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Generalized PSK in Space–Time Coding

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Abstract—**A wireless communication system using multiple antennas promises reliable transmission under Rayleigh flat fading assumptions. Design criteria and practical schemes have been presented for both coherent and noncoherent communication channels. In this paper, we generalize one-dimensional (1-D) phase-shift keying (PSK) signals and introduce space–time constellations from generalized PSK (GPSK) signals based on the complex and real orthogonal designs. The resulting space–time constellations reallocate the energy for each transmitting antenna and feature good diversity products; consequently, their performances are better than some of the existing comparable codes. Moreover, since the maximum-likelihood (ML) decoding of our proposed codes can be decomposed to 1-D PSK signal demodulation, the ML decoding of our codes can be implemented in a very efficient way.**

Index Terms—**Diversity, multiple antennas, orthogonal designs, phase-shift keying (PSK), space–time coding.**

I. Introduction and Model

Multiple antenna communication systems have been actively studied recently [1], [6], [7], [14]. By exploiting the temporal and spatial diversity at both the transmitter and receiver sides, this kind of system can increase the channel capacity compared to single-antenna communication systems, consequently promising more reliable data transmission for high-rate applications.

We investigate a communication system under Rayleigh flat fading assumptions and, we further assume that fading is quasi-static over a time period of length T. Denote by M or N the number of transmitting or receiving antennas, respectively. Let ρ represent the expected signal-to-noise ratio (SNR) at each receiving antenna. For the system above, the basic equation between the received signal R, which is a $T \times N$ matrix, and the transmitted signal Φ, which is chosen from a $T \times M$ matrix constellation $\mathcal{V} = \{\Phi_1, \Phi_2, \cdots, \Phi_L\}$ (L is the constellation size), is given through

$$R = \sqrt{\frac{\rho}{M}}\Phi H + W$$

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where the $M \times N$ matrix H accounts for the multiplicative complex Gaussian fading coefficients and the $T \times N$ matrix W accounts for the additive white Gaussian noise (AWGN). The entries $h_{m,n}$ of the matrix H as well as the entries $w_{t,n}$ of the matrix W are assumed to have a statistically independent complex normal distribution $\mathcal{CN}(0,1)$. One can verify that the transmission rate is determined by L and T as

$$\mathtt{R} = \frac{\log_2(L)}{T}.$$

When the fading coefficients are unknown to the transmitter but known to the receiver, it is proven [2], [15] that the channel capacity will increase linearly with $\min\{M, N\}$. Since H is known to the receiver, for a received signal R, the maximum-likelihood (ML) decoder will take the following evaluation to resolve the most likely sent signal (codeword):

$$\hat{\Phi} = \arg\min_{\Phi \in \mathcal{V}} \left\| R - \sqrt{\frac{\rho}{M}}\Phi H \right\|$$

where $\|\cdot\|$ represents Frobenius norm. Let $P_{\Phi_l, \Phi_{l'}}$ denote the probability that the ML decoder mistakes Φ_l for $\Phi_{l'}$. The upper bound of this error probability has been derived in [14]

$$P_{\Phi_l, \Phi_{l'}} \leq \frac{1}{2}\prod_{m=1}^{M}\left[1 + \frac{\rho T}{4M}\delta_m^2(\Phi_l - \Phi_{l'})\right]^{-N}$$

where $\delta_m(\Phi_l - \Phi_{l'})$ is the mth singular value of $\Phi_l - \Phi_{l'}$. Through the analysis of the above upper bound, design criteria for designing codes for the coherent channel with ideal channel-state information (CSI) are proposed in [14] as follows.

The Rank Criterion: In order to achieve the maximum diversity MN, the matrix $B(\Phi_l, \Phi_l') = \Phi_l - \Phi_{l'}$ has to have full rank for any codewords Φ_l and $\Phi_{l'}$. If $B(\Phi_l, \Phi_{l'})$ has minimum rank r over the set of two tuples of distinct codewords, then a diversity rN is achieved.

The Determinant Criterion: Suppose that a diversity benefit of rN is our target. The minimum of rth roots of the sum of determinants of all $r \times r$ principal cofactors of $A(\Phi_l, \Phi_{l'}) = B^*(\Phi_l, \Phi_{l'})B(\Phi_l, \Phi_{l'})$ taken over all pairs of distinct codewords Φ_l and $\Phi_{l'}$ corresponds to the coding advantage, where r is the rank of $A(\Phi_l, \Phi_{l'})$.

Note that, for the case when $T = M$ and full diversity is achieved, the above design criteria can be simplified as follows.

Construct a constellation of matrix $\mathcal{V} = \{\Phi_1, \Phi_2, \cdots, \Phi_L\}$ such that the *diversity product* [6]

$$\prod \mathcal{V} = \min_{l \neq l'} \frac{1}{2} \left|\det(\Phi_l - \Phi_{l'})\right|^{\frac{1}{M}}$$

is as large as possible.

Orthogonal designs have been investigated for the constellation construction for coherent channels. Using a complex orthogonal design, Alamouti [1] proposed a very simple transmitter diversity scheme with two transmitting antennas. A very interesting coding scheme from the real orthogonal designs is presented in [13]. Applying the similar idea as in [1], the authors also explain how the symmetric structure of the orthogonal design codes leads to a much simpler decoding algorithm. Other dimensional orthogonal designs and nonsquare orthogonal designs have been discussed in [8], [10], and [12]. However, no explicit constellations have been studied in detail in these works.

For a slow fading channel, for instance, a fixed wireless communication system, the coherent assumption is reasonable because the transmitter can send pilot signals which allow the receiver to estimate the fading coefficients accurately. However, in certain situations, due to the difficulty of measuring the fading coefficients under the practical communication environment (for instance, the limited resources or a fast fading link) the above coherent model would be questionable. Because of this, researchers put their efforts into the case when the channel is noncoherent, i.e., the CSI is not known by either the transmitter or the receiver.

In [7], Hochwald and Marzetta study unitary space–time modulation for a noncoherent channel. We will use the same notations as in the coherent case, so the basic equation will be still the same:

$$R = \sqrt{\frac{\rho}{M}} \Phi H + W.$$

However, in noncoherent scenarios, it is assumed that the receiver does not know the exact values of the entries of H (other than their statistical distribution). Another difference is that the signal constellation $\mathcal{V} := \{\Phi_1, \Phi_2, \cdots, \Phi_L\}$ has unitary constraints: $\Phi_k^* \Phi_k = I_M$ for $k = 1, 2, \cdots, L$. The last equation simply states that the columns of Φ_k form a "unitary frame," i.e., the column vectors all have unit length in the complex vector space $\mathbb{C}^T$ and the vectors are pairwise orthogonal. The scaled matrices $\sqrt{T}\Phi_k$ represent the codewords used during the transmission.

The decoding task asks for the computation of the most likely sent codeword $\hat{\Phi}$ given the received signal R. Under the assumption of above model, the ML decoder will have to compute

$$\hat{\Phi} = \arg \max_{\Phi \in \{\Phi_1, \Phi_2, \cdots, \Phi_L\}} \|R^* \Phi\|$$

for each received signal R (see [7]).

It has been shown in [7] that the pairwise probability of mistaking Φ_l for $\Phi_{l'}$ using ML decoding satisfies

$$P_{\Phi_l, \Phi_{l'}} \leq \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{\left(\frac{\rho T}{M}\right)^2 \left(1 - \delta_m^2\left(\Phi_l^* \Phi_{l'}\right)\right)}{4(1 + \frac{\rho T}{M})}\right]^{-N}$$

where $\delta_m(\Phi_l^* \Phi_{l'})$ is the m-th singular value of $\Phi_l^* \Phi_{l'}$. An important special case occurs when $T = 2M$. In this situation it is customary to represent the unitary matrix Φ_k in the form:

$$\Phi_k = \frac{\sqrt{2}}{2} \begin{pmatrix} I_M \\ \Psi_k \end{pmatrix}. \tag{1}$$

Note that, by the definition of Φ_k, the matrix Ψ_k is a $M \times M$ unitary matrix. In most of the literature mentioned above, researchers focus their attention on constellations having the special form (1). A differential modulation scheme is discussed in [6], and this special form is used. In this scheme, one does not send the identity matrix (the upper part of the signal) every time. Instead of sending

$$\begin{pmatrix} I_M \\ \Psi_1 \end{pmatrix}, \begin{pmatrix} I_M \\ \Psi_2 \end{pmatrix}, \begin{pmatrix} I_M \\ \Psi_3 \end{pmatrix}, \cdots$$

one sends

$$I_M, \Psi_1, \Psi_2\Psi_1, \Psi_3\Psi_2\Psi_1, \cdots.$$

This increases the transmission rate by a factor of 2 to

$$\mathtt{R} = \frac{\log_2(L)}{M} = 2\frac{\log_2(L)}{T}.$$

Letting Ψ_τ and R_τ denote the sent and received signals at time τ, respectively, the ML decoder of the above differential space time modulation scheme will have to compute

$$\hat{\Psi}_\tau = \arg \min_{\Psi \in \mathcal{V}} \|R_\tau - \Psi R_{\tau-1}\|.$$

For the constellations with the special form (1), the pairwise error probability satisfies

$$P_{\Phi_l, \Phi_{l'}} \leq \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{\rho^2 \delta_m^2(\Psi_l - \Psi_{l'})}{4(1 + 2\rho)}\right]^{-N}. \tag{2}$$

As explained in [6], at high SNR scenarios, the right-hand side of the above inequality is governed by the diversity product

$$\prod \mathcal{V} = \min_{l \neq l'} \frac{1}{2} \left|\det(\Psi_l - \Psi_{l'})\right|^{\frac{1}{M}}.$$

Thus, the unitary differential modulation design criterion for a noncoherent channel is to choose a constellation $\mathcal{V}$ such that $\prod \mathcal{V}$ is as large as possible. We call a constellation $\mathcal{V}$ fully diverse if $\prod \mathcal{V} > 0$.

