unstructured perturbations is not true when the perturbations are block structured. For a limited class of problems, quadratic stability in the face of structured complex perturbations is equivalent to scaled $\|\cdot\|_\infty$ norms, and hence $\|\cdot\|_\infty$ synthesis techniques, coupled with diagonal constant scalings, can be used to design quadratically stable systems.

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On the Norms Used in Computing the Structured Singular Value

NAM-KIU TSING

Abstract—**Different norms are considered to replace the Euclidean norm in an algorithm given by Fan and Tits (*IEEE Trans. Automat. Contr.*, vol. 33, pp. 284–289, 1988) which is used for the computation of the structured singular value of any matrix. It is shown that the l_1 norm is the best possible norm in a certain sense.**

Recently, there has been a considerable amount of interest in the study of the structured singular value, the concept of which was originated by Doyle [1] and is used as a tool for the analysis and synthesis of feedback systems with structured uncertainties (e.g., see [2], [3] and their references).

Let M be an $n \times n$ complex matrix, and $\mathcal{K} = (k_1, \cdots, k_m)$ an m tuple of positive integers which satisfies $\sum_{i=1}^m k_i = n$. For $i = 1, \cdots, m$, denote the ith-block-projection matrix by $P_i =$ block diag $(O_{k_1}, \cdots, O_{k_{i-1}}, I_{k_i}, O_{k_{i+1}}, \cdots, O_{k_m})$ where O_k and I_k are the zero matrix and identity matrix, respectively, of order $k \times k$ for any positive integer k. Then the structured singular value of M with respect to the block structure $\mathcal{K}$ is the nonnegative scalar

$$\mu(M) = \max_{x \in \partial B} \{\|Mx\| : \|P_i x\| \, \|Mx\| = \|P_i Mx\|, i = 1, \cdots, m\}$$

where $\|\cdot\|$ denotes the Euclidean (l_2) norm in $\mathcal{C}^n$ and ∂B the corresponding unit sphere. One major issue in the study of $\mu(M)$ is the computation of it. In their paper [2], the authors devise an algorithm [2, Algorithm 1], which we shall explain immediately, to compute $\mu(M)$. They first define, for any real number α, the Hermitian matrices

$$A_i(\alpha) = \alpha P_i - M^H P_i M, \qquad \text{for } i = 1, \cdots, m$$

and the m-form numerical range associated with $A_1(\alpha), \cdots, A_m(\alpha)$:

$$W(\alpha) = \{(v_1, \cdots, v_m) \in \Re^m : \exists x \in \partial B \text{ such that } v_i = x^H A_i(\alpha) x \text{ for all } i\}.$$

A function $c(\cdot) : \Re \to \Re$, which depends on M and $\mathcal{K}$, is then defined by

$$c(\alpha) = \min \{\|v\| : v \in W(\alpha)\}$$

where $\|\cdot\|$ again denotes the Euclidean norm (on $\Re^m$ this time). Then they show [2, Corollary 1 and Proposition 1] that, for any matrix M with structure $\mathcal{K}$, $c(\cdot)$ satisfies

$$c(\cdot) \text{ is continuous} \tag{1}$$

$$c(\mu^2) = 0, \; c(\alpha) > 0 \qquad \text{for all } \alpha > \mu^2 \tag{2}$$

and

$$c(\alpha + s) \le c(\alpha) + s \qquad \text{for all } s \ge 0 \text{ and real } \alpha \tag{3}$$

where $\mu = \mu(M)$. The initial step in [2, Algorithm 1] is to set $\alpha_0 = \bar{\sigma}^2(M)$ where $\bar{\sigma}(M)$ is the largest singular value of M. The iteration step is to set $\alpha_{k+1} = \alpha_k - c(\alpha_k)$ for $k = 0, 1, 2, \cdots$. The authors show in [2, Theorem 2] that, since $c(\cdot)$ satisfies (1)–(3), Algorithm 1 will generate a monotonic decreasing sequence $\{\alpha_k\}$ with $\lim_{k \to \infty} \alpha_k = \mu^2$. The structured singular value μ can thus be obtained. They also remark in the footnote that the Euclidean norm in the definition of $c(\cdot)$ can be replaced by the l_1 norm to get the strongest version of Proposition 1. The purpose of the present note is to elaborate on this remark.

