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Steinness of universal covers of certain compact Kähler manifolds

By *Ngaiming Mok* at Hong Kong

Very little is known about the complex structure of universal covers $\tilde{X}$ of projective-algebraic manifolds X. According to the Shafarevich Conjecture, such complex manifolds $\tilde{X}$ should always be holomorphically convex. Nonetheless, there is up to this point no overwhelming supporting evidence for this conjecture. A weakened conjecture is the statement that the holomorphic convexity of $\tilde{X}$ is a property of the homotopy type of X. (We may reasonably extend the latter conjecture to include the larger class of compact Kähler manifolds.) In this vein we consider in the present article sufficient topological conditions on a compact Kähler manifold X guaranteeing that the universal cover $\tilde{X}$ is a Stein manifold. We study compact Kähler manifolds whose fundamental groups $\pi_1(X)$ admit non-trivial discrete representations into semisimple Lie groups of the noncompact type. We prove the Steinness of their universal covering spaces $\tilde{X}$ by imposing an additional homological condition on the representations. More precisely, we prove

Main Theorem. *Let G be a semisimple Lie group of the noncompact type and $K \subset G$ be a maximal compact Lie group. Let $\Gamma \subset G$ be a Zariski-dense torsion-free discrete subgroup and denote by N the corresponding Riemannian locally symmetric space $\Gamma \backslash G/K$. Let X be a compact Kähler manifold whose fundamental group $\pi_1(X)$ admits a representation $\Phi : \pi_1(X) \to G$ such that $\Phi(\pi_1(X)) = \Gamma$. If furthermore the induced homomorphism $\Phi_* : H_2(X, \mathbb{R}) \to H_2(N, \mathbb{R})$ is injective, then the universal covering space $\tilde{X}$ of X is a Stein manifold.*

Our result includes as a special case the following situation. Let Ω be a bounded symmetric domain and Γ be a torsion-free discrete group of holomorphic isometries such that $\Omega/\Gamma = N$ is compact. Let X_0 be a complex submanifold of N of dimension at least 3 which is a complete intersection of smooth hyperplane sections (with respect to some projective embedding of N). If X is a compact Kähler manifold homotopic to X_0, then by the Lefschetz Theorem $\pi_1(X)$ is isomorphic to $\pi_1(N) = \Gamma \subset G = \mathrm{Aut}_0(\Omega)$ and the induced homomorphism $H_2(X, \mathbb{R}) \to H_2(N, \mathbb{R})$ is an isomorphism. In this special case our Main Theorem can also be deduced from the work of Simpson [Si] on complex variations of Hodge structures.

Here $\tilde{N} = G/K$, equipped with a canonical metric induced by the Killing form of the Lie algebra of G, is a Riemannian symmetric manifold of the noncompact type. In particular, it is of nonpositive Riemannian sectional curvature. This entitles us to use the method of harmonic maps. We consider a harmonic map $f: X \to N$ inducing Φ and its lifting $F: \tilde{X} \to \tilde{N} = G/K$ to universal covers. In the ensuing discussion we assume for simplicity that $\Phi: \pi_1(X) \to G$ is faithful, so that $F: \tilde{X} \to G/K$ is proper.

By Grauert's solution to the Levi Problem [G], $\tilde{X}$ is Stein if and only if there exists on $\tilde{X}$ a strictly plurisubharmonic exhaustion function. From the $\partial\bar{\partial}$-Bochner-Kodaira formula of Siu [S 2] and Sampson [Sa] and comparison theorems in Differential Geometry it follows readily that the pull-back φ by $F: \tilde{X} \to G/K$ of the square of a geodesic distance function on the symmetric space G/K is a real-analytic plurisubharmonic exhaustion function. To invoke Grauert's result the trouble is that φ is in general only weakly plurisubharmonic. The key step of our proof of the Main Theorem consists of the following proposition on the structure of the Levi form $L(\varphi) = i\partial\bar{\partial}\varphi$. In a nutshell it says that $\tilde{X}$ can be decomposed into locally closed complex-analytic subvarieties in such a way that the zero eigenspaces of $L(\varphi)$ are transverse to each piece of the decomposition.

Proposition 1. *Let X be as in the Main Theorem and $\tilde{X}$ be its universal cover. Assume that $\Phi: \pi_1(X) \to G$ is injective and let φ be the real-analytic plurisubharmonic exhaustion function on $\tilde{X}$ as defined in* (2.1). *Then there exists a finite set $\{S_1, \ldots, S_p = X\}$ of complex-analytic subvarieties of X satisfying the following properties*:

(1) $S_1 \subset S_2 \subset \cdots \subset S_p = X$;

(2) *each irreducible component of S_i is a proper subvariety of an irreducible component of S_{i+1} for $i < p$;*

(3) *S_i contains the singular locus of S_{i+1} for $i < p-1$;*

(4) *denoting by $\tilde{S}_i$ the inverse image of S_i with respect to the universal covering map $\tilde{X} \to X$, $\varphi|_{\tilde{S}_{i+1} - \tilde{S}_i}$ is strictly plurisubharmonic for $i < p$.*

Decomposing X into a union of complex manifolds as given in Proposition 1, we could invoke Narasimhan's generalization [N 1] of Grauert's solution to the Levi Problem to prove the Steinness of X. To do this on each $X_c = \{\varphi < c\}$ we have to modify the weakly plurisubharmonic exhaustion function $(c - \varphi)^{-1}$ to get a strictly plurisubharmonic exhaustion function. We do this inductively on $X_c \cap S_i$. The key induction step consists of applying a theorem of Siu's [S 1] asserting that a Stein space in any complex space admits a Stein neighborhood. Since $X \cap S_i$ may be singular we need the solution of the Levi problem in the case of singular spaces, as given in [N 1].

For $\Phi: \pi_1(X) \to G$ not necessarily injective we can replace $\tilde{X}$ by an intermediate regular covering space $X^\#$ corresponding to $\operatorname{Ker}\Phi$ to prove that $X^\#$ is Stein. We can then conclude the Steinness of $\tilde{X}$ by invoking Stein [St] on covering spaces of Stein spaces.