The normalized square complex orthogonal design codes and real orthogonal design codes can be used in noncoherent scenarios also. The *diversity sum* [4], [9] of a unitary constellation $\mathcal{V} = \{\Phi_1, \Phi_2, \cdots, \Phi_L\}$ is defined as

$$\sum \mathcal{V} = \min_{l \neq l'} \sqrt{1 - \frac{\|\Phi_l^* \Phi_{l'}\|_F^2}{M}}.$$

It has been shown that, in terms of performance, diversity product dominates in the high SNR region, while diversity sum governs the low SNR region. Interestingly enough, unitary codes from orthogonal designs will have exactly the same diversity product as diversity sum. This special attribute implies that codes from orthogonal designs with large diversity product (or sum) will promise uniformly good performance at low and high SNR regions.

In this paper, we will show how to construct space–time codes from these schemes using generalized phase-shift keying (GPSK) signals. This paper is organized as follows. In Section II, three series of two-dimensional (2-D) GPSK constellations from complex orthogonal designs will be presented. An algebraic calculation will show that they have larger diversity products than the original orthogonal design constellations. As a consequence of larger diversity, the ML decoding of a GPSK constellation gives better performance. In Section III, we explicitly construct GPSK constellations from the real orthogonal designs. Fast decoding algorithms for the proposed square constellations will be presented in Sections II and III in detail. Our proposed constellation in Sections II and III will be presented for noncoherent channels; however, without any doubts, the resulting codes can be used for coherent channels as well. We then explain briefly how to construct nonsquare GPSK constellations from nonsquare orthogonal designs in Section IV. Finally, in Section V, we will give the conclusions.

II. GPSK Constellations from Square Complex Orthogonal Designs

A very simple yet interesting complex orthogonal scheme is described in [1]. We will consider the normalized version of this proposed code, i.e., every element of this code is a matrix given by

$$\mathcal{O}(a,b)=\frac{1}{\sqrt{2}}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where $|a|^2=|b|^2=1$. Observe that $\mathcal{O}(a,b)$ is a unitary matrix and a constellation $\mathcal{O}(n)$ with size $L=n^2$ is obtained by letting a and b range over the set of nth roots of unity

$$\vartheta=\left\{1, e^{\frac{2\pi i}{n}}, \cdots, e^{\frac{2\pi i(n-1)}{n}}\right\}$$

namely the entries of the matrix are chosen from scaled one-dimensional (1-D) PSK signal set ϑ. This would be the most commonly implemented Alamouti's scheme. The diversity product of the constellation $\mathcal{O}(n)$ is

$$\prod \mathcal{O}(n)=\frac{\sqrt{2}}{2}\sin\frac{\pi}{n}.$$

Note that $\mathcal{O}(n)$ is similar to ϑ in the sense that all of the elements in $\mathcal{O}(n)$ have unit energy and every pair of elements differ only by the phases. $\mathcal{O}(n)$ is a subset of the special unitary group

$$SU(2)=\left\{\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \middle| |a|^2+|b|^2=1\right\}.$$

In this section, we will present three series of unitary constellations as finite subsets of $SU(2)$.

The basic principle to decode the constellation $\mathcal{O}(n)$ has been discussed in [1]. We are going to describe this decoding process in another way in more detail and generalize it to our constellations. Consider a noncoherent wireless communication system modulated by $\mathcal{O}(n)$ with two transmitting antennas and N receiving antennas and assume that the differential unitary space–time modulation [6] is used. Let X and Y denote the received matrices at time block τ and $\tau+1$, respectively, then the ML decoder will perform the following decoding task

$$(\hat{a},\hat{b})=\arg\min_{a,b\in\vartheta}\left\|Y-\frac{1}{\sqrt{2}}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}X\right\|^2.$$

With simple matrix manipulations, one can check that

$$\begin{aligned}(\hat{a},\hat{b})&=\arg\min_{a,b\in\vartheta}\sum_{i=1}^{N}\left\|\begin{pmatrix} Y_{1i} \\ Y_{2i}\end{pmatrix}-\frac{1}{\sqrt{2}}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} X_{1i} \\ X_{2i}\end{pmatrix}\right\|^2\\
&=\arg\min_{a,b\in\vartheta}\sum_{i=1}^{N}\left\|\begin{pmatrix} Y_{1i} \\ Y_{2i}^*\end{pmatrix}-\frac{1}{\sqrt{2}}\begin{pmatrix} X_{1i} & X_{2i} \\ X_{2i}^* & -X_{1i}^* \end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}\right\|^2\\
&=\arg\min_{a,b\in\vartheta}\sum_{i=1}^{N}\frac{2}{|X_{1i}|^2+|X_{2i}|^2}\\
&\quad\times\left\|\frac{1}{\sqrt{2}}\begin{pmatrix} X_{1i}^* & X_{2i} \\ X_{2i}^* & -X_{1i} \end{pmatrix}\begin{pmatrix} Y_{1i} \\ Y_{2i}^*\end{pmatrix}\right.\\
&\qquad\left.-\frac{1}{2}\left(|X_{1i}|^2+|X_{2i}|^2\right)\begin{pmatrix} a \\ b\end{pmatrix}\right\|^2\\
&=\arg\min_{a,b\in\vartheta}\sum_{i=1}^{N}\frac{2}{|X_{1i}|^2+|X_{2i}|^2}\\
&\quad\times\left(\left|\frac{1}{\sqrt{2}}X_{1i}^*Y_{1i}+\frac{1}{\sqrt{2}}X_{2i}Y_{2i}^*-\frac{1}{2}\left(|X_{1i}|^2+|X_{2i}|^2\right)a\right|^2\right.\\
&\qquad+\left|\frac{1}{\sqrt{2}}X_{2i}^*Y_{1i}+\frac{1}{\sqrt{2}}X_{1i}Y_{2i}^*\right.\\
&\qquad\left.\left.-\frac{1}{2}\left(|X_{1i}|^2+|X_{2i}|^2\right)b\right|^2\right).\end{aligned}$$

Further algebraic simplifications show that ML decoding is very simple:

$$(\hat{a},\hat{b})=\arg\max_{a,b\in\vartheta}\left(Re\left(a\sum_{i=1}^{N}Z_i^*\right)+Re\left(b\sum_{i=1}^{N}W_i^*\right)\right) \tag{3}$$

where $Z_i=X_{1i}^*Y_{1i}+X_{2i}Y_{2i}^*$ and $W_i=X_{2i}^*Y_{1i}+X_{1i}Y_{2i}^*$. Since a and b are independent of each other (a and b can be chosen freely in the set ϑ), the evaluations above amount to

$$\hat{a}=\arg\max_{a\in\vartheta}Re\left(a\sum_{i=1}^{N}Z_i^*\right)$$

and

$$\hat{b}=\arg\max_{b\in\vartheta}Re\left(b\sum_{i=1}^{N}W_i^*\right).$$

Rewrite a as $a=e^{2\pi ji/n}$ and b as $b=e^{2\pi ki/n}$ and let

$$\lfloor r\rceil=\left\lfloor r+\frac{1}{2}\right\rfloor$$

i.e., $\lfloor r\rceil$ denotes the smaller of the closest integers to r. The ML decoder will take the following simple form:

$$\hat{j}=\left\lfloor\frac{n\arg Z}{2\pi}\right\rceil,\quad \hat{k}=\left\lfloor\frac{n\arg W}{2\pi}\right\rceil, \tag{4}$$

where $Z=\sum_{i=1}^{n}Z_i$ and $W=\sum_{i=1}^{n}W_i$.

