

Research Article

Sharp Integral Inequalities Involving High-Order Partial Derivatives

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The main purpose of the present paper is to establish some new sharp integral inequalities involving higher-order partial derivatives. Our results in special cases yield some of the recent results on Agarwal, Wirtinger, Poincaré, Pachpatte, Smith, and Stredulinsky's inequalities and provide some new estimates on such types of inequalities.

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1. Introduction

Inequalities involving functions of n independent variables, their partial derivatives, integrals play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1–10]. Especially, in view of wider applications, inequalities due to Agarwal, Opial, Pachpatte, Wirtinger, Poincaré and et al. have been generalized and sharpened from the very day of their discover. As a matter of fact, these now have become research topic in their own right [11–14]. In the present paper, we will use the same method of Agarwal and Sheng [15], establish some new estimates on these types of inequalities involving higher-order partial derivatives. We further generalize these inequalities which lead to result sharper than those currently available. An important characteristic of our results is that the constant in the inequalities are explicit.

2. Main results

Let R be the set of real numbers and R^n the n -dimensional Euclidean space. Let E, E' be a bounded domain in R^n defined by $E \times E' = \prod_{i=1}^n [a_i, b_i] \times [c_i, d_i], i = 1, \dots, n$. For $x_i, y_i \in R, i = 1, \dots, n, (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ is a variable point in $E \times E'$ and

$dx dy = dx_1 \cdots dx_n dy_1 \cdots dy_n$. For any continuous real-valued function $u(x, y)$ defined on $E \times E'$, we denote by $\int_E \int_{E'} u(x, y) dx dy$ the $2n$ -fold integral

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{c_1}^{d_1} \cdots \int_{c_n}^{d_n} u(x_1, \dots, x_n, y_1, \dots, y_n) dx_1 \cdots dx_n dy_1 \cdots dy_n, \quad (2.1)$$

and for any $(x, y) \in E \times E'$, $\int_{E(x)} \int_{E'(x)} u(s, t) ds dt$ is the $2n$ -fold integral

$$\int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \int_{c_1}^{y_1} \cdots \int_{c_n}^{y_n} u(s_1, \dots, s_n, t_1, \dots, t_n) dx_1 \cdots dx_n dt_1 \cdots dt_n. \quad (2.2)$$

We represent by $F(E \times E')$ the class of continuous functions $u(x, y) : E \times E' \rightarrow \mathbb{R}$ for which $D^{2n}u(x, y) = D_1 \cdots D_{2n}u(x, y)$, where

$$D_1 = \frac{\partial}{\partial x_1}, \dots, D_n = \frac{\partial}{\partial x_n}, D_{n+1} = \frac{\partial}{\partial y_1}, \dots, D_{2n} = \frac{\partial}{\partial y_n} \quad (2.3)$$

exists and that for each i , $1 \leq i \leq n$,

$$u(x, y)|_{x_i=a_i} = 0, \quad u(x, y)|_{y_i=c_i} = 0, \quad u(x, y)|_{x_i=b_i} = 0, \quad u(x, y)|_{y_i=d_i} = 0, \quad (i = 1, \dots, n) \quad (2.4)$$

the class $F(E \times E')$ is denoted as $G(E \times E')$.

Theorem 2.1. Let $\mu \geq 0$, $\lambda \geq 1$ be given real numbers, and let $p(x, y) \geq 0$, $(x, y) \in E \times E'$ be a continuous function. Further, let $u(x, y) \in G(E \times E')$. Then, the following inequality holds

$$\begin{aligned} & \int_E \int_{E'} p(x, y) |u(x, y)|^\mu dx dy \\ & \leq \int_E \int_{E'} p(x, y) q(x, y, \lambda, \mu) dx dy \left(\int_E \int_{E'} |D^{2n}u(x, y)|^\lambda dx dy \right)^{\mu/\lambda}, \end{aligned} \quad (2.5)$$

where

$$q(x, y, \lambda, \mu) = \left(\frac{1}{2^{n+1}} \prod_{i=1}^n [(x_i - a_i)(b_i - x_i)(y_i - c_i)(d_i - y_i)]^{(\lambda-1)/2} \right)^{\mu/\lambda}. \quad (2.6)$$