Suppose we consider any norm $N(\cdot)$ on $\Re^m$ instead of the Euclidean norm. Similar to the case of $c(\cdot)$, we may define a function $c_N : \Re \to \Re$ (which also depends on M and $\mathcal{K}$) by

$$c_N(\alpha) = \min \{N(v) : v \in W(\alpha)\}.$$

It is not hard to see that $c_N(\cdot)$ always satisfies (1) and (2); and if in addition $c_N(\cdot)$ satisfies (3) also, then the function $c(\cdot)$ in Algorithm 1 can be replaced by $c_N(\cdot)$. Let

$$\mathfrak{N} = \{N(\cdot) : N(\cdot) \text{ is a norm on } \Re^m \text{ and } c_N(\cdot) \text{ satisfies condition (3) for all matrices } M \text{ with structure } \mathcal{K}\}.$$

Then Algorithm 1 will work with $c(\cdot)$ being replaced by any $c_N(\cdot)$ where $N(\cdot) \in \mathfrak{N}$. In view of the iteration step of the algorithm, we may want to choose a norm $N_0(\cdot) \in \mathfrak{N}$ such that

$$c_{N_0}(\alpha) \ge c_N(\alpha) \qquad \text{for all } N(\cdot) \in \mathfrak{N} \text{ and } \alpha > \mu^2$$

so that the resulting algorithm has the fastest convergent rate and is thus the most efficient. The following result shows that $\|\cdot\|_1$, i.e., the l_1 norm on $\Re^m$ defined by

$$\|v\|_1 = \sum_{i=1}^m |v_i| \qquad \text{for all } v = (v_1, \cdots, v_m) \in \Re^m,$$

will give such a "best possible" norm.

Theorem: Let $\mathcal{K} = (k_1, \cdots, k_m)$ be a given block structure. Then

a) $\|\cdot\|_1 \in \mathfrak{N}$, and

b) for any $N(\cdot) \in \mathfrak{N}$, $\|v\|_1 \ge N(v)$ for all $v \in \Re^m$, so that

$$c_1(\alpha) := \min \{\|v\|_1 : v \in W(\alpha)\} \ge c_N(\alpha)$$

for any real α and complex matrix M with block structure $\mathcal{K}$.

Proof:

a) The proof of the fact that $c_1(\cdot)$ satisfies (3) is similar to (and

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simpler than) that of $c(\cdot)$ given in the proof of [2, Proposition 1]. Hence, $\|\cdot\|_1 \in \mathfrak{N}$.

b) We prove by contradiction. Suppose $N(\cdot) \in \mathfrak{N}$, but $N(v) > \|v\|_1$ for some $v \in \mathbb{R}^m$. Let $\{e_1, \cdots, e_m\}$ be the standard basis for $\mathbb{R}^m$. Without loss of generality, we may assume

$$N(e_1) = 1 + q > \|e_1\|_1 = 1;$$

for if $N(e_i) \le \|e_i\|_1$ for all $i = 1, \cdots, m$, then

$$N(v) = N\left(\sum_{i=1}^m v_i e_i\right) \le \sum_{i=1}^m |v_i| N(e_i) \le \sum_{i=1}^m |v_i| \, \|e_i\|_1 = \|v\|_1$$

for all $v = (v_1, \cdots, v_m) \in \mathbb{R}^m$, which contradicts the assumption. Let

$$L = \{v \in \mathbb{R}^m : v_i + w_2 v_2 + \cdots + w_m v_m = 1\}$$

be a supporting hyperplane to the ball in $\mathbb{R}^m$ with respect to $N(\cdot)$ with radius $(1+q)$. As e_1 is on L, we have