TABLE I
DIVERSITY PRODUCT OF $\mathcal{V}_1(n)$

n	r	$\frac{\sqrt{2}}{2}\sin\frac{\pi}{n}=\prod\mathcal{O}(n)$	$\mathtt{R}(\mathcal{O}(n))$	$\frac{1}{2}\sqrt{(\frac{\sqrt{2}}{2}-r)^2+(\frac{\sqrt{2}}{2}-\sqrt{1-r^2})^2}$	$\prod\mathcal{V}_1(n)$	$\mathtt{R}(\mathcal{V}_1(n))$
4	0.259	0.5	2.000	0.259	0.259	2.500
6	0.284	0.353	2.585	0.246	0.246	3.085
8	0.321	0.271	3.000	0.227	0.227	3.500
10	0.360	0.219	3.322	0.209	0.209	3.822
12	0.386	0.183	3.585	0.193	0.183	4.085

Assume that a communication channel is modulated by 1-D PSK signals ϑ. Let $a = e^{2\pi ji/n} \in \vartheta$ denote the sent signal at one time and Z denote the corresponding corrupted signal. The ML decoding will look for the closest signal $\hat{a} = e^{2\pi\hat{j}i/n}$ to a in the signal set ϑ, i.e.,

$$\hat{j} = \left\lfloor \frac{n \arg Z}{2\pi} \right\rceil. \tag{5}$$

Comparing (4) and (5), one can conclude that the decoding of space–time code $\mathcal{O}(n)$ is decomposable and can be implemented by taking 1-D PSK demodulation twice. The normalized Alamouti codes $\mathcal{O}(n)$'s admit very simple decoding algorithms, since roughly only $4N$ complex multiplications and $4N$ complex additions are needed. The main objective of this paper is to introduce such decomposable codes based on orthogonal designs for coherent and noncoherent channels. We call such codes *generalized PSK (GPSK) constellations*. Instead of allocating the same energy to every transmitting antenna, we attempt to exploit the transmit diversity and improve the power allocation. Compared to $\mathcal{O}(n)$, these codes have larger diversity for relatively large transmission rate, which promise better performances with ML decoding. Due to the symmetry from GPSK signals, these codes are also decomposable, and therefore fast decoding algorithms can be also applied to these codes. In the following three subsections, we present the constructions of these three series of constellations.

A. Construction 1

Let n be an even number and let $0 < r < \sqrt{2}/2$ be the root of the following equation:

$$\left(\frac{\sqrt{2}}{2} - r\right)^2 + \left(\frac{\sqrt{2}}{2} - \sqrt{1-r^2}\right)^2 = 4r^2 \sin^2 \frac{2\pi}{n}. \tag{6}$$

Consider the following sets of the scaled 1-D PSK signals:

$$A_1(n) = \left\{ \frac{\sqrt{2}}{2e^{i\frac{2\pi}{n}k}} | k = 0, 1, \cdots, n-1 \right\}$$
$$A_2(n) = \left\{ re^{i\frac{4\pi}{n}k} \middle| k = 0, 1, \cdots, \frac{n}{2} - 1 \right\}$$
$$A_3(n) = \left\{ \sqrt{1-r^2} e^{i\frac{2\pi}{n}k} | k = 0, 1, \cdots, n-1 \right\}.$$

Consider the following subsets of $SU(2)$:

$$C_1(n) = \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_1(n),\ b \in A_1(n) \right\}$$
$$C_2(n) = \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_2(n),\ b \in A_3(n) \right\}$$
$$C_3(n) = \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_3(n),\ b \in A_2(n) \right\}.$$

Theorem 2.1: Let

$$\mathcal{V}_1(n) := C_1(n) \bigcup C_2(n) \bigcup C_3(n)$$

where $\mathcal{V}_1(n)$ is a fully diverse constellation of $2n^2$ elements with diversity product

$$\prod \mathcal{V}_1(n) = \min\left\{ \frac{\sqrt{2}}{2}\sin\frac{\pi}{n}, \frac{1}{2}\sqrt{\left(\frac{\sqrt{2}}{2} - r\right)^2 + \left(\frac{\sqrt{2}}{2} - \sqrt{1-r^2}\right)^2} \right\}.$$

Proof: See Appendix A. □

Table I shows how the diversity product of $\mathcal{V}_1(n)$ compares with the diversity product of $\mathcal{O}(n)$ when $n \le 12$.

Corollary 2.2: For $n \ge 12$

$$\prod \mathcal{V}_1(n) = \frac{\sqrt{2}}{2}\sin\frac{\pi}{n}.$$

Proof: For $n \ge 12$

$$\begin{aligned} &\frac{1}{2}\sqrt{\left(\frac{\sqrt{2}}{2} - r\right)^2 + \left(\frac{\sqrt{2}}{2} - \sqrt{1-r^2}\right)^2} \\ &= r\sin\frac{2\pi}{n} \\ &= 2r\sin\frac{\pi}{n}\cos\frac{\pi}{n} \\ &\ge 2 \times 0.386\cos\frac{\pi}{12}\sin\frac{\pi}{n} \\ &\ge \frac{\sqrt{2}}{2}\sin\frac{\pi}{n}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \prod \mathcal{V}_1(n) &= \min\left\{ \frac{\sqrt{2}}{2}\sin\frac{\pi}{n}, \frac{1}{2}\sqrt{\left(\frac{\sqrt{2}}{2} - r\right)^2 + \left(\frac{\sqrt{2}}{2} - \sqrt{1-r^2}\right)^2} \right\} \\ &= \frac{\sqrt{2}}{2}\sin\frac{\pi}{n}. \end{aligned}$$

□

The above corollary indicates that, for $n \ge 12$, the GPSK constellation $\mathcal{V}_1(n)$ has the same diversity product as the orthogonal constellation $\mathcal{O}(n)$, while it has twice as many elements: $\mathcal{V}_1(n)$ has $2n^2$ elements whereas $\mathcal{O}(n)$ has n^2 elements.

Similar to the case for $\mathcal{O}(n)$, the ML decoding of $\mathcal{V}_1(n)$ boils down to (3). However we can not separate the estimation of a, b using the simple Formula (4), because generally a and b are not independent anymore. If we restrict the decoding evaluation in a particular $C_i(n)$, then $\arg(a)$ and $\arg(b)$ can be chosen freely.

Namely, within the restricted searching area $C_i(n)$, a and b are independent of each other. Assuming that the originally sent codeword falls in $C_1(n)$, one can use (4) to resolve a candidate $(\hat{a}_1, \hat{b}_1)$

$$\hat{j} = \left\lfloor \frac{n \arg Z}{2\pi} \right\rceil, \quad \hat{k} = \left\lfloor \frac{n \arg W}{2\pi} \right\rceil.$$

Similarly, we can have candidates $(\hat{a}_i, \hat{b}_i)$ for $C_i(n)$, where $i = 2, 3$. The final ML decoder will resolve the most likely sent codeword

$$(\hat{a}, \hat{b}) = \arg \max_{\hat{a}_i, \hat{b}_i} (Re(aZ^*) + Re(bW^*)).$$

Again, we conclude that the decoding of $\mathcal{V}_1(n)$ is decomposable, and therefore $\mathcal{V}_1(n)$ admits a simple decoding. The above evaluations require roughly $4N$ complex multiplications and $4N$ complex additions, therefore the decoding of the GPSK constellation $\mathcal{V}_1(n)$ has the same complexity as that of $\mathcal{O}(n)$.

B. Construction 2

Let $n = 2m$ and consider the following sets consisting of scaled 1-D PSK signals:

$$\begin{aligned}
A_1(n) &= \left\{ re^{i\frac{2\pi}{m}j} | j = 0, 1, \cdots, m-1 \right\} \\
A_2(n) &= \left\{ \sqrt{1-r^2} e^{i(\frac{2\pi}{m}j + \frac{\pi}{m})} | j = 0, 1, \cdots, m-1 \right\} \\
A_3(n) &= \left\{ \sqrt{1-r^2} e^{i\frac{2\pi}{m}j} | j = 0, 1, \cdots, m-1 \right\} \\
A_4(n) &= \left\{ re^{i(\frac{2\pi}{m}j + \frac{\pi}{m})} | j = 0, 1, \cdots, m-1 \right\}
\end{aligned}$$

where

$$r = \frac{1}{\sqrt{2\sin^2\frac{\pi}{m} + 2\sqrt{2}\sin\frac{\pi}{m} + 2}}.$$

Based on the above signal sets, construct the following subsets of $SU(2)$:

$$\begin{aligned}
C_1(n) &= \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_1(n), \ b \in A_2(n) \right\} \\
C_2(n) &= \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_2(n), \ b \in A_1(n) \right\} \\
C_3(n) &= \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_3(n), \ b \in A_4(n) \right\} \\
C_4(n) &= \left\{ \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} | a \in A_4(n), \ b \in A_3(n) \right\}.
\end{aligned}$$

Theorem 2.3: Let

$$\mathcal{V}_2(n) := C_1(n) \bigcup C_2(n) \bigcup C_3(n) \bigcup C_4(n)$$

where $\mathcal{V}_2(n)$ is a fully diverse constellation of n^2 elements with diversity product

$$\prod \mathcal{V}_2(n) = \min \left\{ r \sin\frac{2\pi}{n}, \sin\frac{\pi}{n} \right\}.$$

TABLE II
DIVERSITY PRODUCT OF $\mathcal{V}_2(n)$

n	r	$r\sin\frac{2\pi}{n}$	$\sin\frac{\pi}{n}$	$\prod \mathcal{V}_2(n)$	$\prod \mathcal{O}(n)$	$\mathrm{R}(\mathcal{O}(n)) = \mathrm{R}(\mathcal{V}_2(n))$
4	0.383	0.383	0.707	0.383	0.5	2.000
6	0.410	0.355	0.500	0.355	0.354	2.585
8	0.447	0.316	0.355	0.316	0.271	3.000
10	0.479	0.282	0.309	0.282	0.219	3.322
12	0.505	0.253	0.259	0.253	0.183	3.585
14	0.527	0.229	0.222	0.222	0.157	3.807

Proof: See Appendix B. □

Table II shows how the diversity product of $\mathcal{V}_2(n)$ compares with that of $\mathcal{O}(n)$ when $n \leq 14$.