Proof. For the set $\{1, \dots, n\}$, let $\pi = A \cup B$, $\pi' = A' \cup B'$ be partitions, where $A = (j_1, \dots, j_k)$, $B = (j_{k+1}, \dots, j_n)$, $A' = (i_1, \dots, i_k)$, and $B' = (i_{k+1}, \dots, i_n)$ are such that $\text{card } A = \text{card } A' = k$ and $\text{card } B = \text{card } B' = n - k$, $0 \leq k \leq n$. It is clear that there are 2^{n+1} such partitions. The set of all such partitions we will denote as Z and Z' , respectively. For fixed partition π , π' and $x \in E$, $y \in E'$, we define

$$\int_{E_\pi(x)} \int_{E'_{\pi'}(y)} u(s, t) ds dt = \int_{A(x)} \int_{B(x)} \int_{A'(y)} \int_{B'(y)} u(s, t) ds dt, \quad (2.7)$$

where $\int_{A(x)}, \int_{A'(y)}$ denote the k -fold integral, $\int_{B(x)}, \int_{B'(y)}$ represent the $(n-k)$ -fold integral. Thus from the assumptions it is clear that for each $\pi \in Z, \pi' \in Z'$

$$|u(x, y)| \leq \int_{E_\pi(x)} \int_{E_{\pi'}(y)} |D^{2n}u(s, t)| ds dt. \quad (2.8)$$

In view of Hölder integral inequality, we have

$$\begin{aligned} |u(x, y)| &\leq \left(\prod_{i \in A} (x_i - a_i) \prod_{i \in B} (b_i - x_i) \prod_{i \in A'} (y_i - c_i) \prod_{i \in B'} (d_i - y_i) \right)^{(\lambda-1)/\lambda} \\ &\quad \times \left(\int_{E_\pi(x)} \int_{E_{\pi'}(y)} |D^{2n}u(s, t)|^\lambda ds dt \right)^{1/\lambda}. \end{aligned} \quad (2.9)$$

A multiplication of these 2^{n+1} inequalities and an application of the Arithmetic-Geometric mean inequality give

$$\begin{aligned} |u(x, y)|^\mu &\leq \left(\prod_{i=1}^n [(x_i - a_i)(b_i - x_i)(y_i - c_i)(d_i - y_i)]^{(\lambda-1)/2} \right)^{\mu/\lambda} \\ &\quad \times \left(\prod_{\pi \in Z, \pi' \in Z'} \left(\int_{E_\pi(x)} \int_{E_{\pi'}(y)} |D^{2n}u(s, t)|^\lambda ds dt \right)^{1/2^{n+1}} \right)^{\mu/\lambda} \\ &\leq \left(\frac{1}{2^{n+1}} \prod_{i=1}^n [(x_i - a_i)(b_i - x_i)(y_i - c_i)(d_i - y_i)]^{(\lambda-1)/2} \right)^{\mu/\lambda} \\ &\quad \times \left(\sum_{\pi \in Z, \pi' \in Z'} \int_{E_\pi(x)} \int_{E_{\pi'}(y)} |D^{2n}u(s, t)|^\lambda ds dt \right)^{\mu/\lambda} \\ &= q(x, y, \lambda, \mu) \left(\int_E \int_{E'} |D^{2n}u(s, t)|^\lambda ds dt \right)^{\mu/\lambda}. \end{aligned} \quad (2.10)$$

Now, multiplying both the sides of (2.10) by $p(x, y)$ and integrating the resulting inequality on $E \times E'$, we have

$$\int_E \int_{E'} p(x, y) |u(x, y)|^\mu dx dy \leq \int_E \int_{E'} p(x, y) q(x, y, \lambda, \mu) dx dy \left(\int_E \int_{E'} |D^{2n}u(s, t)|^\lambda ds dt \right)^{\mu/\lambda}, \quad (2.11)$$

where

$$q(x, y, \lambda, \mu) = \left(\frac{1}{2^{n+1}} \prod_{i=1}^n [(x_i - a_i)(b_i - x_i)(y_i - c_i)(d_i - y_i)]^{(\lambda-1)/2} \right)^{\mu/\lambda}. \quad (2.12)$$