$$\min_{v \in L} N(v) = N(e_1) = 1 + q.$$

Note that some of the w_i's may be zero. However, for any $\epsilon > 0$, we can choose nonzero $w_2', \cdots, w_m' \in \mathbb{R}$ which are arbitrary close to $w_2, \cdots, w_m$, respectively, such that if

$$L' = \{v \in \mathbb{R}^m : v_1 + w_2' v_2 + \cdots + w_m' v_m = 1\}$$

then

$$\min_{v \in L'} N(v) > N(e_1) - \epsilon.$$

Now choose $\lambda, \beta_1, \cdots, \beta_m \in \mathbb{R}$ such that $\lambda - \beta_1^2 = 1$ and $\lambda - \beta_i^2 = 1/w_i'$ for $i = 2, \cdots, m$. Then, by defining

$$M = \sum_{i=1}^m \beta_i P_i \tag{4}$$

(which depends on ϵ because the β_i's depend on ϵ), we have, for any real α,

$$\begin{aligned} A_i(\alpha) &= \alpha P_i - \left(\sum_{j=1}^m \beta_j P_j\right) P_i \left(\sum_{j=1}^m \beta_j P_j\right) \\ &= (\alpha - \beta_i^2) P_i \qquad \text{for } i = 1, \cdots, m \end{aligned}$$

and

$$\begin{aligned} c_N(\alpha) &= \min\{N(v) : v_i = x^H A_i(\alpha) x, \quad i = 1, \cdots, m, x \in \partial B\} \\ &= \min\{N(v) : v_i = (\alpha - \beta_i^2) a_i, \quad a_i \ge 0, i = 1, \cdots, m, \\ &\qquad a_1 + \cdots + a_m = 1\}. \end{aligned}$$

Hence,

$$\begin{aligned} c_N(\lambda - 1) &= \min\{N(v) : v_i = (\lambda - 1 - \beta_i^2) a_i, \quad a_i \ge 0, \\ &\qquad i = 1, \cdots, m, a_1 + \cdots + a_m = 1\} \\ &= \min\{N(v) : v_1 = 0, v_i = (\lambda - 1 - \beta_i^2) a_i, \quad a_i \ge 0, \\ &\qquad i = 2, \cdots, m, a_1 \ge 0, a_1 + \cdots + a_m = 1\}. \end{aligned}$$

By putting $(a_1, \cdots, a_m) = e_1$, we get

$$c_N(\lambda - 1) = N(0) = 0.$$

Also,

$$\begin{aligned} c_N(\lambda) &= \min\{N(v) : v_i = (\lambda - \beta_i^2) a_i, \quad a_i \ge 0, i = 1, \cdots, m, \\ &\qquad a_1 + \cdots + a_m = 1\} \\ &= \min\{N(v) : v_1 = a_1 \ge 0, v_i = a_i / w_i', \quad a_i \ge 0, \\ &\qquad i = 2, \cdots, m, a_1 + \cdots + a_m = 1\} \\ &= \min\{N(v) : v_1 + w_2' v_2 + \cdots + w_m' v_m = 1, \quad v_1 \ge 0, \\ &\qquad w_i' v_i \ge 0, i = 2, \cdots, m\} \\ &\ge \min\{N(v) : v \in L'\} \\ &> N(e_1) - \epsilon \\ &= (1 + q) - \epsilon \\ &= (1 + q) + c_N(\lambda - 1) - \epsilon. \end{aligned}$$

As $\epsilon > 0$ is arbitrary, we may choose $0 < \epsilon < q$ so that, for the matrix M defined in (4), we have

$$c_N((\lambda - 1) + 1) = c_N(\lambda) > c_N(\lambda - 1) + 1.$$

As a result, $c_N(\cdot)$ does not satisfy (3) for this matrix M, and hence, $N(\cdot) \notin \mathfrak{N}$. Thus, if $N(\cdot) \in \mathfrak{N}$, then we must have

$$\|v\|_1 \ge N(v) \qquad \text{for all } v \in \mathbb{R}^m,$$

and hence,

$$\begin{aligned} c_1(\alpha) &= \min\{\|v\|_1 : v \in W(\alpha)\} \\ &\ge \min\{N(v) : v \in W(\alpha)\} \\ &= c_N(\alpha) \qquad \text{for any real } \alpha. \end{aligned}$$