To prove Proposition 1 we have to show that the zero eigenspaces of the Levi form $L(\varphi)$ must be transverse to the sets on which φ is weakly plurisubharmonic. We argue by contradiction. Assuming that Proposition 1 fails, we construct non-trivial 2-dimensional homology classes which are collapsed to zero under the harmonic map, contradicting with the hypothesis that $\Phi_* : H_2(X, \mathbb{R}) \to H_2(N, \mathbb{R})$ is injective. For the construction of such homology classes we use the notion of semi-Kähler structures as developed in Mok [M 1], [M 2]. In other words, we use meromorphic foliations equipped with compatible degenerate Kähler metrics. These semi-Kähler structures arise from harmonic maps via the use of the $\partial\bar{\partial}$-Bochner-Kodaira formula. The argument leading to the construction of the 2-dimensional homology classes can be summarized in the following more general proposition, applied to non-singular models of subvarieties of X.

Proposition 2. *Let S be a compact Kähler manifold and $g: S \to N$ be a non-constant harmonic map of S into a Riemannian manifold (N, h) of nonpositive sectional curvature in the complexified sense. Suppose the $g^* T_N^{\mathbb{C}}$-valued 1-form ∂g is everywhere degenerate and the kernel is generically of dimension ℓ. Then, there is a holomorphic foliation $\mathcal{G}$ on S defined outside a complex-analytic subvariety $V \subset S$ of codimension $\geqq 2$, such that $\mathcal{G}_x$ agrees with* $\operatorname{Ker}(\partial g(x))$ *on a dense open subset of S. Furthermore, there exists a 2ℓ-dimensional homology class ξ such that $g_* \xi = 0$, where ξ is represented by a current $\tilde{\Theta}$ of bi-dimension (ℓ, ℓ) such that outside of V, $\tilde{\Theta}$ is a measured foliated cycle Θ defined by leaves of $\mathcal{G}$.*

We will start in § 1 with recalling some basic facts about harmonic maps of compact Kähler manifolds into Kähler manifolds of nonpositive curvature in the *complexified* sense. This will lead to the existence of a meromorphic foliation together with a compatible Kähler semi-metric. In § 2 we explain first of all how to construct a weakly plurisubharmonic exhaustion function φ on $\tilde{X}$ such that the kernel of the Levi form $L(\varphi)$ is related to the meromorphic foliation. We analyze the kernel of $L(\varphi)$ and obtain a decomposition of $\tilde{X}$ consisting of pieces on which φ is strictly plurisubharmonic, proving Proposition 1. The main difficulty is to establish Proposition 2 using semi-Kähler structures. In § 3 we will invoke Narasimhan's theorem [N 1] to prove by induction the Steinness of $\tilde{X}$. Finally, along the same line of arguments, we prove a strengthened version of our Main Theorem in the case of compact Kähler surfaces X by dropping the homological condition on second Betti groups. We prove in this case the holomorphic convexity of the intermediate regular covering space $X^\#$ corresponding to $\operatorname{Ker} \Phi$.

§ 1. Harmonic maps on compact Kähler manifolds

(1.1) The $\partial\bar{\partial}$-Bochner-Kodaira formula. Let (X, g) be a compact Kähler manifold and (N, h) be a Riemannian manifold. We consider harmonic maps $f: (X, g) \mapsto (N, h)$. On (X, g) write $(g_{\alpha\bar{\beta}})$ for the metric tensor in local holomorphic coordinates (z_α); $(g^{\alpha\bar{\beta}})$ for the conjugate inverse of $(g_{\alpha\bar{\beta}})$. Then, f is harmonic if and only if it satisfies the Laplace-Beltrami equation

$$\sum_{\alpha, \beta, j, k} g^{\alpha\bar{\beta}} \left[\frac{\partial^2 f^i}{\partial z_\alpha \partial \bar{z}_\beta} + {}^N\Gamma^i_{jk} \frac{\partial f^j}{\partial z_\alpha} \frac{\partial f^k}{\partial \bar{z}_\beta} \right] = 0$$

for each i. Here $({}^N\Gamma^i_{jk})$ are the Riemann-Christoffel symbols of (N, h). The term inside the parenthesis defines components of the complex Hessian $\nabla\bar{\partial} f$ of f. f is said to be pluriharmonic if and only if $\nabla\bar{\partial} f \equiv 0$. For harmonic maps $f: (X, g) \to (N, h)$ we have the following $\partial\bar{\partial}$-Bochner-Kodaira formula of Siu [S 2] and Sampson [Sa]. In what follows X and N need not satisfy the hypotheses of the Main Theorem.

Proposition (Siu [S2], Sampson [Sa]). *Let (X, g) be a compact Kähler manifold and (N, h) be a Riemannian manifold. Let $f: (X, g) \to (N, h)$ be a smooth harmonic map. Then,*

$$\int_X \|\nabla \bar{\partial} f\|^2 + H(\partial f \otimes \partial f; \overline{\partial f \otimes \partial f}) = 0 ,$$

where $H(\cdot, \bar{\cdot})$ is some Hermitian bilinear form defined by the curvature tensor of (N, h). Suppose furthermore that (N, h) is of nonpositive sectional curvature in the complexified sense. Then, $H(\partial f \otimes \partial f; \overline{\partial f \otimes \partial f}) \geqq 0$. As a consequence both terms inside the integrand vanish identically. In particular, f is pluriharmonic.

Here (N, h) is said to be of nonpositive curvature in the *complexified* sense if and only if $R^N_{AB\bar{B}\bar{A}} \leqq 0$ for the curvature tensor R^N of N and for *complexified* tangent vectors A and B. Here and in what follows $df = \partial f + \bar{\partial} f$ where ∂f is an $f^* T_N^{\mathbb{C}}$-valued $(1,0)$-form and $\bar{\partial} f$ is an $f^* T_N^{\mathbb{C}}$-valued $(0, 1)$-form.