□

Remark 2.2. Taking for $p(x, y) = 1$ in (2.5), (2.5) reduces to

$$\int_E \int_{E'} |u(x, y)|^\mu dx dy \leq K'_0 \left(\int_E \int_{E'} |D^{2n}u(x, y)|^\lambda dx dy \right)^{\mu/\lambda}, \quad (2.13)$$

where

$$K'_0 = \left(\left(\frac{1}{2} \right)^{\mu/\lambda} B^2 \left(1 + \frac{\mu}{2} - \frac{\mu}{2\lambda}, 1 + \frac{\mu}{2} - \frac{\mu}{2\lambda} \right) \right)^n \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{1+\mu-\mu/\lambda}, \quad (2.14)$$

and B is the Beta function.

Taking for $\lambda = \mu = 2$ in (2.13) reduces to

$$\int_E \int_{E'} |u(x, y)|^2 dx dy \leq \left(\frac{\pi^2}{8}\right)^n M'^2 \left(\int_E \int_{E'} |D^{2n} u(x, y)|^2 dx dy\right), \quad (2.15)$$

where

$$M' = \prod_{i=1}^n \frac{(b_i - a_i)(d_i - c_i)}{4}. \quad (2.16)$$

Let $u(x, y)$ reduce to $u(x)$ in (2.15) and with suitable modifications, then (2.15) becomes the following two Wirting type inequalities:

$$\int_E |u(x)|^2 dx \leq \left(\frac{\pi}{4}\right)^n M^2 \left(\int_E |D^n u(x)|^2 dx\right), \quad (2.17)$$

where

$$M = \prod_{i=1}^n \frac{(b_i - a_i)}{2}. \quad (2.18)$$

Similarly

$$\int_E |u(x)|^4 dx \leq \left(\frac{3\pi}{16}\right)^n M^4 \left(\int_E |D^n u(x)|^4 dx\right), \quad (2.19)$$

where M is as in (2.17).

For $n = 2$, the inequalities (2.17) and (2.19) have been obtained by Smith and Stredulinsky [16], however, with the right-hand sides, respectively, multiplies $(4/\pi)^2$ and $(16/3\pi)^4$. Hence, it is clear that inequalities (2.17) and (2.19) are more strengthened.

Remark 2.3. Let $u(x, y)$ reduce to $u(x)$ in (2.5) and with suitable modifications, then (2.5) becomes the following result:

$$\int_E p(x) |u(x)|^\mu dx \leq \int_E p(x) q(x, \lambda, \mu) dx \left(\int_E |D^n u(x)|^\lambda dx\right)^{\mu/\lambda}, \quad (2.20)$$

where

$$q(x, \lambda, \mu) = \left(\frac{1}{2^n} \prod_{i=1}^n [(x - a_i)(b_i - x_i)]^{(\lambda-1)/2}\right)^{\mu/\lambda}. \quad (2.21)$$

This is just a new result which was given by Agarwal and Sheng [15].

Theorem 2.4. Let $p(x, y) \geq 0$, $(x, y) \in E \times E'$ be a continuous function. Further, let for $k = 1, \dots, r$, $\mu_k \geq 0$, $\lambda_k \geq 1$, be given real numbers such that $\sum_{k=1}^r (\mu_k / \lambda_k) = 1$, and $u_k(x, y) \in G(E \times E')$. Then the following inequality holds

$$\begin{aligned} & \int_E \int_{E'} p(x, y) \prod_{k=1}^r |u_k(x, y)|^{\mu_k} dx dy \\ & \leq \int_E \int_{E'} p(x, y) \prod_{k=1}^r q(x, y, \lambda_k, \mu_k) dx dy \sum_{k=1}^r \frac{\mu_k}{\lambda_k} \int_E \int_{E'} |D^{2n} u_k(x, y)|^{\lambda_k} dx dy. \end{aligned} \quad (2.22)$$

