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Research Article

On Inverse Hilbert-Type Inequalities

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This paper deals with new inverse-type Hilbert inequalities. Our results in special cases yield some of the recent results and provide some new estimates on such types of inequalities.

1. Introduction

Considerable attention has been given to Hilbert inequalities and Hilbert-type inequalities and their various generalizations by several authors including Handley et al. [1], Minzhe and Bicheng [2], Minzhe [3], Hu [4], Jichang [5], Bicheng [6], and Zhao [7, 8]. In 1998, Pachpatte [9] gave some new integral inequalities similar to Hilbert inequality (see [10, page 226]). In 2000, Zhao and Debnath [11] established some inverse-type inequalities of the above integral inequalities. This paper deals with some new inverse-type Hilbert inequalities which provide some new estimates on such types of inequalities.

2. Main results

Theorem 2.1. *Let $0 < p_i \leq 1$ $(i = 1, \ldots, n)$ and $r \leq 0$. Let $\{a_{i,m_i}\}$ be n positive sequences of real numbers defined for $m_i = 1, 2, \ldots, k_i$, where k_i $(i = 1, \ldots, n)$ are natural numbers, define $A_{i,m_i} = \sum_{s_i=1}^{m_i} a_{i,s_i}$, and define $A_{i,0} = 0$. Then for $p^{-1} + q^{-1} = 1$, $p < 0$ or $0 < p < 1$, one has*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} A_{i,m_i}^{p_i}}{\left((1/n)\sum_{i=1}^{n} m_i^r\right)^{n/(pr)}} \geq \prod_{i=1}^{n} p_i k_i^{1/p} \left(\sum_{m_i=1}^{k_i} (k_i - m_i + 1)\left(a_{i,m_i} A_{i,m_i}^{p_i - 1}\right)^q \right)^{1/q}. \tag{2.1}$$

Proof. By using the following inequality (see [10, page 39]):

$$h_i a_{i,m_i}^{h_i-1}(a_{i,m_i}-b_{i,m_i}) \le a_{i,m_i}^{h_i} - b_{i,m_i}^{h_i} \le h_i b_{i,m_i}^{h_i-1}(a_{i,m_i}-b_{i,m_i}), \tag{2.2}$$

where $a_{i,m_i} > 0$, $b_{i,m_i} > 0$, and $0 \le h_i \le 1$ $(i = 1,2,\dots,n)$, we obtain that

$$\begin{gathered} A_{i,m_i+1}^{p_i} - A_{i,m_i}^{p_i} \ge p_i A_{i,m_i+1}^{p_i-1}(A_{i,m_i+1} - A_{i,m_i}) = p_i a_{i,m_i+1} A_{i,m_i+1}^{p_i-1}, \\ \sum_{m_i=0}^{k_i-1} A_{i,m_i+1}^{p_i} - A_{i,m_i}^{p_i} = A_{i,k_i}^{p_i} \ge \sum_{m_i=0}^{k_i-1} p_i a_{i,m_i+1} A_{i,m_i+1}^{p_i-1} = p_i \sum_{m_i=1}^{k_i} a_{i,m_i} A_{i,m_i}^{p_i-1}, \end{gathered} \tag{2.3}$$

thus

$$A_{i,m_i}^{p_i} \ge p_i \sum_{s_i=1}^{m_i} a_{i,s_i} A_{i,s_i}^{p_i-1}. \tag{2.4}$$

From inequality (2.4) and in view of the following mean inequality and inverse Hölder's inequality [10, page 24], we have

$$\prod_{i=1}^{n} m_i^{1/n} \ge \left(\frac{1}{n}\sum_{i=1}^{n} m_i^r\right)^{1/r}, \tag{2.5}$$

$$\frac{\prod_{i=1}^{n} A_{i,m_i}^{p_i}}{\left((1/n)\sum_{i=1}^{n} m_i^r\right)^{n/(pr)}} \ge \prod_{i=1}^{n} p_i \left(\sum_{s_i=1}^{m_i}\left(a_{i,s_i}A_{i,s_i}^{p_i-1}\right)^q\right)^{1/q}. \tag{2.6}$$