(1.2) Meromorphic foliations and compatible Kähler semi-metrics. One application of the Bochner-Kodaira formula for harmonic maps is to show that f is either holomorphic or anti-holomorphic under certain assumptions on the curvature tensor of the target manifold. This was the first application in Siu [S2] for the problem of strong rigidity. Short of that we can still affirm that f is pluriharmonic and that it leads to holomorphic foliations. We state what we need in the following proposition, for which the method of proof relies on an argument of Siu [S3] based on verifying the Frobenius condition.

Proposition 3 (Carlson-Toledo [CT]). *In the statement of the $\partial\bar{\partial}$-Bochner-Kodaira formula suppose (N, h) is a Riemannian symmetric space of the noncompact type. Then, $f^* T_N^{\mathbb{C}}$ can be endowed a holomorphic structure such that ∂f becomes a holomorphic section with values in $\mathrm{Hom}(T_X, f^* T_N^{\mathbb{C}})$. In particular, where ∂f is of constant rank, the distribution $x \to \mathrm{Ker}(\partial f)(x))$ defines an integrable holomorphic distribution and thus a holomorphic foliation.*

The distribution $x \to \mathrm{Ker}((\partial f)(x))$ gives rise to a meromorphic foliation $\mathscr{F}$ on X. This means that there exists a complex-analytic subvariety $V \subset X$ of complex codimension $\geqq 2$ such that $\mathscr{F}$ defines a holomorphic foliation on $X - V$. There is a d-closed semipositive $(1,1)$-form ω on X compatible with $\mathscr{F}$ on a dense open set of X in such a way that ω can be intepreted at generic points as defining a Kähler form on a local complex submanifold transverse to the foliation. ω is defined as follows. The Riemannian semi-metric, as a symmetric 2-tensor field, decomposes into types in terms of the complex structure of X. The $(1,1)$-part then defines a Hermitian semi-metric, which one can show to be a Kähler semi-metric in the sense that the corresponding $(1,1)$-form ω is d-closed, as a consequence of the pluriharmonicity of f (cf. Mok [M1]). $\mathscr{F}_S$ and ω are compatible on a dense open set U on X in the sense that for $x \in U$, the kernel of the semipositive $(1,1)$-form $\omega(x)$ is the same as $\mathrm{Ker}((\partial f)(x))$. As in Mok [M2] we call $(X, \mathscr{F}, \omega, V)$ a semi-Kähler structure. We will also use the terminology concerning semi-Kähler structures as developed in [M1], [M2].

(1.3) We will need to consider restrictions of pluriharmonic maps $f: X \to N$ to possibly singular irreducible subvarieties $S \subset X$. Consider the restriction $f|_S: S \to N$. We replace S by a non-singular Kähler model $\hat{S}$ obtained by desingularizing S and denote by

$\nu : \hat{S} \to S$ the desingularization. The composite map $f \circ \nu : \hat{S} \to N$ is smooth and pluriharmonic outside a complex-analytic subvariety so that it is globally pluriharmonic on $\hat{S}$. On the other hand, the representation $\Phi|_{\pi_1(S)} : \pi_1(S) \to \pi_1(N)$ gives rise canonically to a representation $\Psi : \pi_1(\hat{S}) \to \pi_1(N)$. By the uniqueness theorem for harmonic maps of Hartmann [H], $f \circ \nu : \hat{S} \to N$ is up to geodesic translations on N the unique harmonic representative for the representation $\Psi : \pi_1(\hat{S}) \to \pi_1(N)$. For the purpose of proving factorization theorems on irreducible subvarieties using harmonic maps, passing to desingularizations we can always reduce to the case of compact Kähler manifolds.

§ 2. Decomposition of $\tilde{X}$

(2.1) Let (X, g) be a compact Kähler manifold satisfying the hypothesis of the Main Theorem. For the ensuing discussion we make the additional simplifying assumption that the representation $\Phi : \pi_1(X) \to G$ is faithful. To prove that the universal cover $\tilde{X}$ is Stein we are going to use the real-analytic plurisubharmonic exhaustion function φ to decompose $\tilde{X}$ into a union of locally closed complex submanifolds as given in Proposition 1.

Write $N = \Gamma \backslash G/K$ and write (N, h) for the corresponding Riemannian locally symmetric manifold. Let $f : (X, g) \to (N, h)$ be a harmonic map inducing $\Phi : \pi_1(X) \to \pi_1(N) = \Gamma$ on fundamental groups. The existence of f follows from Eells-Sampson [ES] in the case of cocompact $\Gamma \subset G$ and from Corlette [C] in the general case. Let $F : \tilde{X} \to \tilde{N} = G/K$ be the lifting to universal covering spaces.

Let $d(\cdot, \cdot)$ denote the geodesic distance function on the Riemannian symmetric space $\tilde{N} = G/K$ and write $r(\tilde{x})$ for $d(o; \tilde{x})$, where o denotes an arbitrary base point on G/K. Let φ be the smooth function defined by $\varphi(\tilde{p}) = r^2(F(\tilde{p}))$ for $\tilde{p} \in \tilde{X}$. Since $F : \tilde{X} \to G/K$ is actually pluriharmonic and the canonical metric on G/K is real-analytic, φ is real-analytic. On a neighborhood of $\tilde{p}$ denote by (z_α) a system of local holomorphic coordinates. Denote by (u_i) a system of normal geodesic coordinates at $F(\tilde{p})$. From the pluriharmonicity of F it follows that

$$(*) \qquad \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}(\tilde{p}) = \sum_{i,j} \frac{\partial^2}{\partial u_i \partial u_j} r^2(F(\tilde{p})) \frac{\partial u^i}{\partial z_\alpha} \frac{\partial u^j}{\partial \bar{z}_\beta} .$$

Recall that G/K, equipped with the canonical metric, is of nonpositive Riemannian sectional curvature. By comparing G/K with the flat Euclidean space of the same real dimension we conclude that r^2 is strictly convex on G/K. It follows from $(*)$ that φ is plurisubharmonic on $\tilde{X}$. Furthermore, if η is a complexified tangent vector of type $(1,0)$ at $\tilde{p}$, the formula shows that η lies in the kernel of the Levi form $L(\varphi) = i\partial\bar{\partial}\varphi$ if and only if it lies on the kernel of ∂F. From the discussion on semi-Kähler structures in (1.2), it follows also that the kernel of $L(\varphi)$ agrees with that of the lifted semi-Kähler form $\tilde{\omega}$.

