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Some new nonlinear retarded sum-difference inequalities with applications

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Abstract

The main objective of this paper is to establish some new retarded nonlinear sum-difference inequalities with two independent variables, which provide explicit bounds on unknown functions. These inequalities given here can be used as handy tools in the study of boundary value problems in partial difference equations.

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1 Introduction

Being important tools in the study of differential, integral, and integro-differential equations, various generalizations of Gronwall inequality [1,2] and their applications have attracted great interests of many mathematicians (cf. [3-16], and the references cited therein). Recently, Agarwal et al. [3] studied the inequality

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t,s) w_i(u(s)) ds, \quad t_0 \le t < t_1.$$

Cheung [17] investigated the inequality

$$u^p(x,y) \le a + \frac{p}{p-q} \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} g_1(s,t) u^q(s,t) \mathrm{d}t \mathrm{d}s + \frac{p}{p-q} \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} g_2(s,t) u^q(s,t) \psi(u(s,t)) \mathrm{d}t \mathrm{d}s.$$

Agarwal et al. [18] obtained explicit bounds to the solutions of the following retarded integral inequalities:

$$\varphi(u(t)) \le c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi\left(u(s)\right) + g_i(s)]\mathrm{d}s,$$

$$\varphi(u(t)) \le c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi_1\left(u(s)\right) + g_i(s)\varphi_2\left(logu(s)\right)]\mathrm{d}s,$$

$$\varphi(u(t)) \le c + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)L\left(s, u(s)\right) + g_i(s)u(s)]\mathrm{d}s,$$

where c is a constant, and Chen et al. [19] did the same for the following inequalities:

$$\psi(u(x,y)) \le c + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} f(s,t)\varphi(u(s,t))\mathrm{d}t\mathrm{d}s,$$

$$\psi(u(x,y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} g(s,t)u(s,t)\mathrm{d}t\mathrm{d}s + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} f(s,t)u(s,t)\varphi(u(s,t))\mathrm{d}t\mathrm{d}s,$$

$$\psi(u(x,y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} g(s,t)w(u(s,t))\mathrm{d}t\mathrm{d}s + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} f(s,t)w(u(s,t))\varphi(u(s,t))\mathrm{d}t\mathrm{d}s,$$

where c is a constant.

Along with the development of the theory of integral inequalities and the theory of difference equations, more attentions are drawn to some discrete versions of Gronwall type inequalities (e.g., [20-22] for some early works). Some recent works can be found, e.g., in [6,23-25] and some references therein. Found in [26], the unknown function u in the fundamental form of sum-difference inequality

$$u(n) \le a(n) + \sum_{s=0}^{n-1} f(s)u(s)$$

can be estimated by $u(n) \le a(n)\prod_{s=0}^{n-1}(1+f(s))$. In [6], the inequality of two variables

$$u^2(m,n) \le c^2 + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u(s,t)w\left(u(s,t)\right)$$

was discussed, and the result was generalized in [23] to the inequality

$$u^p(m,n) \le c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^q(s,t)w\left(u(s,t)\right).$$

In this paper, motivated mainly by the works of Cheung [17,23], Agarwal et al. [3,18], and Chen et al. [19], we shall discuss upper bounds of the function $u(m, n)$ satisfying one of the following general sum-difference inequalities

$$\psi(u(m,n)) \le a(m,n) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} w\left(u(\alpha_i(s),\beta_i(t))\right)[f_i(s,t)\varphi\left(u(\alpha_i(s),\beta_i(t))\right) + g_i(s,t)], \tag{1.1}$$

$$\psi(u(m,n)) \le a(m,n) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} w\left(u(\alpha_i(s),\beta_i(t))\right)[f_i(s,t)\varphi_1\left(u(\alpha_i(s),\beta_i(t))\right) + g_i(s,t)\varphi_2\left(logu(\alpha_i(s),\beta_i(t))\right)], \tag{1.2}$$

$$\psi(u(m,n)) \le a(m,n) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} w\left(u(\alpha_i(s),\beta_i(t))\right)[f_i(s,t)L\left(s,t,u(\alpha_i(s),\beta_i(t))\right) + g_i(s,t)u\left(\alpha_i(s),\beta_i(t)\right)], \tag{1.3}$$