Taking the sum of both sides of (2.6) over m_i from 1 to $k_i(1,2,\dots,n)$ first and then using again inverse Hölder's inequality, we obtain that

$$\begin{aligned} \sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} A_{i,m_i}^{p_i}}{\left((1/n)\sum_{i=1}^{n} m_i^r\right)^{n/(pr)}} &\ge \prod_{i=1}^{n} p_i \left(\sum_{m_i=1}^{k_i}\left(\sum_{s_i=1}^{m_i}\left(a_{i,s_i}A_{i,s_i}^{p_i-1}\right)^q\right)^{1/q}\right) \\ &\ge \prod_{i=1}^{n} p_i k_i^{1/p}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(a_{i,s_i}A_{i,s_i}^{p_i-1}\right)^q\right)^{1/q} \\ &= \prod_{i=1}^{n} p_i k_i^{1/p}\left(\sum_{s_i=1}^{k_i}(k_i - s_i + 1)\left(a_{i,s_i}A_{i,s_i}^{p_i-1}\right)^q\right)^{1/q} \\ &= \prod_{i=1}^{n} p_i k_i^{1/p}\left(\sum_{m_i=1}^{k_i}(k_i - m_i + 1)\left(a_{i,m_i}A_{i,m_i}^{p_i-1}\right)^q\right)^{1/q}. \end{aligned} \tag{2.7}$$

This completes the proof. □

Remark 2.2. Taking $n = 2,\ q = -2,\ r = -1$ to (2.1), (2.1) becomes

$$\begin{aligned} \sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2} \frac{A_{1,m_1}^{p_1}A_{2,m_2}^{p_2}}{\left(m_1^{-1}+m_2^{-1}\right)^{-3}} &\ge 8p_1p_2(k_1k_2)^{3/2}\left(\sum_{m_1=1}^{k_1}(k_1 - m_1 + 1)\left(a_{1,m_1}A_{1,m_1}^{p_1-1}\right)^{-2}\right)^{-1/2} \\ &\quad \times \left(\sum_{m_2=1}^{k_2}(k_2 - m_2 + 1)\left(a_{2,m_2}A_{2,m_2}^{p_2-1}\right)^{-2}\right)^{-1/2}. \end{aligned} \tag{2.8}$$

This is just an inverse form of the following inequality which was proven by Pachpatte [9]:

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^pB_n^q}{m+n}\le\frac{1}{2}pq(kr)^{1/2}\left(\sum_{m=1}^{k}(k-m+1)\left(a_mA_m^{p-1}\right)^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)\left(b_nB_n^{q-1}\right)^2\right)^{1/2}. \tag{2.9}$$

Theorem 2.3. *Let $\{a_{i,m_i}\}$, A_{i,m_i}, k_i, p, and q be as defined in Theorem 2.1. Let $\{p_{i,m_i}\}$ be n positive sequences for $m_i=1,2,\dots,k_i$ $(i=1,2,\dots,n)$. Set $P_{i,m_i}=\sum_{s_i=1}^{m_i}p_{i,s_i}$ $(i=1,2,\dots,n)$. Let ϕ_i $(i=1,2,\dots,n)$ be n real-valued nonnegative, concave, and supermultiplicative functions defined on $R_+=[0,+\infty)$. Then,*

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(A_{i,m_i})}{\left((1/n)\sum_{i=1}^{n}m_i^r\right)^{n/(pr)}}\ge M(k_1,k_2,\dots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{a_{i,m_i}}{p_{i,m_i}}\right)\right)^q\right)^{1/q}, \tag{2.10}$$

where

$$M(k_1,k_2,\dots,k_n)=\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^p\right)^{1/p}. \tag{2.11}$$