(2.2) In this section we assume Proposition 2 on the construction of homology classes and deduce Proposition 1. We observe first of all that Proposition 1 is an immediate consequence of

Proposition 1′. *Let X be a compact Kähler manifold satisfying the hypothesis of the Main Theorem and φ be the plurisubharmonic exhaustion function as defined in* (2.1). *Let*

$S \subset X$ be a positive-dimensional irreducible complex-analytic subvariety. Then, there exists a proper complex-analytic subvariety $T \subset S$ which contains the singularities of S, such that the holomorphic tensor $\partial f|_S \in \Gamma(X, \mathrm{Hom}(T_S, f^* T_N^{\mathbb{C}}))$ is injective on $S - T$. As a consequence, letting $\tilde{V}$ denote the preimage of V in $\tilde{X}$ under the universal covering map $\tilde{X} \to X$ for any subvariety $V \subset X$, the plurisubharmonic function φ is strictly plurisubharmonic on $\tilde{S} - \tilde{T}$.

We proceed to deduce Proposition 1′ from Proposition 2. To this end we argue by contradiction. As explained in (1.3) for the proof of Proposition 1′ we may assume that S is non-singular. The failure of Proposition 1′ means equivalently that

$$\partial f|_S \in \Gamma(X, \mathrm{Hom}(T_S, f^* T_N^{\mathbb{C}}))$$

has a non-trivial kernel everywhere on S. Let ℓ be the generic rank of $\mathrm{Ker}(\partial f|_S)$. Assuming Proposition 2 we will have a 2ℓ-dimensional homology class ξ which is collapsed to zero under the harmonic map f. For $\ell = 1$ we will have a contradiction with the hypothesis that $f_* : H_2(X, \mathbb{R}) \to H_2(N, \mathbb{R})$ is injective in the Main Theorem. For $\ell \geqq 2$ and X projective-algberaic by slicing by hyperplane sections we may always replace S by some S' such that $\partial f|_{S'}$ is generically of rank 1. In general, we have

Lemma 1. *Let X be a compact Kähler manifold satisfying the hypothesis of the Main Theorem. Then, X is projective-algebraic.*

For the proof of Lemma 1 we need the following result, which will be referred to as the Factorization Theorem.

Theorem (Mok [M1]). *Let X be a compact Kähler manifold with fundamental group $\pi_1(X) = \Gamma$ and let G be a semisimple Lie group of the noncompact type. Let $\Phi : \pi_1(X) \to G$ be a discrete Zariski-dense representation. Then, there exists a finite unramified covering X' of X and a modification $\hat{X} \to X'$ of X', a non-singular projective-algebraic variety Z of the general type, a surjective holomorphic map $\sigma : \hat{X} \to Z$ with connected fibers and a representation $\Xi : \pi_1(Z) \to G$ such that $\Phi = \Xi \circ \sigma_*$ on $\pi_1(X')$, where $\sigma_* : \pi_1(\hat{X}) \to \pi_1(Z)$ is induced by σ and $\pi_1(\hat{X})$ is canonically identified with $\pi_1(X')$.*

Proof of Lemma 1. By Moišezon's Theorem a compact Kähler manifold X is projective-algebraic if and only of it is Moišezon. This is in particular the case if X is of the general type. For X satisfying the hypothesis of the Main Theorem using the Factorization Theorem it suffices to show that $\sigma : \hat{X} \to Z$ has generically zero-dimensional fibers. Otherwise let $E_0 \subset \hat{X}$ be a generic fiber, of dimension $m > 0$, so that $\sigma(E_0)$ is a point b while the image E of E_0 on X under the composite map $\hat{X} \to X' \to X$ is still of dimension m. Let $h : Z \to N$ be a harmonic map inducing $\Xi : \pi_1(Z) \to \pi_1(N) = \Gamma$ on fundamental groups. Denote by $\hat{f} : \hat{X} \to N$ the lifting of $f : X \to N$ to $\hat{X}$. We may choose h such that $\hat{f} = h \circ \sigma$, so that $\hat{f}(E_0) = h(\sigma(E_0)) = h(b)$ is a point. As $\hat{f}(E_0) = f(E)$ the (m, m)-cycle $E \subset X$ is collapsed to a point under f. We claim that the image of $H_2(E, \mathbb{R})$ in $H_2(X, \mathbb{R})$ is non-zero. To see this let $\hat{E} \to E$ be a desingularization of E. It suffices to observe that for the composite map $\nu : \hat{E} \to X$, $\nu_*(H_2(\hat{E}, \mathbb{R})) \neq 0$, which follows from duality and the fact that $\nu^* \alpha \neq 0$ for any Kähler class $\alpha \in H^2(X, \mathbb{R})$. As a consequence, $\mathrm{Ker}(f_* : H_2(X, \mathbb{R}) \mapsto H_2(N, \mathbb{R}))$ is non-zero, contradicting the hypothesis of the Main Theorem.

From Lemma 1 we deduce that Proposition 1 is a consequence of Proposition 2, the proof of which will be given in the next section.

(2.3) It remains to prove Proposition 2. The only thing yet unproved is the last statement asserting the existence of $\xi \in H_{2\ell}(S, \mathbb{R})$ such that $g_* \xi = 0$. We adopt now the notations of Proposition 2 and consider a harmonic map $g: S \to N$, where S is a compact Kähler manifold and $N = \Gamma \backslash G/K$. (In other words, we replace S as N in Proposition 1′ by a desingularization and denote the lifted map by g and the lifted foliation by $\mathscr{G}$.)