Corollary 2.4: For $n \geq 14$

$$\prod \mathcal{V}_2(n) = \sin\frac{\pi}{n}.$$

Proof: According to the definition of r

$$2(1 - 2r\sqrt{1-r^2}) = 4r^2 \sin^2\frac{\pi}{m}.$$

One can also check that

$$\begin{aligned}
4r^2\sin^2\frac{\pi}{m} &= 16r^2\sin^2\frac{\pi}{2m}\cos^2\frac{\pi}{2m} \\
&= 4r^2\cos^2\frac{\pi}{2m}\left(4\sin^2\frac{\pi}{2m}\right) \\
&= \frac{4\cos^2\frac{\pi}{2m}}{2\sin^2\frac{\pi}{m} + 2\sqrt{2}\sin\frac{\pi}{m} + 2}\left(4\sin^2\frac{\pi}{2m}\right) \\
&\geq \frac{4\cos^2\frac{\pi}{14}}{2\sin^2\frac{\pi}{7} + 2\sqrt{2}\sin\frac{\pi}{7} + 2}\left(4\sin^2\frac{\pi}{2m}\right) \\
&\geq 4\sin^2\frac{\pi}{2m}.
\end{aligned}$$

Consequently, one concludes that

$$\prod \mathcal{V}_2(n) = \min\left\{ r\sin\frac{\pi}{m}, \sin\frac{\pi}{2m} \right\} = \sin\frac{\pi}{n}.$$

□

For $n \geq 14$, the constellation $\mathcal{V}_2(n)$ has as many elements as the constellation $\mathcal{O}(n)$ does, however, its diversity product is larger than that of $\mathcal{O}(n)$ by a factor of $\sqrt{2}$, i.e., $\prod \mathcal{V}_2(n) = \sqrt{2} \prod \mathcal{O}(n)$. The corollary indicates that we could allocate different power to the transmitting antennas to achieve more reliable transmission, while still keeping the total energy. Similar to the case for $\mathcal{V}_1(n)$, one can apply exactly the same algorithm to achieve the ML decoding for $\mathcal{V}_2(n)$. It can be easily seen that these two algorithms have the same complexity.

C. Construction 3

We will take further efforts to explore the subsets of $SU(2)$. In the following, we are going to describe a series of unitary constellation $\mathcal{V}_3(n)$. For the sample program implemented to construct $\mathcal{V}_3(n)$, we refer to [3]. Now, for given integers $n > 0$ and $0 \leq k \leq n$, we define

$$N_0 = 1,$$
$$N_k = \left\lfloor \frac{\pi}{\arcsin \frac{\sin \frac{\pi}{4n}}{\cos \frac{(n-k)\pi}{2n}}} \right\rfloor, \; k = 1, 2, \cdots, n$$
$$M_k = \left\lfloor \frac{\pi}{\arcsin \frac{\sin \frac{\pi}{4n}}{\sin \frac{(n-k)\pi}{2n}}} \right\rfloor,$$
$$k = 0, 1, \cdots, n-1, \quad M_n = 1$$
$$C_k(n) = \left\{ \begin{pmatrix} a_{k,j} & b_{k,l} \\ -b_{k,l}^* & a_{k,j}^* \end{pmatrix} \middle| a_{k,j} = \cos \frac{(n-k)\pi}{2n} e^{i\frac{2j\pi}{N_k}}, \right.$$
$$j = 0, \cdots, N_k - 1, b_{k,l} = \sin \frac{(n-k)\pi}{2n} e^{i\frac{2l\pi}{M_k}},$$
$$\left. l = 0, \cdots, M_k - 1 \right\}.$$

Theorem 2.5: Let

$$\mathcal{V}_3(n) = \bigcup_{k=0}^{n} C_k(n)$$

where $\mathcal{V}_3(n)$ is a fully diverse constellation of $\sum_{k=0}^{n} M_k N_k$ elements with diversity product

$$\prod \mathcal{V}_3(n) = \sin \frac{\pi}{4n}.$$

Proof: See Appendix C □

Corollary 2.6: When $n \to \infty$, $\mathcal{V}_3(n)$ will have $O(n^3)$ elements and have the diversity product $O(1/n)$.

Proof: See Appendix D. □

This corollary indicates that, asymptotically, $\mathcal{V}_3(n)$ will have a much better diversity product compared to $\mathcal{O}(n)$, $\mathcal{V}_1(n)$, and $\mathcal{V}_2(n)$, because the other three constellations asymptotically will have $O(n^2)$ elements and have diversity product $O(1/n)$. Observe that $SU(2)$ in fact can be viewed as the three-dimensional (3-D) unit sphere. Finding a constellation with the optimal diversity product can be converted to a sphere packing problem on this unit sphere. For n points on a 3-D unit sphere, the largest minimum distance one can hope for is asymptotically $O(1/n^{1/3})$. Therefore, when n becomes large, asymptotically $\mathcal{V}_3(n)$ is the best constellation over all the subsets of $SU(2)$. Table III shows the diversity product of constellation $\mathcal{V}_3(n)$.

The ML decoding of $\mathcal{V}_3(n)$ is also decomposable, and it roughly requires $4N + 2n$ complex multiplications and $4N + 2n$ complex additions; therefore, asymptotically, it requires $O(N) + O(L^{1/3})$ complex multiplications and additions. One can see from Table III that, for reasonably small values of n, the constellation size grows rapidly. For example, when the constellation size L is already huge (10770), n is still reasonably small (13). Thus, even if we are dealing with a high rate constellation, the decoding process of $\mathcal{V}_3(n)$ is still very simple.

We compare different codes from GPSK signals with a transmission rate of around 4.5 in Fig. 1. One can see that $\mathcal{V}_3(5)$ outperforms other constellations even with the highest transmission rate. Of course, the decoding of $\mathcal{V}_3(5)$ is a little more complex than other constellations; however, the sacrifice in decoding efficiency is worthwhile for the remarkably gained performance.

TABLE III
DIVERSITY PRODUCT OF $\mathcal{V}_3(n)$

n	size	diversity product	$\mathtt{R}(\mathcal{V}_3(n))$
3	124	0.259	3.477
4	293	0.195	4.097
5	582	0.156	4.592
6	974	0.131	4.964
7	1640	0.112	5.340
8	2438	0.098	5.626
9	3510	0.087	5.889
10	4898	0.078	6.129
11	6516	0.071	6.335
12	8433	0.065	6.521
13	10770	0.060	6.697

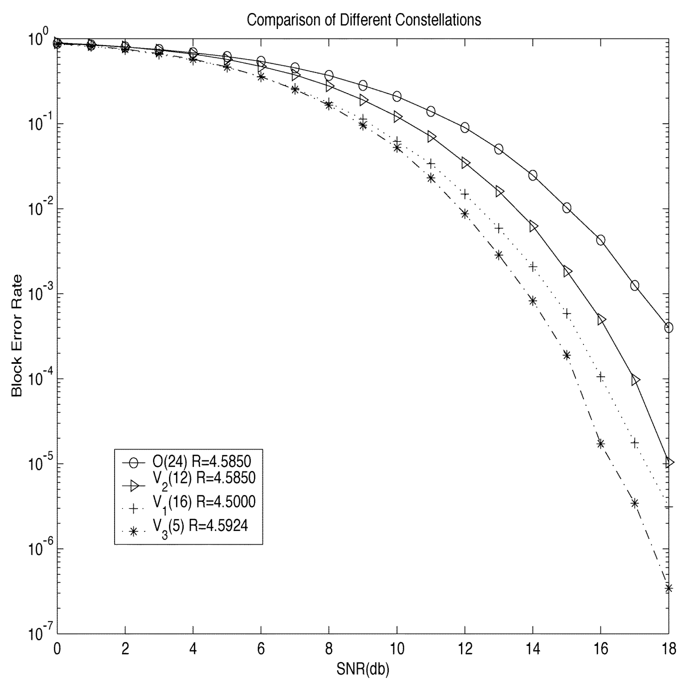

Fig. 1. Performance of different codes from GPSK signals, $M = 2, N = 12$.