for $(m, n) \in [m_0, m_1) \cap \mathbb{N}_+ \times [n_0, n_1) \cap \mathbb{N}_+$, where $a(m, n)$, $b(m, n)$ are nonnegative and nonde-creasing functions in each variable. Inequalities (1.1), (1.2), and (1.3) are the discrete versions of Agarwal et al. [18] and Chen et al. [19]. They not only generalized the forms with one variable into the ones with two variables but also extended the constant 'c' out of integral into a function '$a(m, n)$'. These inequalities will play an important part in the study on difference equation. To illustrate the action of their inequalities, we also gave an example of boundary value problem in partial difference equation.

2 Main result

Throughout this paper, k, m_0, m_1, n_0, n_1 are fixed natural numbers. $\mathbb{N}_+ := \{1, 2, 3, \ldots\}$, $I := [m_0, m_1] \cap \mathbb{N}_+$, $I_m := [m_0, m] \cap \mathbb{N}_+$, $J := [n_0, n_1] \cap \mathbb{N}_+$, $J_n := [n_0, n] \cap \mathbb{N}_+$, $\mathbb{R}_+ := [0, \infty)$. For functions $s(m)$, $z(m, n)$, $m, n \in \mathbb{N}$, their first-order (forward) differences are defined by $\Delta s(m) = s(m + 1) - s(m)$, $\Delta_1 z(m, n) = z(m + 1, n) - z(m, n)$ and $\Delta_2 z(m, n) = z(m, n + 1) - z(m, n)$. Obviously, the linear difference equation $\Delta x(m) = b(m)$ with initial condition $x(m_0) = 0$ has solution $\sum_{s=m_0}^{m-1} b(s)$. For convenience, in the sequel, we define $\sum_{s=m_0}^{m_0-1} b(s) = 0$. We make the following assumptions:

(H_1) $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing with $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$;

(H_2) $a, b : I \times J \to (0, \infty)$ are nondecreasing in each variable;

(H_3) $w, \phi, \phi_1, \phi_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$ are nondecreasing with $w(0) > 0$, $\phi(r) > 0$, $\phi_1(r) > 0$ and $\phi_2(r) > 0$ for $r > 0$;

(H_4) $\alpha_i: I \to I$ and $\beta_i: J \to J$ are nondecreasing with $\alpha_i(m) \le m$ and $\beta_i(n) \le n$, $i = 1, 2, \ldots, k$;

(H_5) $f_i, g_i: I \times J \to \mathbb{R}_+$, $i = 1, 2, \ldots, k$.

Theorem 1. *Suppose* (H_1- H_5) *hold and* $u(m, n)$ *is a nonnegative function on* $I \times J$ *satisfying* (1.1). *Then, we have*

$$u(m,n) \le \psi^{-1}\left[W^{-1}\left(\Phi^{-1}\left(A(m,n)\right)\right)\right] \tag{2.1}$$

for all $(m, n) \in I_{M_1} \times J_{N_1}$, *where*

$$W(r) := \int_1^r \frac{ds}{w(\psi^{-1}(s))} \text{ for } r > 0; \quad W(0) := \lim_{r \to 0^+} W(r), \tag{2.2}$$

$$\Phi(r) := \int_1^r \frac{ds}{\varphi(\psi^{-1}(W^{-1}(s)))} \text{ for } r > 0; \quad \Phi(0) := \lim_{r \to 0^+} \Phi(r), \tag{2.3}$$

$$A(m,n) := \Phi\left(W(a(m,n)) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} g_i(s,t)\right) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t), \tag{2.4}$$

and $(M_1, N_1) \in I \times J$ *is arbitrarily chosen such that*

$$A(M_1, N_1) \in Dom(\Phi^{-1}),\ \Phi^{-1}(A(M_1, N_1)) \in Dom(W^{-1}). \tag{2.5}$$