In the case where $\mathscr{G}$ is holomorphic, the measured foliated cycle Θ to be constructed is equivalently (by Poincaré duality) a closed $(m-\ell, m-\ell)$-current which can be defined locally as follows. Let $U \cong D \times D'$ be a distinguished polycylinder on S such that $\{z\} \times D'$ corresponds to a leaf of the holomorphic foliation $\mathscr{G}|_U$, so that D is a parameter space for the space of local leaves. The semi-Kähler form ω corresponds to a closed (1,1)-form $\bar{\omega}$ on D which is almost everywhere positive definite. We write $\bar{\theta}$ for the corresponding generically non-degenerate metric on D, and $S_t^0 := g^{-1}(t)$; $t \in g(S)$; for a level set of g. By Sard's Theorem we may choose t such that S_t^0 is smooth and such that g is a submersion onto its image on a neighborhood of S_t^0. Denote by S_t any connected component of such a choice of S_t^0. As S_t is saturated with respect to the foliation $\mathscr{G}$, we can identify $U \cap S_t$ with $\Sigma \times D'$ for some real submanifold $\Sigma \subset D$. Perturbing t slightly if necessary we may furthermore assume that for any distinguished polycylinder $U \cong D \times D' \subset S - V$ adapted to $\mathscr{G}$, the restriction $\bar{\theta}|_\Sigma$ is almost everywhere non-degenerate so that its volume form defines a non-trivial positive measure on Σ which we denote symbolically by $d\lambda$. The definition of a current is local. The closed positive $(m-\ell, m-\ell)$-current Θ can be defined as follows. Let η be a smooth (ℓ, ℓ)-form on U with compact support. Then, by definition

$$(*) \qquad \Theta(\eta) = \int_\Sigma \left(\int_{L_z} \eta \right) d\lambda(z),$$

where L_z denotes the leaf corresponding to $\{z\} \times D'$. It is clear that this definition does not depend on the choice of U in the sense that for any two overlapping distinguished polycylinders U and U' and for any $U'' \subset U \cap U'$, the currents defined on U and U' restrict to the same current on U''. When $\mathscr{G}$ is holomorphic, this defines a closed positive $(m-\ell, m-\ell)$-current Θ with support on the smooth level set S_t of g. In the general case there exists a complex-analytic subvariety $V \subset S$ of complex codimension at least 2 such that $\mathscr{G}$ is holomorphic on $S - V$, for a generic choice of S_t we can certainly assume that $S_t \cap (S - V)$ is non-empty and that as a consequence by the same definition $(*)$ we have defined a non-trivial closed positive $(m-\ell, m-\ell)$-current on $S - V$, which we will continue to denote by Θ. The problem is therefore to prove that Θ can be extended trivially as a closed positive current on S. If that can be done, it is immediate that the trivially extended current $\tilde{\Theta}$ represents by Poincaré duality a non-zero class ξ in $H_{2\ell}(S, \mathbb{R})$ which is mapped to zero under g. (This is the case because of the obvious fact that the pull-back of an (ℓ, ℓ)-form is zero on S_t since g maps S_t into a point, and because we are taking the trivial extension $\tilde{\Theta}$.) We remark that the fact that V is of complex codimension at least 2 is not sufficient to imply the extendibility of Θ since ℓ is arbitrary. In place of Remmert-Stein-type extension theorems we need a Bishop-type extension theorem, which in the case of closed positive currents is furnished by the following theorem of Skoda [Sk].

Theorem (Skoda [Sk]). *Let U be a domain in $\mathbb{C}^m$ and $E \subset U$ be a complex-analytic subvariety. Let T be a closed positive (p,p)-current on U for some p, $1 \leqq p \leqq m$. Suppose T is of finite total mass on U with respect to the Euclidean Kähler form β, i.e.,*

$$\int_{U-E} T \wedge \beta^{m-p} < \infty .$$

Then, T can be trivially extended as a closed positive (p,p)-current on U.

Recall that S is a projective-algebraic manifold and in particular Kähler. Denote by μ some Kähler form on S. To apply Skoda's Theorem it suffices to show that

$$(\#) \qquad \int_{S-V} \Theta \wedge \mu^{m-p} < \infty .$$

From the definition of Θ this follows readily from the following lemma.

Lemma 2. *We use notations as in the preceding paragraph. Denote by χ the Kähler metric on S whose Kähler form is $\mu + \omega$. Denote by dv the positive measure on the smooth level set S_t defined by the restriction $\chi|_{S_t}$ of χ to S_t. Let $U \cong D \times D'$ be a distinguished polycylinder on S adapted to $\mathscr{G}$. Then,*

$$\int_{\Sigma} \left(\int_{L_z} \frac{\mu^{\ell}}{\ell!} \right) d\lambda(z) < \int_{S_t \cap U} dv .$$

Proof. The proof of Lemma 2 is analogous to that of Mok [M1], Lemma 4, p. 573, where the statement, with some obvious modification, already applies to the situation when S_t is assumed to be a complex submanifold. The proof for the general case where S_t and Σ are only smooth manifolds is similar, except for the fact that the volume forms involved are no longer exterior powers of Kähler forms. Applying Fubini's Theorem the lemma reduces readily to proving the following statement in linear algebra:

(S) Let P, Q be real r-by-r symmetric matrices such that P is positive definite and Q is positive semi-definite of rank $t < r$, and P, Q are of the form

$$P = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}, \quad Q = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

where D is positive definite. Then

$$\det(P+Q) = \det \begin{bmatrix} A+D & B \\ B^t & C \end{bmatrix} > \det(D) \cdot \det(C) .$$

We proceed to prove Statement (S). Without loss of generality we may replace D by a diagonal matrix with diagonal entries $a_1, \ldots, a_t$. Write $\{e_1, \ldots, e_r\}$ for the canonical basis of $\mathbb{R}^r$ and write $\varepsilon = e_1 \wedge \cdots \wedge e_r$. Write $\mu_1, \ldots, \mu_r$ for the column vectors of P and $v_1, \ldots, v_t, 0, \ldots, 0$ for the column vectors of Q. We have

$$\mu_1 \wedge \cdots \wedge \mu_r = \det(P) \cdot \varepsilon, \quad \text{etc.},$$

so that

$$\det(P+Q)\cdot\varepsilon = (\mu_1+\nu_1)\wedge\cdots\wedge(\mu_t+\nu_t)\wedge\mu_{t+1}\wedge\cdots\wedge\mu_r$$
$$=\nu_1\wedge\cdots\wedge\nu_t\wedge\mu_{t+1}\wedge\cdots\wedge\mu_r+\mu_1\wedge\cdots\wedge\mu_t\wedge\mu_{t+1}\wedge\cdots\wedge\mu_r+\Sigma\,,$$