We compare the GPSK constellations with the Cayley codes [5] in Fig. 2. One can see that $\mathcal{V}_1(44)$ with transmission rate $\mathtt{R} = 5.9594$ has already a gain of approximately 2 dB compared to the Cayley code, and the performances of $\mathcal{V}_3(9)$ with $\mathtt{R} = 5.8886$ and $\mathcal{V}_3(10)$ with $\mathtt{R} = 6.1290$ are even more remarkable. Note that all of the GPSK constellations admit the presented simple decoding algorithms. $\mathcal{V}_1(44)$ or $\mathcal{V}_2(64)$ only need about eight complex multiplications and eight complex additions to decode one codeword, and $\mathcal{V}_3(9)$ or $\mathcal{V}_3(10)$ need approximately 28 complex multiplications and 28 complex additions to decode.

D. Higher Dimension Construction

Apparently, the idea of "power relocation" can be applied to other higher dimensional complex orthogonal designs. For instance, one can adjust the parameters such as amplitudes, phases

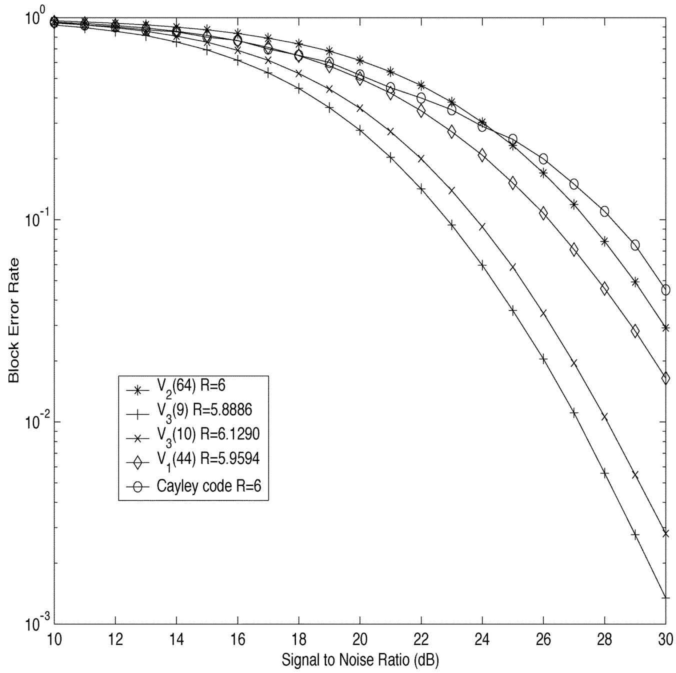

Fig. 2. Performances of GPSK constellations and a Cayley code, $M = 2$, $N = 2$.

to carve constellations with large diversity product from the following 8×8 complex orthogonal design:

$$\begin{pmatrix} z_1 & z_2 & z_3 & 0 & z_4 & 0 & 0 & 0 \\ -z_2^* & z_1^* & 0 & z_3 & 0 & z_4 & 0 & 0 \\ z_3^* & 0 & -z_1^* & z_2 & 0 & 0 & z_4 & 0 \\ 0 & z_3^* & -z_2^* & -z_1 & 0 & 0 & 0 & z_4 \\ z_4^* & 0 & 0 & 0 & -z_1^* & z_2 & -z_3 & 0 \\ 0 & z_4^* & 0 & 0 & -z_2^* & -z_1 & 0 & -z_3 \\ 0 & 0 & z_4^* & 0 & -z_3^* & 0 & z_1 & z_2 \\ 0 & 0 & 0 & z_4^* & 0 & -z_3^* & -z_2^* & z_1^* \end{pmatrix}.$$

The problem of constructing an eight-dimensional (8-D) complex orthogonal code from the design above with the maximal diversity product is equivalent to the complex vector packing problem with the constraint

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1.$$

Instead of going through more tedious derivations (adjusting many parameters) to obtain more technical results, we shall present a universal algorithm to generate complex packing vectors with good minimum distance. Consequently, we obtain a constellation with a large diversity product.

We first illustrate the basic idea using the 8×8 example design above. Since the distance of two complex vectors satisfies

$$\sum_{j=1}^{4} |z_j - w_j|^2 \geq \sum_{j=1}^{4} (|z_j| - |w_j|)^2$$

we can pack the amplitude vector ($|z_1|$, $|z_2|$, $|z_3|$, $|z_4|$) first, and then pack the argument vector ($\arg(z_1)$, $\arg(z_2)$, $\arg(z_3)$, $\arg(z_4)$). To explain this implementation in more detail, we represent z using the polar coordinates

$$\begin{aligned} z_1 &= \cos\theta_1 e^{i\gamma_1} \\ z_2 &= \sin\theta_1 \cos\theta_2 e^{i\gamma_2} \\ z_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 e^{i\gamma_3} \\ z_4 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 e^{i\gamma_4}. \end{aligned}$$

Let θ_j run over $m_j + 1$ evenly distributed discrete values from 0 to $\pi/2$

$$\theta_j \in \left\{ k\frac{\pi}{2m_j} | k = 0, 1, \cdots, m_j \right\}$$

and let γ_j run over n_j evenly distributed discrete values from 0 to 2π

$$\gamma_j \in \left\{ k\frac{2\pi}{n_j} | k = 0, 1, \cdots, n_j - 1 \right\}.$$

Now we have a finite set of complex vectors.

Let $\mathbf{v} = (v_1, v_2, v_3, v_4)$ and $\mathbf{w} = (w_1, w_2, w_3, w_4)$ be two distinct resulting complex vectors. If $|v_1| \neq |w_1|$, one can check that

$$|\mathbf{v} - \mathbf{w}|^2 \geq 4\sin^2\left(\frac{\pi}{(4m_1)}\right).$$

If $|v_1| = |w_1|$ but $|v_2| \neq |w_2|$, we will have

$$|\mathbf{v} - \mathbf{w}|^2 \geq 4\left(1 - |v_1|^2\right)\sin^2\left(\frac{\pi}{(4m_2)}\right).$$

In the case that $|v_1| = |w_1|$ and $|v_2| = |w_2|$ but $|v_3| \neq |w_3|$, similar algebraic calculations will give a lower bound

$$|\mathbf{v} - \mathbf{w}|^2 \geq 4\left(1 - |v_1|^2 - |v_2|^2\right)\sin^2\left(\frac{\pi}{(4m_3)}\right).$$

We further consider the case $|v_j| = |w_j|$ for $j = 1, 2, 3, 4$. In this case, if $v_j \neq w_j$ for some j, we can have

$$|\mathbf{v} - \mathbf{w}|^2 \geq 4|v_j|^2\sin^2\left(\frac{\pi}{n_j}\right).$$

It is easy to check that the lower bound can be reached for some special pair of z, w in all the cases. Based on the 8-D real orthogonal design [cf. (9)], assign $s_{2j-1} = Re(z_j)$ and $s_{2j} = Im(z_j)$ for $j = 1, 2, 3, 4$. Now we have a finite unitary constellation whose diversity product is

$$\min_{\mathbf{v},\mathbf{w}} \frac{\sqrt{|\mathbf{v} - \mathbf{w}|}}{2}.$$

However, if we choose m_j, n_j arbitrarily, often we will end up having a constellation with a small diversity. In the sequel, we are going to add suitable constraints to the choices of m_j and n_j to guarantee that we have a large diversity product. The algorithm will be presented in a general setting so that it can generate packing vectors

$$\mathbf{z} = (z_1, z_2, \cdots, z_k)$$

with polar coordinate representation

$$\begin{aligned} z_1 &= \cos\theta_1 e^{i\gamma_1} \\ z_2 &= \sin\theta_1\cos\theta_2 e^{i\gamma_2} \\ &\vdots \\ z_{k-1} &= \sin\theta_1\sin\theta_2\cdots\cos\theta_{k-1}e^{i\gamma_{k-1}} \\ z_k &= \sin\theta_1\sin\theta_2\cdots\sin\theta_{k-1}e^{i\gamma_k}. \end{aligned}$$

For the sample program of this algorithm, we refer to [3].