Proof. From assumption (H_2) and the inequality (1.1), we have

$$\begin{aligned}\psi(u(m,n)) \le\ & a(M,n) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} w(u(\alpha_i(s), \beta_i(t))) \\ & \cdot [f_i(s,t)\varphi\left(u(\alpha_i(s), \beta_i(t))\right) + g_i(s,t)]\end{aligned} \tag{2.6}$$

for all $(m, n) \in I_M \times J$, where $m_0 \le M \le M_1$ is a natural number chosen arbitrarily. Define a function $\eta(m, n)$ by the right-hand side of (2.6). Clearly, $\eta(m, n)$ is positive and nondecreasing in each variable, with $\eta(m_0, n) = a(M, n) > 0$. Hence (2.6) is equivalent to

$$u(m,n) \le \psi^{-1}(\eta(m,n)) \tag{2.7}$$

for all $(m, n) \in I_M \times J$. By ($H4$) and the monotonicity of w, ψ^{-1} and η, we have, for all $(m, n) \in I_M \times J$,

$$\begin{aligned}\Delta_1\eta(m,n) &= b(M,n)\sum_{i=1}^{k}\sum_{t=n_0}^{n-1} w(u(\alpha_i(m), \beta_i(t)))[f_i(m,t)\varphi(u(\alpha_i(m), \beta_i(t))) + g_i(m,t)] \\ &\le w(\psi^{-1}(\eta(m,n)))b(M,n)\sum_{i=1}^{k}\sum_{t=n_0}^{n-1}[f_i(m,t)\varphi(\psi^{-1}(\eta(m,t))) + g_i(m,t)].\end{aligned} \tag{2.8}$$

On the other hand, by the monotonicity of w and ψ^{-1},

$$W(\eta(m+1,n)) - W(\eta(m,n)) = \int_{\eta(m,n)}^{\eta(m+1,n)} \frac{ds}{w(\psi^{-1}(s))} \le \frac{\Delta_1\eta(m,n)}{w(\psi^{-1}(\eta(m,n)))}. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\begin{aligned}& W(\eta(m+1,n)) - W(\eta(m,n)) \\ & \le b(M,n)\sum_{i=1}^{k}\sum_{t=n_0}^{n-1}\left[f_i(m,t)\varphi\left(\psi^{-1}(\eta(m,t))\right) + g_i(m,t)\right]\end{aligned} \tag{2.10}$$

for (m, n), $(m + 1, n) \in I_M \times J$. Keeping n fixed and substituting m with s in (2.10), and then summing up both sides over s from m_0 to $m - 1$, we get

$$\begin{aligned} W(\eta(m,n)) &\le W(\eta(m_0,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[f_i(s,t)\varphi\left(\psi^{-1}(\eta(s,t))\right) + g_i(s,t)\right] \\ &= W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_i(s,t)\varphi\left(\psi^{-1}(\eta(s,t))\right) + g_i(s,t)] \\ &\le c(M,n) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)\varphi\left(\psi^{-1}(\eta(s,t))\right) \end{aligned} \tag{2.11}$$

for $(m, n) \in I_M \times J$, where

$$c(M,n) = W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} g_i(s,t). \tag{2.12}$$

Now, define a function $\Gamma(m, n)$ by the right-hand side of (2.11). Clearly, $\Gamma(m, n)$ is positive and nondecreasing in each variable, with $\Gamma(m_0, n) = c(M, n) > 0$. Hence (2.11) is equivalent to

$$\eta(m,n) \le W^{-1}(\Gamma(m,n)) \tag{2.13}$$

for all $(m,n) \in I_M \times J_{N_1}$, where N_1 is defined in (2.5). By ($H4$) and the monotonicity of ϕ, ψ^{-1}, W^{-1} and Γ , we have, for all $(m,n) \in I_M \times J_{N_1}$,

$$\begin{aligned} \Delta_1\Gamma(m,n) &= b(M,n)\sum_{i=1}^{k}\sum_{t=n_0}^{n-1} f_i(m,t)\varphi(\psi^{-1}(\eta(m,t))) \\ &\le b(M,n)\varphi(\psi^{-1}(W^{-1}(\Gamma(m,n))))\sum_{i=1}^{k}\sum_{t=n_0}^{n-1} f_i(m,t). \end{aligned} \tag{2.14}$$