where Σ is the sum of a finite number of exterior products, each of which is obtained from $\mu_1\wedge\cdots\wedge\mu_t\wedge\mu_{t+1}\wedge\cdots\wedge\mu_r$ by replacing s of the first t column vectors μ_i, $1\leqq i\leqq t$, by ν_i for some s, $1\leqq s<t$. Any such exterior product in Σ is then of the form $\det(R)\cdot\varepsilon$ where R is obtained from P by replacing s of the first t column vectors of P by those of Q. Since we assume that D is a diagonal matrix with diagonal entries $a_1,\ldots,a_t$; $a_i>0$; it is clear that $\det(R)=a_{i(1)}\cdots a_{i(s)}\det(T)$ for some positive definite matrix T. From this it follows readily that

$$\det(P+Q)>\det(D)\cdot\det(C)+\det(P)>\det(D)\cdot\det(C)\,,$$

proving Statement (S).

Lemma 2 follows readily from Statement (S), applied pointwise to the integrands.

We proceed to complete the proof of Proposition 2. Since Θ is an integral of currents defined by the leaves of the holomorphic foliation $\mathscr{G}|_{S-V}$, its mass with respect to the background Kähler metric $\mu+\omega$ on S is locally the integral of the corresponding masses of the leaves L_z with respect to the transverse measure $d\lambda(z)$. It follows from Lemma 2 that the total mass is finite and that by the Theorem of Skoda cited above the trivial extension $\tilde{\Theta}$ of Θ from $S-V$ to S exists and defines a non-zero closed positive current. $\tilde{\Theta}$ defines a non-zero cohomology class in $H^{2(m-\ell)}(S,\mathbb{R})$ and by Poincaré duality a non-zero homology class $\xi\in H_{2\ell}(S,\mathbb{R})$. To show that $g_*\xi$ is zero it is equivalent to show that the direct image under g of $\tilde{\Theta}$ as a current is zero. We have $g_*(\tilde{\Theta})(\sigma)=\tilde{\Theta}(g^*\sigma)$. Let $U\cong D\times D'$ be a distinguished polycylinder adapted to $\mathscr{G}$ and $\varrho\geqq 0$ be a smooth function on U with compact support. Since the leaves of $\mathscr{G}|_{S-V}$ are mapped to a point and $\tilde{\Theta}$ is the trivial extension of Θ, we have

$$\tilde{\Theta}(\varrho\cdot g^*\sigma)=\int_{\Sigma}\left(\int_{L_z}g^*\sigma\right)d\lambda(z)=0$$

as $g^*\sigma|_{L_z}\equiv 0$. By an argument using partition of unity it follows that $g_*(\tilde{\Theta})=\tilde{\Theta}(g^*\sigma)=0$. Equivalently $g_*\xi=0$ and the proof of Proposition 2 is completed.

In the next section we will use the decomposition of $\tilde{X}$ as given in Proposition 1 to prove that $\tilde{X}$ is Stein. The proof will rely on Narasimhan's generalization [N 1] to complex spaces of Grauert's solution to the Levi problem [G], together with Siu's theorem [S 1] that a Stein subvariety of a complex space always admits a Stein neighborhood.

§ 3. Solution of the Levi problem on $\tilde{X}$ with the weakly plurisubharmonic exhaustion function φ

(3.1) We recall first of all Grauert's solution to the Levi problem by means of the bumping technique. We have

Theorem (Grauert [G]). *Let Z be a complex manifold on which there is a twice continuously differentiable strictly plurisubharmonic exhaustion function ψ. Then, Z is a Stein manifold.*

To prove the Steinness of $\tilde{X}$ in the Main Theorem, we will make use of the (in general weakly) plurisubharmonic exhaustion function φ of Proposition 1 and the decomposition given there. We will proceed by induction. Since the complex-analytic subvarieties $\tilde{S}_1 \subset \tilde{S}_2 \subset \cdots \subset \tilde{S}_p = \tilde{X}$ are possibly singular for $\tilde{S}_i$ with $i < p$, we will need the generalization to complex spaces of Grauert's solution to the Levi problem, as given in Narasimhan [N1]. For the statement we need the notion of strictly plurisubharmonic functions on a complex space. Let U be a domain in some $\mathbb{C}^N$ and $A \subset U$ be a complex subvariety. Let $\psi : A \to [-\infty, \infty)$ be an upper semi-continuous function on A. We say that ψ is plurisubharmonic on A if and only if for any $x \in A$ there exists a neighborhood U_x of x in U such that $\psi = \tilde{\psi}|_{A \cap U_x}$ on $A \cap U_x$ for some continuous plurisubharmonic function $\tilde{\psi}$ on $A \cap U_x$. ψ is said to be strictly plurisubharmonic if for any $x \in A$ we can choose U_x and $\tilde{\psi}$ such that $\tilde{\psi} - \varepsilon \sum_{i=1}^{N} |z_i|^2$ is plurisubharmonic on U_x, where $(z_1, \ldots, z_N)$ are the Euclidean coordinates of $\mathbb{C}^N$. There is an *a priori* weaker notion of "weakly plurisubharmonic functions" on complex spaces, where an upper semi-continuous function $\psi : A \to [-\infty, \infty)$ is said to be "weakly plurisubharmonic" if and only if for any holomorphic map f of the unit disk Δ into A, $\psi \circ f$ is either subharmonic or identically $-\infty$. In Fornaess-Narasimhan [FN], §5 it was proven that this notion of "weakly plurisubharmonic" functions on complex spaces is equivalent to the former notion of plurisubharmonic functions. We have

Theorem (Narasimhan [N1]). *Let Z be a complex space on which there is a continuous strictly plurisubharmonic exhaustion function ψ. Then, Z is a Stein space.*

A related result from Narasimhan [N1] that we will need is

Proposition (Narasimhan [N1]). *Let Z be a Stein space and η be a plurisubharmonic function on X. Then, for any real number c the open subset $Z_c = \{x \in Z : \eta(x) < c\} \subset Z$ is Runge in Z.*