Algorithm I: (given the input n)

1. Fix $m_1 = n$ and let θ_1 run over $n+1$ even distributed discrete values from 0 to $\pi/2$ (consequently $|z_1|$ runs over $n+1$ discrete values from 0 to 1).
2. For any fixed $|z_1|, |z_2|, \cdots, |z_{i-1}|$, take m_i to be the largest integer such that

$$4\big(1-|z_1|^2-|z_2|^2-\cdots-|z_{i-1}|^2\big)\sin^2\!\left(\frac{\pi}{(4m_i)}\right)\geq 4\sin^2\!\left(\frac{\pi}{(4n)}\right).$$

   Let θ_i run over $m_i + 1$ evenly distributed discrete values from 0 to $\pi/2$. Increase i by 1, repeat step 2 until $i = k+1$.
3. For any fixed $|z_j|$, take n_j to be the largest integer such that

$$4|z_j|^2\sin^2\left(\frac{\pi}{n_j}\right) \geq 4\sin^2\left(\frac{\pi}{(4n)}\right).$$

   Let γ_j run over n_j evenly distributed discrete values from 0 to 2π.

The steps above result in a finite set of complex vectors $\mathbf{z} = (z_1, z_2, \cdots, z_k)$. For $k = 4$, we obtain an 8-D complex orthogonal design constellation with diversity product $\sin(\pi/(4n))$. Recall that, for the packing problem with n points on a $2k-1$–dimensional unit sphere, asymptotically the largest minimum distance one can hope for is $O(1/n^{2k-1})$. The following theorem indicates that the algorithm produce optimal packing vectors when $n \to \infty$. Although the proof of this theorem is quite tedious, the basic idea is very similar to Corollary 2.6, thus we skip the proof.

Theorem 2.7: Asymptotically, Algorithm I will generate $O(n^{2k-1})$ complex packing vectors with the minimum distance $O(1/n)$.

The analysis similar to the one for $\mathcal{V}_3(n)$ shows that the constellations corresponding to the packing vectors generated by Algorithm I are also decomposable, and therefore they admit fast decoding as well. To decode one codeword, $O(M^2N) + O(L^{(k-1)/(2k-1)})$ complex multiplications and additions are needed.

III. GPSK Constellations From Square Real Orthogonal Designs

In [13], space–time block coding has been discussed for coherent channels. We show that the normalized version of the real orthogonal codes can be used in unitary space time modulation for noncoherent scenarios too. We are going to present explicit implementations of the real orthogonal designs, which feature simple decoding algorithms. For simplicity we describe our idea using a four-dimensional (4-D) real orthogonal design

$$S = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4 & -s_3 & s_2 & s_1 \end{pmatrix}. \tag{7}$$

One natural idea is to transform a two-dimensional (2-D) complex orthogonal code $\mathcal{O}(n)$ to a 4-D real orthogonal code. Recall that $\mathcal{O}(n)$ is a set of matrices

$$\mathcal{O}(a,b) = \frac{1}{\sqrt{2}}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

with a and b ranging over $\vartheta = \{1, e^{2\pi i/n}, \cdots, e^{2\pi i(n-1)/n}\}$. Let $s_1 = Re(a)$, $s_2 = Im(a)$, $s_3 = Re(b)$, and $s_4 = Im(b)$, we will have a 4-D real orthogonal code with the same diversity product as $\mathcal{O}(n)$.

This code also features a simple decoding algorithm. The simple decoding of real orthogonal design codes has been explained in [13]; thus, in the following, we will discuss the decoding process in more detail such that it can be generalized to our constellations straightforwardly. First, we introduce some notations to be used. The superscript T will be the notation for a matrix transpose. For two vectors F and G, let $F \cdot G$ denote the dot product of F and G. For a vector F, F_j denotes the jth element of F. For a matrix F, F_j denotes the jth column vector. Let B be the mapping

$$B(x_1,x_2,x_3,x_4) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & x_4 & -x_3 \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4 & x_3 & -x_2 & -x_1 \end{pmatrix}$$

and immediately one checks that

$$S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B(x_1,x_2,x_3,x_4)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}. \tag{8}$$

Assume that the differential unitary space time modulation is used for a wireless communication system with M transmitting antennas and N receiving antennas. Let X and Y denote the received $M \times N$ matrices at time block $\tau - 1$ and τ, respectively. The ML decoder will make the following estimation:

$$\hat{S} = \arg\min_S \|Y - SX\|^2.$$

We separate X and Y into the real parts and the imaginary parts with

$$X = F + iG, \quad Y = P + iQ.$$

Then we have

$$\hat{S} = \arg\min_S \left(\|P - SF\|^2 + \|Q - SG\|^2\right)$$
$$= \arg\min_S \left(\sum_{j=1}^N \|P_j - SF_j\|^2 + \sum_{j=1}^N \|Q_j - SG_j\|^2\right).$$

Utilizing (8), we have

$$\hat{S} = \arg\min_{s_1,s_2,s_3,s_4} \left(\sum_{j=1}^N \left\|P_j - B(F_j)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}\right\|^2 + \sum_{j=1}^N \left\|Q_j - B(G_j)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}\right\|^2\right).$$

Since $B(F_j)$ and $B(G_j)$ are orthogonal matrices, simple algebraic manipulations will lead the above evaluation to

$$\arg\min_{s_1,s_2,s_3,s_4} \left(\sum_{j=1}^N \frac{1}{|F_j|^2}\left\|B^T(F_j)P_j - |F_j|^2\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}\right\|^2 + \sum_{j=1}^N \frac{1}{|G_j|^2}\left\|B^T(G_j)Q_j - |G_j|^2\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}\right\|^2\right)$$
$$= \arg\max_{s_1,s_2,s_3,s_4} (s_1, s_2, s_3, s_4) \cdot \left(\sum_{j=1}^N B^T(F_j)P_j + \sum_{j=1}^N B^T(G_j)Q_j\right)$$
$$= \arg\max_{s_j} \sum_{j=1}^N s_j U_j$$

where $U_j = \sum_{j=1}^N B^T(F_j)P_j + \sum_{j=1}^N B^T(G_j)Q_j$. Since the construction of our code is based on $\mathcal{O}(n)$ as described above, one can check that (s_1, s_2) in fact is independent of (s_3, s_4). Rewrite $s_1 + is_2 = (1/\sqrt{2})e^{2k\pi i/n}$ and $s_3 + is_4 = (1/\sqrt{2})e^{2l\pi i/n}$. Then, from

$$\hat{S} = \arg\max_{s_1,s_2,s_3,s_4} \left((s_1, s_2)\cdot(U_1, U_2)^T + (s_3, s_4)\cdot(U_3, U_4)^T\right)$$

we conclude that the ML decoding of this GPSK constellation is decomposable and can be boiled down to the following simple form:

$$\hat{k} = \left\lfloor \frac{2\pi \arg(U_1 + iU_2)}{n} \right\rceil, \quad \hat{l} = \left\lfloor \frac{2\pi \arg(U_3 + iU_4)}{n} \right\rceil.$$

Generally speaking, the use of the proposed codes above for a wireless communication system with M transmitting antennas and N receiving antennas will take $8M^2N$ real multiplications and $8M^2N$ real additions to decode one codeword, which is very simple.

Apparently, we can apply the same idea for $\mathcal{V}_i(n)$ $(i = 1, 2, 3)$ to construct GPSK real orthogonal constellations. The ML decoder of the corresponding codes will take similar approaches as in the $\mathcal{O}(n)$ case, except that one should notice that (s_1, s_2) is not independent of (s_3, s_4) anymore. In this case, one can restrict the searching area to be the subsets $C_i(n)$ "locally" and apply the similar techniques as we did in the complex orthogonal design case to achieve the ML decoding, and then the corresponding codes will admit simple decoding algorithms too. Also, we conclude that codes from $\mathcal{V}_1(n)$ or $\mathcal{V}_2(n)$ are of the same complexity as codes from $\mathcal{O}(n)$. For the codes from $\mathcal{V}_3(n)$, the complexity increases for high transmission rates; however, they will have more pronounced performances.

In the sequel, we are going to present a series of 8-D GPSK real orthogonal constellation $\mathcal{V}_4(n)$. Consider the 8-D orthogonal design

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ -s_2 & s_1 & s_4 & -s_3 & s_6 & -s_5 & -s_8 & s_7 \\ -s_3 & -s_4 & s_1 & s_2 & s_7 & s_8 & -s_5 & -s_6 \\ -s_4 & s_3 & -s_2 & s_1 & s_8 & -s_7 & s_6 & -s_5 \\ -s_5 & -s_6 & -s_7 & -s_8 & s_1 & s_2 & s_3 & s_4 \\ -s_6 & s_5 & -s_8 & s_7 & -s_2 & s_1 & -s_4 & s_3 \\ -s_7 & s_8 & s_5 & -s_6 & -s_3 & s_4 & s_1 & -s_2 \\ -s_8 & -s_7 & s_6 & s_5 & -s_4 & -s_3 & s_2 & s_1 \end{pmatrix}. \tag{9}$$

Similar to the 4-D real orthogonal constellation case, one can obtain 8-D unitary codes by transforming $V(n) \times W(n)$, where V or W denotes any one of $\mathcal{O}$, $\mathcal{V}_1$, $\mathcal{V}_2$, or $\mathcal{V}_3$ and $\times$ denotes the Cartesian product. For this implementation, s_i can be assigned to be the scaled version of the real or imaginary part of "a" or "b" as in the 4-D case. However, as in the complex orthogonal case, $\mathcal{V}_3(n)$ motivates us to explore more densely packed constellations.