Proof. Setting $\mu = \mu_k$, $\lambda = \lambda_k$ and $u(x, y) = u_k(x, y)$, $1 \leq k \leq r$ in (2.10), multiplying the r inequalities, and applying the extended Arithmetic-Geometric means inequality,

$$\prod_{k=1}^r a_k^{\mu_k/\lambda_k} \leq \sum_{k=1}^r \frac{\mu_k}{\lambda_k} a_k, \quad a_k \geq 0, \quad (2.23)$$

to obtain

$$\begin{aligned} \prod_{k=1}^r |u_k(x, y)|^{\mu_k} &\leq \prod_{k=1}^r q(x, y, \lambda_k, \mu_k) \left(\int_E \int_{E'} |D^{2n} u_k(s, t)|^{\lambda_k} ds dt \right)^{\mu_k/\lambda_k} \\ &\leq \prod_{k=1}^r q(x, y, \lambda_k, \mu_k) \sum_{k=1}^r \frac{\mu_k}{\lambda_k} \int_E \int_{E'} |D^{2n} u_k(s, t)|^{\lambda_k} ds dt. \end{aligned} \quad (2.24)$$

Now multiplying both sides of (2.24) by $p(x, y)$ and then integrating over $E \times E'$, we obtain (2.22). \square

Corollary 2.5. *Let the conditions of Theorem 2.4 be satisfied. Then the following inequality holds*

$$\int_E \int_{E'} p(x, y) \prod_{k=1}^r |u_k(x, y)|^{\mu_k} dx dy < K'_1 \int_E \int_{E'} p(x, y) dx dy \sum_{k=1}^r \frac{\mu_k}{\lambda_k} \int_E \int_{E'} |D^{2n} u_k(x, y)|^{\lambda_k} dx dy, \quad (2.25)$$

where

$$K'_1 = \left(\frac{1}{2^{n+1}} \right)^{\sum_{k=1}^r \mu_k} \prod_{i=1}^n [(b_i - a_i)(b_i - a_i)]^{-1 + \sum_{k=1}^r \mu_k}. \quad (2.26)$$

This is just a general form of the following inequality which was established by Agarwal and Sheng [15]:

$$\int_E p(x) \prod_{k=1}^r |u_k(x)|^{\mu_k} dx < K_1 \int_E p(x) dx \sum_{k=1}^r \frac{\mu_k}{\lambda_k} \int_E |D^n u_k(x)|^{\lambda_k} dx, \quad (2.27)$$

where

$$K_1 = \left(\frac{1}{2^n} \right)^{\sum_{k=1}^r \mu_k} \prod_{i=1}^n (b_i - a_i)^{-1 + \sum_{k=1}^r \mu_k}. \quad (2.28)$$

Remark 2.6. For $p(x, y) = 1$, the inequality (2.22) becomes

$$\int_E \int_{E'} \prod_{k=1}^r |u_k(x, y)|^{\mu_k} dx dy \leq K'_2 \sum_{k=1}^r \frac{\mu_k}{\lambda_k} \int_E \int_{E'} |D^{2n} u_k(x, y)|^{\lambda_k} dx dy, \quad (2.29)$$

where

$$K'_2 = \left(\frac{1}{2} B^2 \left(\frac{1 + \sum_{k=1}^r \mu_k}{2}, \frac{1 + \sum_{k=1}^r \mu_k}{2} \right) \right)^n \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{\sum_{k=1}^r \mu_k}. \quad (2.30)$$

For $u(x, y) = u(x)$, the inequality (2.29) has been obtained by Agarwal and Sheng [15].

Theorem 2.7. Let λ and $u(x, y)$ be as in Theorem 2.1, $\mu \geq 1$ be a given real number. Then the following inequality holds

$$\int_E \int_{E'} |u(x, y)|^\lambda dx dy \leq K'_3(\lambda, \mu) \int_E \int_{E'} \|\text{grad } u(x, y)\|_\mu^\lambda dx dy, \quad (2.31)$$

where

$$K'_3(\lambda, \mu) = \frac{1}{2n} B^2 \left(\frac{\lambda+1}{2}, \frac{\lambda+1}{2} \right) K \left(\frac{\lambda}{\mu} \right) \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{\lambda/n}, \quad (2.32)$$

$$\|\text{grad } u(x, y)\|_\mu = \left(\sum_{i=1}^n \left| \frac{\partial^2}{\partial x_i \partial y_i} u(x, y) \right|^\mu \right)^{1/\mu},$$

and where $K(\lambda/\mu) = 1$ if $\lambda \geq \mu$, and $K(\lambda/\mu) = n^{1-\lambda/\mu}$ if $0 \leq \lambda/\mu \leq 1$.

