)$$

where $\deg(Q_1) = 1$, *each term of Q_2 has degree ≥ 2 and does not contain negative powers of z. Then $\psi = \psi_1 \circ \psi_2$ where $\psi_1 : \ x \to x, y \to y + Q_1(x)$, $\psi_2 : \ x \to x, y \to y + Q_2(x)$ and $\varphi\psi_2\varphi^{-1}$ has no negative powers of z to the right.*

Lemma 3.1 together with its left analogue and remark 2.9 imply following statement:

Proposition 3.2. *In the strictly increasing process, consider the step with appearing coefficients not in $F[z]$.*

$$\varphi : \ x \to \ x + P(y), y \to y$$

Let

$$\psi: \ x \to x, y \to y + Q_1(x) + Q_2(x)$$

where $\deg(Q_1) = 1$, *each term of* Q_2 *has degree* ≥ 2 *and does not contain negative powers of* z . *Then* $\psi = \psi_1 \circ \psi_2$ *where* $\psi_1 : \ x \to x, y \to y + Q_1(x)$, $\psi_2 : \ x \to x, y \to y + Q_2(x)$ *and* $\varphi \circ \psi_2 \circ \varphi^{-1}$ *is a* z*-automorphism of* $F\langle x, y, z\rangle$.

Proof. Consider set of elements from $F(z)$ which are coefficients of our monomials. If all valuations of $F(z)$ centered in finite points are positive, then they belong to $F[z]$ and we are done. Due to symmetry, it is enough to consider right coefficients and due to substitution $z \to z+a$ just valuation centered in zero. Then by Lemma 3.1, we are done. □

Proof. It is easy to see that $\psi \circ \varphi \circ \psi^{-1}$ has following form: $x \to x + c_1 R(a'_{21}x + a'_{22}y), y \to c_2 R(a'_{21}x + a'_{22}y)$, where $a'_{ij} = \alpha a_{ij} \in F[z]$ are relatively prime, $\alpha \in F[z]$ is the least common multiple of the denominators of $a_{21}, a_{22} \in F[z]$ and $c_1, c_2 \in F[z]$ such that $c_1 a_{21} + c_2 a_{22} = 0$. Choose $r, s \in F[z]$ such that $ra'_{21} + sa'_{22} = 1$.

Acting the linear automorphism $x \to rx + sy, y \to a'_{21}x + a'_{22}y$ over $F[z]$ to $\psi \circ \varphi \circ \psi^{-1}$, we get an automorphism of the following form: $x \to rx + sy + tR(a'_{21}x + a'_{22}y), y \to a'_{21}x + a'_{22}y$, which is elementarily equivalent to $x \to rx + sy, y \to a'_{21}x + a'_{22}y$. Hence $\psi \circ \varphi \circ \psi^{-1}$ is tame. □

The next proposition is well-known from linear algebra.

Proposition 3.3. *Let* (f, g) *is a* z*-automorphism of* $F[z][x, y]$ *linear in both* x *and* y. *Then it is a tame* z*-automorphism.*

Now we are ready to complete the proof of Theorem 1.1. Suppose a z-automorphism $\varphi = (f, g)$ of $F[z][x, y]$ can be lifted to an automorphism of $F[z] *_F F\langle x, y\rangle$ (i.e. an automorphism of $F\langle x, y, z\rangle$ fixing z), which is decomposed into product of elementary one according to strictly increasing process. The coefficients of elementary operation can be in $F(z)\backslash F[z]$ only for linear terms (see Lemma 2.6 and Remark 2.9) and conjugating non-linear elementary step with respect to the automorphisms corresponding to these terms are z-tame. Hence φ is a product of z-tame automorphisms and z-automorphisms linear in both x and y. Now we are done by Proposition 3.3.

By carefully looking through the above proofs, we actually obtained the following

Theorem 3.4. *An automorphism* (f,g) *in* $Aut_{F[z]}F\langle x,y,z\rangle$, *can be canonically decomposed as product of the following type of automorphisms:*
i) Linear automorphisms in $Aut_{F[z]}F\langle x,y,z\rangle$*;*
ii) Automorphisms which can be obtained by an elementaty automorphism in $Aut_{F[z]}F\langle x,y,z\rangle$ *conjugated by a linear automorphism in* $Aut_{F(z)}F(z) *_F F\langle x,\ y\rangle$.

Theorem 3.4 opens a way to obtain stably tameness of $\mathrm{Aut}_{F[z]}F\langle x,y,z\rangle$, which will be done in a separate paper [2].

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