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A CANCELLATION CONJECTURE FOR FREE ASSOCIATIVE ALGEBRAS

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Abstract. We develop a new method to deal with the Cancellation Conjecture of Zariski in different environments. We prove the conjecture for free associative algebras of rank two. We also produce a new proof of the conjecture for polynomial algebras of rank two over fields of zero characteristic.

1. Introduction and main results

There is a famous

Conjecture 1.1 (Cancellation Conjecture of Zariski). *Let R be an algebra over a field K. If $R[z]$ is K-isomorphic to $K[x_1,\dots,x_n]$, then R is isomorphic to $K[x_1,\dots,x_{n-1}]$.*

Conjecture 1.1 was proved for $n = 2$ by Abhyankar, Eakin and Heizer [1], and Miyanishi [10]. For $n = 3$, the conjecture was proved by Fujita [5], and Miyanishi and Sugie [11] for zero characteristic, and by Russell [12] for arbitrary fields K. For $n \geq 4$, the conjecture remains open to the best of our knowledge. See [4, 6, 7, 8, 9, 14] for Zariski's conjecture and related topics.

Denote by $A * B$ the free product of two K-algebras A and B. In view of Conjecture 1.1, it is natural and interesting to raise

Conjecture 1.2 (Cancellation Conjecture for Free Associative Algebras). *Let R be an algebra over a field K. If $R * K[z]$ is K-isomorphic to $K\langle x_1,\dots,x_n\rangle$, then R is K-isomorphic to $K\langle x_1,\dots,x_{n-1}\rangle$.*

In this paper we develop a new method based on the conditions of algebraic dependence, which can be used in different environments. In particular, by this method we prove Conjecture 1.2 for $n = 2$:

Theorem 1.3. *Let R be an algebra over an arbitrary field K. If $R * K[z]$ is K-isomorphic to $K\langle x,y\rangle$, then R is K-isomorphic to $K[x]$.*

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We also produce a new and simple proof for Conjecture 1.1 for $n = 2$ in the zero characteristic case [1]:

Proposition 1.4. *Let R be an algebra over a field K of zero characteristic. If $R[t]$ is K-isomorphic to $K[x, y]$, then R is isomorphic to $K[x]$.*

2. Preliminaries

Call a set of elements of an associative K-algebra *algebraically dependent* over K if the K-subalgebra generated by the elements is not free on that generating set. To prove the main results, we need well-known necessary and sufficient conditions for algebraic dependence.

Lemma 2.1. *Let K be an arbitrary field, $f, g \in K\langle x_1, \dots, x_n\rangle$. Then f and g are algebraically dependent over K if and only if $[f, g] = 0$, where $[f, g] = fg - gf$ is the commutator of f and g.*

See Corollary 6.7.4, p. 338, Cohn [3].

Lemma 2.2. *Let K be a field of zero characteristic, $f, g \in K[x_1, \dots, x_n]$. Then f and g are algebraically dependent over K if and only if $J_{x_i,x_j}(f, g) = 0$ for all $1 \leq i < j \leq n$, where $J_{x_i,x_j}(f, g)$ is the Jacobian determinant of f and g with respect to x_i and x_j.*

See, for instance, Jie-Tai Yu [15], for a proof.

We also need a description of the subset of all elements of a polynomial or a free associative algebra which are algebraically dependent on a fixed element. The following result is due to Bergman [2]. See also Cohn [3].

Lemma 2.3. *Let K be an arbitrary field, $f \in K\langle x_1, \dots, x_n\rangle - K$, $\mathcal{C}(f)$ the set of all $g \in K\langle x_1, \dots, x_n\rangle$ such that $[f, g] = 0$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K\langle x_1, \dots, x_n\rangle$.*

For polynomial algebras, the analogue of the above result has been obtained by Shestakov and Umirbaev [13]:

Lemma 2.4. *Let K be a field of zero characteristic, $f \in K[x_1, \dots, x_n] - K$, $\mathcal{C}(f)$ the set of all $g \in K[x_1, \dots, x_n]$ such that $J_{x_i,x_j}(f, g) = 0$ for all $1 \leq i < j \leq n$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K[x_1, \dots, x_n]$.*

3. Proofs of the main results

Proof of Theorem 1.3. Let $R * K[z] \cong K\langle x, y\rangle$. The endomorphism of $R * K[z]$ taking z to 0 and acting as the identity on R is not one-to-one. Hence the images v and w of the generators x, y under that endomorphisms are algebraically dependent over K. Obviously R is generated by v, w. By Lemma 2.1, it is easy to deduce that any element $f = f(v, w) \in R$ and v are algebraically dependent over K. By Lemma 2.1 and Lemma 2.3, $R \subset K[u]$ for some $u \in R * K[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$, u_1 contains only monomials occurring in u with z-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, h is a polynomial over K in one variable. Substituting $z = 0$, $f = h(u_0)$. Therefore, $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. This forces $R = K[u_0]$. Therefore, R is K-isomorphic to $K[x]$. $\square$

Proof of Proposition 1.4. As $R[z]$ is K-isomorphic to $K[x,y]$, it is easy to deduce that R has a transcendence degree 1 over K. Therefore, there exists a $g \in R - K$ such that for all $f \in R$, f and g are algebraically dependent over K. By Lemma 2.2 and Lemma 2.4, $R \subset K[u]$ for some $u \in R[t]$. Write $u = u_0 + u_1$, where $u_0 \in R$, u_1 contains only monomials occurring in u with z-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, h is a polynomial over K in one variable. Substituting $z = 0$, $f = h(u_0)$. Therefore, $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. This forces $R = K[u_0]$. Therefore, R is K-isomorphic to $K[x]$. □

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