On the other hand, by the monotonicity of ϕ, ψ^{-1}, and W^{-1}, we have

$$\begin{aligned} \Phi(\Gamma(m+1,n)) - \Phi(\Gamma(m,n)) &= \int_{\Gamma(m,n)}^{\Gamma(m+1,n)} \frac{\mathrm{d}s}{\varphi(\psi^{-1}(W^{-1}(s)))} \\ &\le \frac{\Delta_1\Gamma(m,n)}{\varphi(\psi^{-1}(W^{-1}(\Gamma(m,n))))}. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$\Phi(\Gamma(m+1,n)) - \Phi(\Gamma(m,n)) \le b(M,n)\sum_{i=1}^{k}\sum_{t=n_0}^{n-1} f_i(m,t) \tag{2.16}$$

for $(m,n), (m+1,n) \in I_M \times J_{N_1}$. Keeping n fixed and substituting m with s in (2.16), and then summing up both sides over s from m_0 to $m - 1$, we get

$$\begin{aligned} \Phi(\Gamma(m,n)) &\le \Phi(\Gamma(m_0,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t) \\ &= \Phi(c(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t) \end{aligned} \tag{2.17}$$

for $(m, n) \in I_M \times J_{N_1}$. From (2.12) and (2.17), we have

$$\begin{aligned}\Gamma(m,n) &\leq \Phi^{-1}\left(\Phi(c(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)\right)\\ &= \Phi^{-1}\left[\Phi\left(W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} g_i(s,t)\right)\right.\\ &\quad \left. + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)\right].\end{aligned}\tag{2.18}$$

From (2.7), (2.13), and (2.18), we get

$$\begin{aligned}u(m,n) &\leq \psi^{-1}(\eta(m,n)) \leq \psi^{-1}(W^{-1}(\Gamma(m,n)))\\ &\leq \psi^{-1}\left\{W^{-1}\left[\Phi^{-1}\left(\Phi\left(W(a(M,n))\right.\right.\right.\right.\end{aligned}\tag{2.19}$$

$$\begin{aligned}&+b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} g_i(s,t)\Bigg)\\ &+b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)\Bigg)\Bigg]\Bigg\}\end{aligned}\tag{2.20}$$

for $(m, n) \in I_M \times J_{N_1}$. Let $m = M$, from (2.20), we observe that

$$\begin{aligned}u(M,n) \leq \psi^{-1}&\left\{W^{-1}\left[\Phi^{-1}\left(\Phi\left(W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} g_i(s,t)\right)\right.\right.\right.\\ &\left.\left.\left. +b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1} f_i(s,t)\right)\right]\right\}\end{aligned}\tag{2.21}$$

for all $(M, n) \in I_{M_1} \times J_{N_1}$, where M_1 is defined by (2.5). Since $M \in I_{M_1}$ is arbitrary, from (2.21), we get the required estimate

$$\begin{aligned}u(m,n) \leq \psi^{-1}&\left\{W^{-1}\left[\Phi^{-1}\left(\Phi\left(W(a(m,n)) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} g_i(s,t)\right)\right.\right.\right.\\ &\left.\left.\left. + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s,t)\right)\right]\right\}\end{aligned}$$

for all $(m, n) \in I_{M_1} \times J_{N_1}$. Theorem 1 is proved.

Theorem 2. *Suppose* (H_1 - H_5) *hold and* $u(m, n)$ *is a nonnegative function on* $I \times J$ *satisfying* (1.2). *Then*

(*i*) *if* $\phi_1(u) \geq \phi_2(\log u)$, *we have*

$$u(m,n) \leq \psi^{-1}\left[W^{-1}\left(\Phi_1^{-1}(D_1(m,n))\right)\right]\tag{2.22}$$

for all $(m, n) \in I_{M_1} \times J_{N_2}$,

(*ii*) *if* $\phi_1(u) \leq \phi_2(\log u)$, *we have*

$$u(m,n) \leq \psi^{-1}\left[W^{-1}\left(\Phi_2^{-1}(D_2(m,n))\right)\right]\tag{2.23}$$

for all $(m, n) \in I_{M_3} \times J_{N_3}$, *where*

$$D_j(m,n) := \Phi_j(W(a(m,n))) + b(m,n) \sum_{i=1}^{k} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} [f_i(s,t) + g_i(s,t)];$$
$$\Phi_j(r) := \int_1^r \frac{ds}{\varphi_j(\psi^{-1}(W^{-1}(s)))} \text{ for } r > 0; \quad \Phi_j(0) := \lim_{r \to 0^+} \Phi_j(r); \tag{2.24}$$