The problem of constructing an 8-D real orthogonal code with the maximal diversity product is equivalent to the packing problem on a seven-dimensional (7-D) unit sphere. Therefore, any currently existing results in packing problem on a 7-D unit sphere can be "borrowed" for the real orthogonal code construction. However, decoding such codes using exhaustive search will be unpractical for high transmission rates. In the sequel, we are going to present a series of 8-D GPSK orthogonal codes $\mathcal{V}_4(n)$ featuring simple decoding algorithms.

For $k = 4$, apply Algorithm I to generate a finite set of complex vectors $\mathbf{z} = (z_1, z_2, z_3, z_4)$. If we assign $s_{2j-1} = Re(z_j)$ and $s_{2j} = Im(z_j)$ with j = 1, 2, 3, 4, we obtain an 8-D real orthogonal design constellation $\mathcal{V}_4(n)$ with diversity product $\sin(\pi/(4n))$.

Applying the same analysis for the constellation $\mathcal{V}_3(n)$, we conclude that $\mathcal{V}_4(n)$ is also decomposable and the decoding complexity for one codeword will be $O(N)+O(L^{3/7})$. In Fig. 3, we compare the performance of $\mathcal{V}_4(2)$ of 103 elements with a 2-D constellation of three elements

$$\{I_2, A, B\}$$

where $A = \mathrm{diag}(e^{i2\pi/3}, e^{i2\pi/3})$ and $B = \mathrm{diag}(e^{i4\pi/3}, e^{i4\pi/3})$. It is well known that this 2-D constellation has the optimal diversity product over all the 2-D unitary constellations with three elements. Since a large number of transmitting antennas guarantee that the full diversity at the transmitter side can be utilized more efficiently [see (2)], it is not too surprising to see that the 8-D GPSK constellation performs better in the high SNR region. The

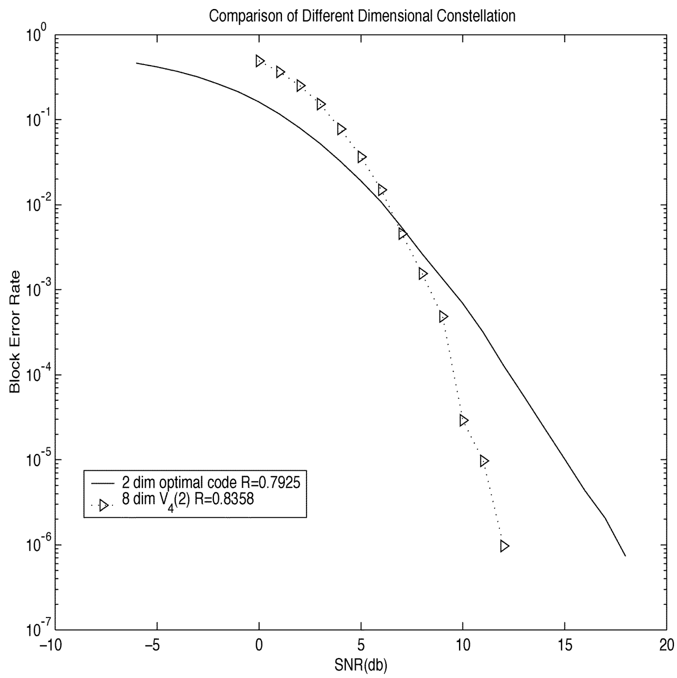

Fig. 3. Performances of a GPSK constellation and a code with optimal diversity product, $N = 2$.

figure shows that, from 6 dB, 8-D constellation begins to outperform the 2-D one. Around the block error rate (BLER) of 10^{-6}, the performance gain is about 5 dB, which is very pronounced considering that the decoding of $\mathcal{V}_4(2)$ is relatively easy.

IV. GPSK Constellations From Nonsquare Orthogonal Designs

In the previous two sections, we constructed GPSK constellations from square orthogonal designs for both coherent and noncoherent multiple antenna system. Nonsquare complex or real orthogonal designs [8], [10], [12] are only meant for coherent detection; however, the square constellation design method can be easily translated to construct nonsquare ones. We shall sketch the idea using a 4 × 3 complex orthogonal design

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ -z_2^* & z_1^* & 0 \\ z_3^* & 0 & -z_1^* \\ 0 & z_3^* & -z_2^* \end{pmatrix}.$$

We require

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = \frac{1}{3} \tag{10}$$

so that any complex orthogonal signal is normalized to have norm 1. Let

$$\mathcal{V} = \{\Phi_1, \Phi_2, \cdots, \Phi_L\}$$

be a set of signals from the complex orthogonal design above with

$$\Phi_i = \begin{pmatrix} \Phi_{i1} & \Phi_{i2} & \Phi_{i3} \\ -\Phi_{i2}^* & \Phi_{i1}^* & 0 \\ \Phi_{i3}^* & 0 & -\Phi_{i1}^* \\ 0 & \Phi_{i3}^* & -\Phi_{i2}^* \end{pmatrix}.$$

The determinant criterion requires that

$$\det(\Phi_i - \Phi_j)^*(\Phi_i - \Phi_j) = |\Phi_{i1} - \Phi_{j1}|^2 + |\Phi_{i2} - \Phi_{j2}|^2 + |\Phi_{i3} - \Phi_{j3}|^2$$

should be as large as possible for every distinct pair i, j. Thus, similar to the square case, we are facing the complex vector packing problem again. One can use Algorithm I to produce such vectors with good minimum distance and the corresponding constellations with large minimum determinant.

Let $\Phi \in \mathcal{V}$ be the sent signal matrix. Let R denote the received matrix. Then, for the coherent detection, the ML decoding amounts to

$$\begin{aligned} \hat{\Phi} &= \arg\min_{\Phi\in\mathcal{V}} \left\| R - \sqrt{\frac{\rho}{M}}\Phi H \right\|^2 \\ &= \arg\min_{\Phi\in\mathcal{V}} \operatorname{tr}\left(R^* - \sqrt{\frac{\rho}{M}} H^*\Phi^*\right)\left(R - \sqrt{\frac{\rho}{M}}\Phi H\right) \\ &= \arg\min_{\Phi\in\mathcal{V}} \operatorname{tr}\left(R^*R - \sqrt{\frac{\rho}{M}} R^*\Phi H \right. \\ &\qquad \left. - \sqrt{\frac{\rho}{M}} H^*\Phi^* R - \frac{\rho}{M} H^*\Phi^*\Phi H\right) \\ &= \arg\max_{\Phi\in\mathcal{V}} \operatorname{tr}(R^*\Phi H + H^*\Phi^* R). \end{aligned}$$

Again, similar to the analysis in the square case, the nonsquare constellations produced by Algorithm I are decomposable and admit a fast decoding algorithm. For nonsquare complex and real orthogonal design with any size, the analysis above can be applied and the corresponding GPSK constellations exist. To decode one codeword, $O(MTN) + O(L^{(k-1)/(2k-1)})$ multiplications and additions are needed, where k denotes the number of symbols used in the orthogonal design (For the example design in this section, it is 3).

For a coherent channel, the signal energy normalization [see (10) for the energy normalization in the example above) is no longer required. One may choose the entries of the constellation matrices from any arbitrary signal set (for example, QAM signal set), therefore the normal energy constraint will hinder us from finding the optimal constellation (in terms of diversity product). However, this constraint does offer to reduce the decoding complexity in certain situations. Note that, without the energy constraint, the term $(\rho/M)H^*\Phi^*\Phi H$ would not disappear in the final expression of the ML decoding evaluation, thus implying that one has to search the most possible signal over the whole chosen signal set. However, a GPSK constellation may process the searching over a much smaller candidate set because of the symmetric structure. For instance, the decoding of one codeword in $\mathcal{V}_3(10)$ will only require 28 simple complex evaluations, while, generally speaking, the comparable constellation using an arbitrary signal sets without energy constraint requires 70 evaluations.

V. Conclusion

The complex and real orthogonal coding schemes admit simple decoding algorithms. Based on these schemes, we generalize 1-D PSK signals and explicitly construct GPSK unitary space time constellations. These constellations can be viewed as higher dimensional generalizations of 1-D PSK signals. Like the 1-D PSK signals and higher dimensional classical orthogonal constellations (such as Alamouti's 2-D constellations), certainly the proposed constellations can be used in coded modulation for multiple-antenna systems (the "irregularity" of GPSK constellation may add minor difficulties in the system design). Based on the connection between the sphere packing problem and finding constellations with large diversity, we point out that GPSK constellations usually do not have optimal diversity. On the other hand, the optimal constellations corresponding to the optimal sphere packing do not necessarily have fast decoding.