Proof. For each fixed i , $1 \leq i \leq n$, in view of

$$u(x, y)|_{x_i=a_i} = 0, \quad u(x, y)|_{y_i=c_i} = 0, \quad u(x, y)|_{x_i=b_i} = 0, \quad u(x, y)|_{y_i=d_i} = 0, \quad (i = 1, \dots, n), \quad (2.33)$$

we have

$$u(x, y) = \int_{a_i}^{x_i} \int_{c_i}^{y_i} \frac{\partial^2}{\partial s_i \partial t_i} u(x, y; s_i, t_i) ds_i dt_i, \quad (2.34)$$

$$u(x, y) = \int_{x_i}^{b_i} \int_{y_i}^{d_i} \frac{\partial^2}{\partial s_i \partial t_i} u(x, y; s_i, t_i) ds_i dt_i,$$

where

$$u(x, y; s_i, t_i) = u(x_1, \dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n, y_1, \dots, y_{i-1}, t_i, y_{i+1}, \dots, y_n). \quad (2.35)$$

Hence from Hölder inequality with indices λ and $\lambda/(1-\lambda)$, it follows that

$$|u(x, y)|^\lambda \leq [(x_i - a_i)(y_i - d_i)]^{\lambda-1} \int_{a_i}^{x_i} \int_{c_i}^{y_i} \left| \frac{\partial^2}{\partial s_i \partial t_i} u(x, y; s_i, t_i) \right|^\lambda ds_i dt_i, \quad (2.36)$$

$$|u(x, y)|^\lambda \leq [(b_i - x_i)(d_i - y_i)]^{\lambda-1} \int_{x_i}^{b_i} \int_{y_i}^{d_i} \left| \frac{\partial^2}{\partial s_i \partial t_i} u(x, y; s_i, t_i) \right|^\lambda ds_i dt_i.$$

Multiplying (2.36), and then applying the Arithmetic-Geometric means inequality, to obtain

$$|u(x, y)|^\lambda \leq \frac{1}{2} [(x_i - a_i)(y_i - c_i)(b_i - x_i)(d_i - y_i)]^{(\lambda-1)/2} \times \int_{a_i}^{x_i} \int_{c_i}^{y_i} \left| \frac{\partial^2}{\partial s_i \partial t_i} u(x, y; s_i, t_i) \right|^\lambda ds_i dt_i, \quad (2.37)$$

and now integrating (2.37) on $E \times E'$, we arrive at

$$\int_E \int_{E'} |u(x, y)|^\lambda dx dy \leq \int_{a_i}^{b_i} \int_{c_i}^{d_i} \frac{1}{2} [(x_i - a_i)(y_i - c_i)(b_i - x_i)(d_i - y_i)]^{(\lambda-1)/2} dx_i dy_i \quad (2.38)$$

$$\times \int_E \int_{E'} \left| \frac{\partial^2}{\partial x_i \partial y_i} u(x, y) \right|^\lambda dx dy.$$

Next, multiplying the inequality (2.38) for $1 \leq i \leq n$, and using the Arithmetic-Geometric means inequality, and in view of the following inequality:

$$\sum_{i=1}^n a_i^\alpha \leq K(\alpha) \left(\sum_{i=1}^n a_i \right)^\alpha, \quad a_i > 0, \quad (2.39)$$

where $K(\alpha) = 1$ if $\alpha \geq 1$, and $K(\alpha) = n^{1-\alpha}$ if $0 \leq \alpha \leq 1$, we get