□

Now the problem remains to devise a method for computing $c_1(\alpha)$ so that the algorithm can be implemented. The computation of $c_1(\alpha)$ for any given M and $\mathcal{K}$ can be, in general, difficult. However, there are existing methods for computing the value

$$c_1'(\alpha) := \min\{\|v\|_1 : v \in co\, W(\alpha)\}$$

where $co\, W(\alpha)$ denotes the convex hull of $W(\alpha)$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^m$. Since the l_∞ norm ($\|\cdot\|_\infty$) is the dual norm of the l_1 norm, we have

$$\begin{aligned} c_1'(\alpha) &= \min_{v \in co\, W(\alpha)} \|v\|_1 \\ &= \min_{v \in co\, W(\alpha)} \max_{\substack{a \in \mathbb{R}^m \\ \|a\|_\infty \le 1}} \langle v, a \rangle. \end{aligned} \tag{5}$$

As $co\, W(\alpha)$ and $\{a \in \mathbb{R}^m : \|a\|_\infty \le 1\}$ are convex sets, and $\langle \cdot, \cdot \rangle$ is bilinear, the max and min in (5) can be interchanged to yield

$$\begin{aligned} c_1'(\alpha) &= \max_{\substack{a \in \mathbb{R}^m \\ \|a\|_\infty \le 1}} \min_{v \in co\, W(\alpha)} \langle v, a \rangle \\ &= \max_{|a_i| \le 1} \min_{x \in \partial B} \sum_{i=1}^m a_i x^H A_i(\alpha) x \\ &= \max_{|a_i| \le 1} \min_{x \in \partial B} x^H \left(\sum_{i=1}^m a_i A_i(\alpha)\right) x \\ &= \max_{|a_i| \le 1} \lambda_{\min} \left(\sum_{i=1}^m a_i A_i(\alpha)\right) \\ &= -\max_{|a_i| \le 1} \lambda_{\max} \left(\sum_{i=1}^m a_i A_i(\alpha)\right) \end{aligned} \tag{6}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote, respectively, the smallest and largest eigenvalue. Several existing algorithms are available for solving the convex problem (6). We refer to a recent paper by Boyd and Yang [4, Sect. 7] discussing the details, including the advantages and disadvantages, of these numerical algorithms. As a result, if $W(\alpha)$ is convex (which is always the case when $m \leq 3$; see [5], for example), $c_1(\alpha)(=c_1'(\alpha))$ could be computed with any of these algorithms. If $m \geq 4$, then $W(\alpha)$ may not be convex. In this case, replacing $c(\cdot)$ by $c_1'(\cdot)$ in [2, Algorithm 1] will result in a sequence $\{\sqrt{\alpha_k}\}$ with limit

$$\mu' = \inf\{\sqrt{\alpha} : c_1'(\beta) > 0 \quad \text{for all } \beta \geq \alpha\},$$

which is clearly an upper bound for $\mu(M)$. This is exactly the same situation as the case of $c(\cdot)$ and $c'(\cdot)$ discussed in [2].