Proof. From the hypotheses and by Jensen's inequality, the means inequality, and inverse Hölder's inequality, we obtain that

$$\begin{aligned}\prod_{i=1}^{n}\phi_i(A_{i,m_i})&=\prod_{i=1}^{n}\phi_i\left(\frac{P_{i,m_i}\sum_{s_i=1}^{m_i}p_{i,s_i}(a_{i,s_i}/p_{i,s_i})}{\sum_{s_i=1}^{m_i}p_{i,s_i}}\right)\ge\prod_{i=1}^{n}\phi_i(P_{i,m_i})\phi_i\left(\frac{\sum_{s_i=1}^{m_i}p_{i,s_i}(a_{i,s_i}/p_{i,s_i})}{\sum_{s_i=1}^{m_i}p_{i,s_i}}\right)\\&\ge\prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\sum_{s_i=1}^{m_i}p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\ge\prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}m_i^{1/p}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^q\right)^{1/q}\\&\ge\left(\frac{1}{n}\sum_{i=1}^{n}m_i^r\right)^{n/(pr)}\prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^q\right)^{1/q}.\end{aligned} \tag{2.12}$$

Dividing both sides of (2.12) by $((1/n)\sum_{i=1}^{n}m_i^r)^{n/(pr)}$ and then taking the sum over m_i $(i=1,2,\dots,n)$ from 1 to k_i (and in view of inverse Hölder's inequality), we have

$$\begin{aligned}\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(A_{i,m_i})}{\left((1/n)\sum_{i=1}^{n}m_i^r\right)^{n/(pr)}}&\ge\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^q\right)^{1/q}\right)\\&\ge\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^p\right)^{1/p}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^q\right)^{1/q}\\&=M(k_1,k_2,\dots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^q\right)^{1/q}\\&=M(k_1,k_2,\dots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{a_{i,m_i}}{p_{i,m_i}}\right)\right)^q\right)^{1/q}.\end{aligned} \tag{2.13}$$

The proof is complete. □

Remark 2.4. Taking $n=2$, $q=-2$, $r=-1$ to (2.10), (2.10) becomes

$$\sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2}\frac{\phi_1(A_{1,m_1})\phi_2(A_{2,m_2})}{\left(m_1^{-1}+m_2^{-1}\right)^{-3}} \geq M(k_1,k_2)\left(\sum_{m_1=1}^{k_1}(k_1-m_1+1)\left(p_{1,m_1}\phi_1\left(\frac{a_{1,m_1}}{p_{1,m_1}}\right)\right)^{-2}\right)^{-1/2} \times\left(\sum_{m_2=1}^{k_2}(k_2-m_2+1)\left(p_{2,m_2}\phi_2\left(\frac{a_{2,m_2}}{p_{2,m_2}}\right)\right)^{-2}\right)^{-1/2}, \tag{2.14}$$

where

$$M(k_1,k_2)=8\left(\sum_{m_1=1}^{k_1}\left(\frac{\phi_1(P_{1,m_1})}{P_{1,m_1}}\right)^{2/3}\right)^{3/2}\left(\sum_{m_2=1}^{k_2}\left(\frac{\phi_2(P_{2,m_2})}{P_{2,m_2}}\right)^{2/3}\right)^{3/2}. \tag{2.15}$$

This is just an inverse of the following inequality which was proven by Pachpatte [9]:

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n} \leq M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left(p_m\phi\left(\frac{a_m}{p_m}\right)\right)^{2}\right)^{1/2} \times\left(\sum_{n=1}^{r}(r-n+1)\left(q_n\psi\left(\frac{b_n}{q_n}\right)\right)^{2}\right)^{1/2}, \tag{2.16}$$

where

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\left(\frac{\phi(P_m)}{P_m}\right)^{2}\right)^{1/2}\left(\sum_{n=1}^{r}\left(\frac{\psi(Q_n)}{Q_n}\right)^{2}\right)^{1/2}. \tag{2.17}$$

Similarly, the following theorem also can be established.