$$\begin{aligned} \int_E \int_{E'} |u(x, y)|^\lambda dx dy &\leq \prod_{i=1}^n \left(\int_{a_i}^{b_i} \int_{c_i}^{d_i} \frac{1}{2} [(x_i - a_i)(y_i - c_i)(b_i - x_i)(d_i - y_i)]^{(\lambda-1)/2} dx_i dy_i \right)^{1/n} \\ &\quad \times \prod_{i=1}^n \left(\int_E \int_{E'} \left| \frac{\partial^2}{\partial x_i \partial y_i} u(x, y) \right|^\lambda dx dy \right)^{1/n} \\ &\leq \frac{1}{2n} \prod_{i=1}^n \left(\int_{a_i}^{b_i} \int_{c_i}^{d_i} \frac{1}{2} [(x_i - a_i)(y_i - c_i)(b_i - x_i)(d_i - y_i)]^{(\lambda-1)/2} dx_i dy_i \right)^{1/n} \\ &\quad \times \sum_{i=1}^n \int_E \int_{E'} \left| \frac{\partial^2}{\partial x_i \partial y_i} u(x, y) \right|^\lambda dx dy \\ &\leq \frac{1}{2n} B^2 \left(\frac{\lambda+1}{2}, \frac{\lambda+1}{2} \right) \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{\lambda/n} \\ &\quad \times \int_E \int_{E'} \|\text{grad } u(x, y)\|_\lambda^\lambda dx dy, \\ &\leq K'_3(\lambda, \mu) \int_E \int_{E'} \|\text{grad } u(x, y)\|_\mu^\lambda dx dy, \end{aligned} \quad (2.40)$$

where

$$\begin{aligned} K'_3(\lambda, \mu) &= \frac{1}{2n} B^2 \left(\frac{\lambda+1}{2}, \frac{\lambda+1}{2} \right) K \left(\frac{\lambda}{\mu} \right) \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{\lambda/n}, \\ \|\text{grad } u(x, y)\|_\mu &= \left(\sum_{i=1}^n \left| \frac{\partial^2}{\partial x_i \partial y_i} u(x, y) \right|^\mu \right)^{1/\mu}, \end{aligned} \quad (2.41)$$

and where $K(\lambda/\mu) = 1$ if $\lambda \geq \mu$, and $K(\lambda/\mu) = n^{1-\lambda/\mu}$ if $0 \leq \lambda/\mu \leq 1$. \square

Remark 2.8. Let $u(x, y)$ reduce to $u(x)$ in (2.31) and with suitable modifications, and let $\lambda \geq 2$, $\mu = 2$, then (2.31) becomes

$$\int_E |u(x)|^\lambda dx \leq K'_3(\lambda, 2) \int_E \|\text{grad } u(x)\|_\mu^\lambda dx. \quad (2.42)$$

This is just a better inequality than the following inequality which was given by Pachpatte [17]

$$\int_E |u(x)|^\lambda dx \leq \frac{1}{n} \left(\frac{\beta}{2} \right)^\lambda \int_E \|\text{grad } u(x)\|_\mu^\lambda dx. \quad (2.43)$$

Because for $\lambda \geq 2$, it is clear that $K'_3(\lambda, 2) < (1/n)(\beta/2)^\lambda$, where $\beta = \max_{1 \leq i \leq n} (b_i - a_i)$.

On the other hand, taking for $\mu = 2, \lambda = 2$ or $\mu = 2, \lambda = 4$ in (2.31) and let $u(x, y)$ reduce to $u(x)$ with suitable modifications, it follows the following Poincaré-type inequalities:

$$\begin{aligned} \int_E |u(x)|^2 dx &\leq \frac{\pi}{16n} \beta^2 \int_E \|\text{grad } u(x)\|_2^2 dx, \\ \int_E |u(x)|^4 dx &\leq \frac{3\pi}{256n} \beta^4 \int_E \|\text{grad } u(x)\|_2^4 dx. \end{aligned} \quad (2.44)$$

The inequalities (2.44) have been discussed in [18] with the right-hand sides, respectively, multiplied by $4/\pi$ and $16/3\pi$. Hence inequalities (2.44) are more strong results on these types of inequalities.

If $\mu \geq \lambda$, in the right sides of (2.31) we can apply Hölder inequality with indices μ/λ and $\mu/(\mu - \lambda)$, to obtain the following corollary.