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All Stabilizing Controllers as Frequency-Shaped State Estimate Feedback

JOHN B. MOORE, KEITH GLOVER, AND ANDREW TELFORD

Abstract—**This paper shows that the class of all strictly proper stabilizing controllers for proper linear plants can be structured as state estimate feedback with frequency shaping in the state estimates and/or in the state estimate feedback law. The selection of where the frequency shaping takes place is at the designer's discretion. The parameterization of the controller class can be in terms of an arbitrary proper stable transfer function, with the closed-loop system transfer functions affine in this transfer function. With constant output feedback permitted in addition to the state estimate feedback, the class of all proper stabilizing controllers can be generated in a like manner. The results of the paper are useful in engineering applications where the states represent physical variables.**

I. Introduction

Consider the stabilizable and detectable linear system with state equations

$$\dot{x} = Ax + Bu, \; y = Cx + Du \tag{1.1}$$

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and transfer function $G \in R_p$ (real rational proper) as

$$G = C(sI - A)^{-1}B + D. \tag{1.2}$$

Consider also a linear constant state estimate feedback controller

$$\dot{\hat{x}} = A\hat{x} + Bu - H(y - \hat{y}), \; \hat{y} = C\hat{x} + Du, \; u = F\hat{x} \tag{1.3}$$

with transfer function $K \in R_{sp}$ (real rational strictly proper)

$$K = -F[sI - (A + BF + HC + HDF)]^{-1}H. \tag{1.4}$$

The controller is known to be stabilizing if and only if

$$[sI - (A + BF)]^{-1}, [sI - (A + HC)]^{-1} \in RH^{\infty} \tag{1.5}$$

where RH^{∞} is the class of all real rational stable transfer functions.

In frequency-shaped estimation and control, the gains H, F are generalized as stabilizable and detectable filters with transfer functions $H, F \in R_p$. The resulting controllers are known to be stabilizing, with all states asymptotically stable for arbitrary initial conditions if and only if [1]

$$F, H \text{ stabilize } G_F, G_H, \text{ respectively}$$

$$G_F = (sI - A)^{-1}B, G_H = C(sI - A)^{-1}. \tag{1.6}$$

(The result is also derived as a byproduct of the theory of this paper.)

In this paper, we show that for the plant G of (1.1), (1.2), the class of all stabilizing controllers (1.3), parameterized in terms of $F, H \in RH^{\infty}$ satisfying (1.6), is the entire class of strictly proper stabilizing controllers for the system (plant) (1.1). Moreover, the entire class can be generated in terms of a stabilizing $F \in R_p$ for G_F where $H \in R_p$ is an arbitrary stabilizing minimum phase "controller" for G_H, and likewise, in terms of a stabilizing $H \in R_p$ for G_H where F is an arbitrary stabilizing minimum phase controller for G_F. With constant output feedback permitted in addition to the state estimate feedback, the class of all proper stabilizing controllers can be generated using a mild variation. Moreover, the parameterizations can be in terms of arbitrary transfer functions $Q_F, Q_H \in RH^{\infty}$, with the closed-loop transfer functions affine in Q_F, Q_H.

The theory on the class of all stabilizing controllers was introduced by Kučera [2] for discrete time and Youla *et al.* [3] for continuous time, and later formulated in an axiomatic framework [4], [5]. The results of this paper build on from, and complement, these characterizations, as do those involving modification to standard state estimate feedback in [1], [7].

The controller structures of this paper have the advantage that they are decomposed into a state estimator and state feedback law, where at the discretion of the designer, generally one or the other or both are frequency shaped. Thus, any stabilizing controller can be viewed in terms of filtered feedback of each state estimate or as direct feedback of each frequency-shaped state estimate. This has appeal in engineering situations where the states represent physical internal variables. For example, knowledge that an effective controller feeds back a low-pass filtered velocity or position estimate could be instructive when improving the design by introducing additional sensors or improving sensor locations. In situations where state estimation is required in addition to control, the results of this paper give useful implementation possibilities. Gain scheduling could be more systematic in the framework of state estimate feedback. This is not to say that state estimate feedback is the best design approach, as illustrated in cases when the frequency shaping in the state estimate feedback cancels out the observer dynamics.

In Section II, known theory [4], [7] for the class of all stabilizing controllers is reviewed and extended for use in subsequent sections. In Section III, the main results of the paper are developed. Some useful related results are summarized in Section IV, and conclusions are drawn in Section V.