Theoretical analysis shows that their decoding procedures are decomposable, i.e., the demodulation of these codes can be boiled down to 1-D PSK demodulation. Therefore, our constellations have very simple decoding procedures. For some of the resulting codes (for example, $\mathcal{V}_1(n)$, $\mathcal{V}_2(n)$), the complexity of ML decoding does not even depend on the transmission rate. We reallocate the power among the antennas (meanwhile keeping the total energy) to improve the diversity product. Numerical experiments show that our codes perform better than some of the currently existing comparable ones. For the sample programs regarding how to construct the proposed constellations, we refer to [3].

Appendix A
Proof for Theorem 1

Proof: Take two elements $A \in C_i$ and $B \in C_j$. Without loss of generality, we further assume $i \leq j$. If $A, B \in C_1(n)$, then

$$|\det(A-B)| \geq \left(\sqrt{2}\sin\frac{\pi}{n}\right)^2 = 2\sin^2\frac{\pi}{n}.$$

If $A \in C_1(n)$, $B \in C_2(n)$ or $A \in C_1(n)$, $B \in C_3(n)$, then

$$|\det(A-B)| \geq \left(\frac{\sqrt{2}}{2} - r\right)^2 + \left(\frac{\sqrt{2}}{2} - \sqrt{1-r^2}\right)^2 = 2 - \sqrt{2}r - \sqrt{2}\sqrt{1-r^2}.$$

If $A \in C_2(n)$ and $B \in C_3(n)$, then

$$|\det(A-B)| \geq 2(\sqrt{1-r^2} - r)^2.$$

If $A, B \in C_3(n)$ or $A, B \in C_2(n)$, then

$$|\det(A-B)| \geq \min\left\{\left(2r\sin\frac{2\pi}{n}\right)^2, \left(2\sqrt{1-r^2}\sin\frac{\pi}{n}\right)^2\right\}.$$

Using the fact that r is the root of (6) and comparing the lower bounds of the diversity product in all of the above cases, the claim in the theorem can be established. □

Appendix B
Proof for Theorem 2.3

Proof: Take two elements $A \in C_i(n)$ and $B \in C_j(n)$. Without loss of generality, we can further assume $i \leq j$. If $i = j$, we have

$$|\det(A-B)| \geq |re^{i\frac{2\pi}{m}k} - re^{i\frac{2\pi}{m}(k+1)}|^2 = 4r^2\sin^2\frac{\pi}{m}.$$

If $A \in C_1(n)$, $B \in C_2(n)$ or $A \in C_3(n)$, $B \in C_4(n)$, one can check that

$$|\det(A-B)| \geq 2\left|re^{i\frac{2\pi}{m}k} - \sqrt{1-r^2}e^{i\left(\frac{2\pi}{m}k+\frac{\pi}{m}\right)}\right|^2 = 2\left(1 - 2r\sqrt{1-r^2}\cos\frac{\pi}{m}\right).$$

Similarly, if $A \in C_1(n)$, $B \in C_3(n)$ or $A \in C_2(n)$, $B \in C_4(n)$, then

$$|\det(A-B)| \geq 2(\sqrt{1-r^2} - r)^2 = 2(1 - 2r\sqrt{1-r^2}).$$

If $A \in C_1(n)$, $B \in C_4(n)$ or $A \in C_2(n)$, $B \in C_3(n)$, then

$$|\det(A-B)| \geq \left|re^{i\frac{2\pi}{m}k} - re^{i\left(\frac{2\pi}{m}+\frac{\pi}{m}\right)}\right|^2 + \left|\sqrt{1-r^2}e^{i\frac{2\pi}{m}k} - \sqrt{1-r^2}e^{i\left(\frac{2\pi}{m}+\frac{\pi}{m}\right)}\right|^2 = 4\sin^2\frac{\pi}{2m}.$$

It follows from the definition of r that

$$2(1 - 2r\sqrt{1-r^2}) = 4r^2\sin^2\frac{\pi}{m}$$

and naturally we will have

$$2\left(1 - 2r\sqrt{1-r^2}\cos\frac{\pi}{m}\right) \geq 2(1 - 2r\sqrt{1-r^2}).$$

Comparing the lower bounds in all the cases and taking the minimum of them, we establish the claim in the theorem. □

Appendix C
Proof for Theorem 2.5

Proof: Pick two distinct elements $A, B \in \mathcal{V}$

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$
$$B = \begin{pmatrix} c & d \\ -d^* & c^* \end{pmatrix}.$$

One can verify that

$$|\det(A-B)| = \det(A-B) = |a-c|^2 + |b-d|^2.$$

So, if $|a| \neq |c|$, then we have

$$|\det(A-B)| = |a-c|^2 + |b-d|^2 \geq (|a| - |c|)^2 + (|b| - |d|)^2 \geq 2 - 2\cos\frac{\pi}{2n}$$

and it can be verified that the equality holds if there is a $k \in \{0, 1, \cdots, n\}$ such that $A \in C_k(n)$ and $B \in C_{k+1}(n)$ or, alternatively, $B \in C_k(n)$ and $A \in C_{k+1}(n)$. In the case that $|a| = |c|$, we will have

$$|\det(A-B)| = |a-c|^2 + |b-d|^2 \geq \max\left\{|a-c|^2, |b-d|^2\right\} \geq 2 - 2\cos\frac{\pi}{2n}.$$

Therefore, for all of the cases, we have

$$|\det(A-B)| \geq 2 - 2\cos\frac{\pi}{2n}.$$

One checks that the lower bound for each case can be reached. So, for constellation $\mathcal{V}_3(n)$, it follows that

$$\prod \mathcal{V}_3(n) = \frac{1}{2}\left(2 - 2\cos\frac{\pi}{2n}\right)^{\frac{1}{2}} = \sin\frac{\pi}{4n}.$$

□

APPENDIX D
PROOF FOR COROLLARY 2.6

Proof: For $k = 1, 2, \cdots, n$,

$$N_k = \left\lfloor \frac{\pi}{\arcsin \frac{\sin \frac{\pi}{4n}}{\cos \frac{(n-k)\pi}{2n}}} \right\rfloor \leq \left\lfloor \frac{\pi}{\arcsin \sin \frac{\pi}{4n}} \right\rfloor \leq 4n.$$

Similarly, for $k = 0, 1, \cdots, n-1$,

$$M_k = \left\lfloor \frac{\pi}{\arcsin \frac{\sin \frac{\pi}{4n}}{\sin \frac{(n-k)\pi}{2n}}} \right\rfloor \leq \left\lfloor \frac{\pi}{\arcsin \sin \frac{\pi}{4n}} \right\rfloor \leq 4n.$$

Hence, we derive an asymptotic upper bound for the cardinality of $\mathcal{V}_3(n)$

$$|\mathcal{V}_3(n)| = \sum_{k=0}^{n} |C_k(n)| = \sum_{k=0}^{n} M_k N_k \leq 2n + 16(n-1)n^2 \leq O(n^3).$$

In the following, we are going to derive an asymptotic lower bound for $|\mathcal{V}_3(n)|$. First pick two real numbers α and β such that

$$0 < \alpha < \beta < 1.$$

For k such that $\alpha n < k < \beta n$ (such k always exists when n is large enough), we have

$$N_k = \left\lfloor \frac{\pi}{\arcsin \frac{\sin \frac{\pi}{4n}}{\cos \frac{(n-k)\pi}{2n}}} \right\rfloor \geq \left\lfloor \frac{\pi}{\arcsin \frac{\sin \frac{\pi}{4n}}{\cos \frac{(1-\alpha)\pi}{2}}} \right\rfloor.$$

We utilize the fact that

$$\sin x \sim x \sim \arcsin x$$

when x is close to 0. When n is large enough, one derives

$$N_k \geq 4 \cos \frac{(1-\alpha)\pi}{2} n.$$

Similarly

$$M_k \geq 4 \sin \frac{(1-\beta)\pi}{2} n.$$

Therefore, we have an asymptotic lower bound for the cardinality of $\mathcal{V}_3(n)$

$$\begin{aligned} |\mathcal{V}_3(n)| &\geq \sum_{\alpha n < k < \beta n} M_k * N_k \\ &\geq 16(\beta - \alpha) \sin \frac{(1-\beta)\pi}{2} \cos \frac{(1-\alpha)\pi}{2} n^3 \geq O(n^3). \end{aligned}$$

Combining the upper and lower bounds, we conclude that $\mathcal{V}_3(n)$ has $O(n^3)$ elements with diversity product $\sin(\pi/(4n)) = O(1/n)$. □

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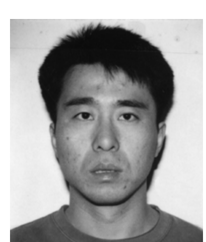

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