Corollary 2.9. *Let the conditions of Theorem 2.7 be satisfied and $\mu \geq \lambda$. Then*

$$\int_E \int_{E'} |u(x, y)|^\lambda dx dy \leq K'_4(\lambda, \mu) \left(\int_E \int_{E'} \|\text{grad } u(x, y)\|_\mu^\mu dx dy \right)^{\lambda/\mu}, \quad (2.45)$$

where

$$K'_4(\lambda, \mu) = K'_3(\lambda, \mu) \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{(\mu-\lambda)/\mu}. \quad (2.46)$$

Remark 2.10. Taking $u(x, y) = u(x)$ and with suitable modifications, the inequality (2.45) reduces to the following result which was given by Agarwal and Sheng [15]:

$$\int_E |u(x)|^\lambda dx \leq K_6(\lambda, \mu) \left(\int_E \|\text{grad } u(x)\|_\mu^\mu dx \right)^{\lambda/\mu}, \quad (2.47)$$

where

$$\begin{aligned} K_6(\lambda, \mu) &= K_5(\lambda, \mu) \prod_{i=1}^n (b_i - a_i)^{(\mu-\lambda)/\mu}, \\ K_5(\lambda, \mu) &= \frac{1}{2n} B\left(\frac{1+\lambda}{2}, \frac{1+\lambda}{2}\right) K\left(\frac{\lambda}{\mu}\right) \prod_{i=1}^n (b_i - a_i)^{\lambda/n}, \end{aligned} \quad (2.48)$$

and $K(\lambda/\mu)$ is as in Theorem 2.7.

Taking $\lambda = 1, \mu = 2$ the inequality (2.45), (2.45) reduces to

$$\left(\int_E \int_{E'} |u(x, y)| dx dy \right)^2 \leq K'_4(1, 2) \int_E \int_{E'} \|\text{grad } u(x, y)\|_2^2 dx dy. \quad (2.49)$$

This is just a general form of the following inequality which was given by Agarwal and Sheng [15].

$$\left(\int_E |u(x)| dx \right)^2 \leq [K_6(1, 2)]^2 \int_E \|\text{grad } u(x)\|_2^2 dx dy. \quad (2.50)$$

Similar to the proof of Theorem 2.7, we have the following theorem.

Theorem 2.11. For $u_k(x, y) \in G(E \times E')$, $\mu_k \geq 1$, $1 \leq k \leq r$. Then the following inequality holds

$$\int_E \int_{E'} \left(\prod_{i=1}^n |u_k(x, y)|^{\mu_k} \right)^{1/r} dx dy \leq K'_5 \int_E \int_{E'} \sum_{k=1}^r \|\text{grad } u_k(x, y)\|_{\mu_k}^{\mu_k} dx dy, \quad (2.51)$$

where

$$K'_5 = \frac{1}{2nr} B^2 \left(\frac{1 + (1/r) \sum_{k=1}^r \mu_k}{2}, \frac{1 + (1/r) \sum_{k=1}^r \mu_k}{2} \right) \prod_{i=1}^n [(b_i - a_i)(d_i - c_i)]^{\sum_{k=1}^r \mu_k / nr}. \quad (2.52)$$

Remark 2.12. Taking $u(x, y) = u(x)$ and with suitable modifications, the inequality (2.51) reduces to the following result:

$$\int_E \left(\prod_{i=1}^n |u_k(x)|^{\mu_k} \right)^{1/r} dx \leq K_9 \int_E \sum_{k=1}^r \|\text{grad } u(x)\|_{\mu_k}^{\mu_k} dx, \quad (2.53)$$

where

$$K_9 = \frac{1}{2nr} B \left(\frac{1 + (1/r) \sum_{k=1}^r \mu_k}{2}, \frac{1 + (1/r) \sum_{k=1}^r \mu_k}{2} \right) \prod_{i=1}^n (b_i - a_i)^{\sum_{k=1}^r \mu_k / nr}. \quad (2.54)$$

In [19], Pachpatte proved the inequality (2.53) for $\mu_k \geq 2$, $1 \leq k \leq r$ with K_9 replaced by $(1/nr)(\beta/2)^{\sum_{k=1}^r \mu_k / r}$, where β is as in Remark 2.8. It is clear that $K_9 < (1/nr)(\beta/2)^{\sum_{k=1}^r \mu_k / r}$, and hence (2.53) is a better inequality than a result of Pachpatte.

Similarly, all other results in [15] also can be generalized by the same way. Here, we omit the details.

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