For our proof of the Steinness of $\tilde{X}$ in the Main Theorem we start with the weakly plurisubharmonic (real-analytic) exhaustion function φ on $\tilde{X}$ as in (2.5), Proposition 3. We are going to construct strictly plurisubharmonic exhaustion functions on

$$\tilde{X}_c = \{x \in \tilde{X} : \varphi(x) < c\}$$

by modifying $\dfrac{1}{c - \varphi}$ by an inductive process using the decomposition of $\tilde{X}$. Together with the proposition above on Rungeness we will be able to conclude the Steinness of $\tilde{X}$. For the modification of $\dfrac{1}{c - \varphi}$ we need the following theorem of Siu:

Theorem (Siu [S1]). *Let Y be a complex space and $A \subset Y$ be a Stein subvariety. Then, there exists a Stein neighborhood V of A in Y.*

In the modification of weakly plurisubharmonic functions $\frac{1}{c-\varphi}$ on $\tilde{X}_c \cap \tilde{S}_i$, $1 \leqq i \leqq p$, we will actually be using restrictions of strictly plurisubharmonic functions defined on tubular neighborhoods of $\tilde{X}_c \cap \tilde{S}_i$ in $\tilde{X}_c$. For this reason the plurisubharmonic functions to be constructed on $\tilde{X}_c \cap \tilde{S}_{i+1}$ will actually be plurisubharmonic in the stronger sense so that the result of Narasimhan [N1] in the original form will be sufficient for our purpose.

(3.2) To prove the Steinness of $\tilde{X}$ in the Main Theorem we will proceed to prove inductively the statements

$(*)_i$ The complex-analytic subvariety $\tilde{S}_i \subset \tilde{X}$ is a Stein space.

Statement $(*)_1$ is valid since $\tilde{S}_1$ is non-singular and φ is strictly plurisubharmonic on $\tilde{S}_1$ by definition. To prove $(*)_i \Rightarrow (*)_{i+1}$, $1 \leqq i \leqq p-1$, we introduce for $1 \leqq j \leqq p$ and for any $c \in \mathbb{R}$ the statement

$(*)_{j,c}$ The complex-analytic subvariety $\tilde{S}_j \cap \tilde{X}_c \subset \tilde{X}_c$ is a Stein space.

By the proposition of Narasimhan [N1] on Rungeness cited in (3.1) we know that $(*)_{j,c}$ implies that $\tilde{S}_j \cap \tilde{X}_{c'}$ is Runge in $\tilde{S}_j \cap \tilde{X}_c$ whenever $c' < c$. It follows readily that Statement $(*)_{j,c}$ for a fixed j and for all $c \in \mathbb{R}$ implies Statement $(*)_j$. (Actually, it is sufficient to know the validity of $(*)_{j,c_k}$ for an increasing and diverging sequence of c_k.) Thus, to prove the Steinness of $\tilde{X}$ it suffices to show $(*)_i \Rightarrow (*)_{i+1,c}$ for any $c \in \mathbb{R}$.

Since $\tilde{S}_i$ is Stein by the induction hypothesis there is a Stein neighborhood U of $\tilde{S}_i$ on $\tilde{X}$, according to the theorem of Siu [S1] cited in (3.1). Let $\varrho \geqq 0$ be a smooth cut-off function on $\tilde{X}$ such that $\varrho \equiv 1$ on some neighborhood of $\tilde{S}_i$ on $\tilde{X}$ and such that $\mathrm{Supp}(\varrho) \subset U$ with $\mathrm{Supp}(\varrho) \cap \tilde{X}_c$ relatively compact in U for any $c \in \mathbb{R}$. Since U is Stein there exists on U a smooth strictly plurisubharmonic function ξ. The function $\varrho\xi$ defined on U can then be extended trivially to a smooth function on $\tilde{X}$.

Fix any $c \in \mathbb{R}$. On $\tilde{S}_{i+1}$, φ is strictly plurisubharmonic wherever $\varrho\xi$ fails to be strictly plurisubharmonic. Since $\tilde{X}_c \subset\subset \tilde{X}$ there exists a positive constant K_c such that $K_c\varphi + \varrho\xi$ is strictly plurisubharmonic on $\tilde{S}_{i+1} \cap \tilde{X}_c$. The continuous function $\frac{1}{c-\varphi} + K_c\varphi + \varrho\xi$ is then strictly plurisubharmonic on $\tilde{S}_{i+1}$ and is an exhaustion function since $K_c\varphi + \varrho\xi$ is bounded. It follows from the theorem of Narasimhan cited in (3.1) that $\tilde{S}_{i+1} \cap \tilde{X}_c$ is Stein, proving $(*)_i \Rightarrow (*)_{i+1,c}$ for all $c \in \mathbb{R}$. With this implication we have established by induction the Steinness of $\tilde{X}$, proving the Main Theorem.

(3.3) In the case of 2 dimensions our method yields only a weakened version of the analogue of the Main Theorem. On the other hand, the result applies to a more general setting, as follows.

Theorem 1. *Let X be a compact surface and G be a semisimple real Lie group of the noncompact type. Suppose there exists a discrete Zariski-dense representation $\Phi: \pi_1(X) \to G$. Let $X^* \to X$ be the regular covering of X corresponding to* Ker Φ. *Then X^* is holomorphically convex.*

Remarks. (i) In the event where $\operatorname{Ker}\Phi$ is finite Theorem 2 asserts that $\tilde{X}$ is itself holomorphically convex, since there is a canonical finite covering $\tilde{X} \to X^{\#}$. In this special case Theorem 2 was also proved by Napier-Ramachandran [N-R], consequence of Theorem 4.8. Their proof relied among other things on: (a) the result [N-R], Theorem 1 asserting in this case that $X^{\#}$ has exactly one end, (b) the existence on $X^{\#}$ of a real-analytic plurisubharmonic exhaustion function and (c) a factorization theorem over Riemann surfaces of Gromov [Gr].