Theorem 2.5. *Let P_{i,m_i}, $\{a_{i,m_i}\}$, $\{p_{i,m_i}\}$, k_i, p, and q be as in Theorem 2.3 and define $A_{i,m_i} = (1/P_{i,m_i})\sum_{s_i=1}^{m_i} p_{i,s_i}a_{i,s_i}$, for $m_i = 1,2,\ldots,k_i$. Let ϕ_i $(i = 1,2,\ldots,n)$ be n real-valued, nonnegative, and concave functions defined on R_+. Then,*

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}P_{i,m_i}\phi_i(A_{i,m_i})}{\left((1/n)\sum_{i=1}^{n}m_i^{r}\right)^{n/(pr)}} \geq \prod_{i=1}^{n}k_i^{1/p}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i(a_{i,m_i})\right)^{q}\right)^{1/q}. \tag{2.18}$$

The proof of Theorem 2.5 can be completed by following the same steps as in the proof of Theorem 2.3 with suitable changes. Here, we omit the details.

Remark 2.6. Taking $n=2$, $q=-2$, $r=-1$ to (2.18), (2.18) becomes

$$\sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2}\frac{P_{1,m_1}P_{2,m_2}\phi_1(A_{1,m_1})\phi_2(A_{2,m_2})}{\left(m_1^{-1}+m_2^{-1}\right)^{-3}} \geq 8(k_1k_2)^{3/2}\left(\sum_{m_1=1}^{k_1}(k_1-m_1+1)\left(p_{1,m_1}\phi_1(a_{1,m_1})\right)^{-2}\right)^{-1/2}\left(\sum_{m_2=1}^{k_2}(k_2-m_2+1)\left(p_{2,m_2}\phi_2(a_{2,m_2})\right)^{-2}\right)^{-1/2}. \tag{2.19}$$

This is just an inverse of the following inequality which was proven by Pachpatte [9]:

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{m+n} \le \frac{1}{2}(kr)^{1/2}\left(\sum_{m=1}^{k}(k-m+1)(p_m\phi(a_m))^2\right)^{1/2}\left(\sum_{n=1}^{r}(r-n+1)(q_n\psi(b_n))^2\right)^{1/2}. \tag{2.20}$$

Remark 2.7. In view of L'Hôpital law, we have the following fact:

$$\lim_{r\to 0}\left(\frac{1}{n}\sum_{i=1}^{n}m_i^r\right)^{n/(pr)} = \exp\left(\frac{n}{p}\lim_{r\to 0}\frac{\ln\left((1/n)\sum_{i=1}^{n}m_i^r\right)}{r}\right) = \exp\left(\frac{n}{p}\lim_{r\to 0}\frac{\sum_{i=1}^{n}m_i^r\ln m_i}{\sum_{i=1}^{n}m_i^r}\right) = (m_1\cdot m_2\cdot\cdots\cdot m_n)^{1/p}. \tag{2.21}$$

Accordingly, in the special case when $n = 2$, $p = 0.1$, and $p_{i,m_i} = 1$, let $r \to 0$, then the inequality (2.18) reduces to the following inequality:

$$\sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2}\frac{\phi_1(A_{1,m_1})\phi_2(A_{2,m_2})}{(m_1m_2)^{-2}} \ge (k_1k_2)^{-1}\left(\sum_{m_1=1}^{k_1}(k_1-m_1+1)(\phi_1(a_{1,m_1}))^{1/2}\right)^2\left(\sum_{m_2=1}^{k_2}(k_2-m_2+1)(\phi_2(a_{2,m_2}))^{1/2}\right)^2. \tag{2.22}$$

This is just a discrete form of the following inequality which was proven by Zhao and Debnath [11]:

$$\int_0^x\int_0^y\frac{\phi(F(s))\psi(G(t))}{(st)^{-2}}ds\,dt \ge (xy)^{-1}\left[\int_0^x(x-s)\{\phi(f(s))\}^{1/2}ds\right]^2\left[\int_0^y(y-t)\{\phi(g(t))\}^{1/2}dt\right]^2. \tag{2.23}$$

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