(ii) For the case where $\Phi : \pi_1(X) \to G$ arises from the hypothesis in the Main Theorem (except for the fact that X_0 and X are of complex dimension 2) we have $X^{\#} = \tilde{X}$. It will follow readily from the proof that the Stein reduction $\tilde{X}_*$ of $\tilde{X}$ is of complex dimension 2. In this case either $\tilde{X}$ is itself Stein or $\tilde{X}_*$ is obtained by blowing down a non-trivial Γ-equivariant complex-analytic subvariety E of $\tilde{X}$ of dimension 1. E is necessarily a countable union of compact complex-analytic subvarieties.

Proof. By the Factorization Theorem there exists a finite unramified cover X' of X and a modification $\hat{X}$ of X', such that the representation Φ, restricted to $\pi_1(\hat{X}) \cong \pi_1(X')$, of finite index in $\pi_1(X)$, factors through some $\Xi : \pi_1(Z) \to G$ via a surjective holomorphic map $\sigma : \hat{X} \to Z$ with connected fibers. Z is either a compact Riemann surface or a projective-algebraic surface of the general type.

In the case Z is a Riemann surface, let $\hat{X}^{\#} \to \hat{X}$ be the regular covering of $\hat{X}$ corresponding to $\operatorname{Ker}\sigma_*$. Then $\sigma : \hat{X} \to Z$ lifts to a proper holomorphic map $\sigma^{\#} : X^{\#} \to \hat{Z}$, where $\tilde{Z}$ is the universal cover of the compact Riemann surface Z of genus $\geqq 2$ and is in particular Stein. Since $\hat{X}^{\#}$ is obtained from some $X'^{\#}$ by modification, where there is a canonical finite covering $X^{\#} \to X'^{\#}$, it follows readily that $X^{\#}$ is also holomorphically convex.

It remains to consider the case where the base space Z of $\theta : \hat{X} \to Z$ is of complex dimension 2. There is a subgroup $\Gamma' \subset \Gamma := \pi_1(X)$ of finite index and a harmonic map $f: X' \to \Phi(\Gamma')\backslash G/K$, representing $\Phi|_{\Gamma'}$, where $X' \to X$ is a finite unramified covering corresponding to Γ', $K \subset G$ is a maximal compact subgroup and $\Phi(\Gamma') \subset G$ is a torsion-free discrete subgroup. We lift f to $f^{\#} : X'^{\#} \to G/K$ where $X'^{\#} \to X'$ is a regular covering corresponding to $\operatorname{Ker}(\Phi|_{\Gamma'})$. $f^{\#}$ is pluriharmonic by the $\partial\bar{\partial}$-Bochner-Kodaira formula of Siu and Sampson. $f^{\#}$ is by definition proper. Write $\varphi(x^{\#})$ for $r^2(0; f^{\#}(x^{\#}))$, defined on $X'^{\#}$, as in (2.1), using now a canonical metric on the Riemannian symmetric manifold G/K. It follows from the properness of $f^{\#}$ that $\varphi : X^{\#} \to \mathbb{R}$ is an exhaustion function. From this as in Napier-Ramachandran [NR], Theorem (4.8), one may apply a result of Diederich-Ohsawa [DO] which implies that a complex surface with a real-analytic plurisubharmonic exhaustion function φ is holomorphically convex provided that φ is strictly plurisubharmonic at one point. Alternatively, one can also adapt the argument of § 3 to conclude that $X^{\#}$ is holomorphically convex, as follows.

The plurisubharmonic exhaustion function φ on $X'^{\#}$ can only fail to be strictly plurisubharmonic on the union of a discrete point set and a complex pure one-dimensional subvariety $E \subset X'^{\#}$. Modifying φ in small neighborhoods of the discrete set of points if necessary we may assume that φ is strictly plurisubharmonic on $X'^{\#} - E$, although the new φ is only smooth and not real-analytic in general. Decompose E into irreducible components. From the fact that φ is an exhaustion function any chain of compact irreducible components of E must be finite. For generic choices of regular values $c' < c$ of φ, $E \cap \{x^\# \in X'^\#: c' < \varphi(x^\#) < c\} := E_{c',c}$ is a Riemann surface without compact irreducible components. Hence $E_{i,c}$ is Stein. By applying the theorem of Siu in (3.1) to $E_{c',c}$ for all such choices of c', c it follows readily that for a generic choice d of regular value of φ, the domain $X_d'^\# = \{x^\# \in X'^\#: \varphi(x^\#) < d\}$ with smooth boundary is a strongly pseudoconvex manifold, from which it follows that $X_d'^\#$ is holomorphically convex. The holomorphic convexity of $X'^\#$ then follows by passing to Stein reductions and by applying the proposition on Rungeness (of Narasimhan [N1]) quoted in (3.1).

Finally, to complete the proof of Theorem 2 it suffices to observe that there is a canonical finite unramified covering $X'^\# \to X^\#$. The holomorphic convexity of $X^\#$ follows readily from the holomorphic convexity of $X'^\#$. The proof of Theorem 2 is complete.

Remarks (on the proof). There is an advantage in the alternative argument proving holomorphic convexity of $X'^\#$ in the second last paragraph. We do not need to use the fact that the original exhaustion function φ is real-analytic but only the fact that the subset on which φ fails to be strictly plurisubharmonic is a complex-analytic subvariety. With this observation the argument generalizes readily to geometrically reductive (in the sense of Labourie [La]) representations of $\pi_1(X)$ on the isometry group of complete Riemannian manifolds with smooth metrics of nonpositive sectional curvature in the *complexified* sense.

Theorem 1, as it stands, is a consequence of the Bochner-Kodaira formula of Siu-Sampson and classical solutions to the Levi problem. The more interesting and difficult problem is to prove holomorphic convexity of $\tilde{X}$ (and not of the intermediate covering space $X^\#$) under the hypothesis of Theorem 1.

For a general discussion on the Shafarevich Conjecture, we refer the reader to Narasimhan [N2]. For the resolution of the Shafarevich Conjecture in special cases and related results, see Napier [Na] and Napier-Ramachandran [NR].

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Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong
e-mail: nmok@hkucc.hku.hk

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