$j = 1, 2$; (M_2, N_2) *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R}_1 := \{(m,n) \in I \times J : D_1(m,n) \in Dom(\Phi_1^{-1}), \Phi_1^{-1}(D_1(m,n)) \in Dom(W^{-1})\}; \tag{2.25}$$

and (M_3, N_3) *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R}_2 := \{(m,n) \in I \times J : D_2(m,n) \in Dom(\Phi_2^{-1}), \Phi_2^{-1}(D_2(m,n)) \in Dom(W^{-1})\}; \tag{2.26}$$

Proof. (*i*) When $\phi_1(u) \geq \phi_2(\log u)$, from inequality (1.2), we have

$$\begin{aligned} \psi(u(m,n)) \leq a(M,n) + b(M,n) \sum_{i=1}^{k} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} w(u(\alpha_i(s), \beta_i(t))) \\ \cdot \left[f_i(s,t)\varphi_1(u(\alpha_i(s), \beta_i(t))) + g_i(s,t)\varphi_2\left(\log(u(\alpha_i(s), \beta_i(t)))\right)\right] \end{aligned} \tag{2.27}$$

for all $(m, n) \in I_M \times J$, where $m_0 \leq M \leq M_2$ is chosen arbitrarily. Let $\Xi(m, n)$ denote the right-hand side of (2.27), which is a positive and nondecreasing function in each variable with $\Xi(m_0, n) = a(M, n)$. Hence (2.27) is equivalent to

$$u(m,n) \leq \psi^{-1}(\Xi(m,n)). \tag{2.28}$$

By (*H*4) and the monotonicity of w, ψ^{-1}, and Ξ, we have, for all $(m, n) \in I_M \times J$,

$$\begin{aligned} \Delta_1 \Xi(m,n) &= b(M,n) \sum_{i=1}^{k} \sum_{t=n_0}^{n-1} w(u(\alpha_i(m), \beta_i(t))) \\ &\quad \cdot \left[f_i(m,t)\varphi_1(u(\alpha_i(m), \beta_i(t))) + g_i(m,t)\varphi_2\left(\log(u(\alpha_i(m), \beta_i(t)))\right)\right] \\ &\leq b(M,n) w\left(\psi^{-1}(\Xi(m,n))\right) \\ &\quad \cdot \sum_{i=1}^{k} \sum_{t=n_0}^{n-1} \left[f_i(m,t)\varphi_1\left(\psi^{-1}(\Xi(m,t))\right) + g_i(m,t)\varphi_2\left(\log(\psi^{-1}(\Xi(m,t)))\right)\right] \end{aligned} \tag{2.29}$$

for all $(m, n) \in I_M \times J$. Similar to the process from (2.9) to (2.11), we obtain

$$\begin{aligned} W(\Xi(m,n)) &\leq W(\Xi(m_0,n)) + b(M,n) \sum_{i=1}^{k} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[f_i(s,t)\varphi_1(\psi^{-1}(\Xi(s,t))) \right. \\ &\quad \left. + g_i(s,t)\varphi_2\left(\log(\psi^{-1}(\Xi(s,t)))\right)\right] \\ &= W(a(M,n)) + b(M,n) \sum_{i=1}^{k} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[f_i(s,t)\varphi_1(\psi^{-1}(\Xi(s,t))) \right. \\ &\quad \left. + g_i(s,t)\varphi_2\left(\log(\psi^{-1}(\Xi(s,t)))\right)\right] \\ &\leq W(a(M,n)) \\ &\quad + b(M,n) \sum_{i=1}^{k} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[f_i(s,t) + g_i(s,t)\right] \varphi_1(\psi^{-1}(\Xi(s,t))) \end{aligned} \tag{2.30}$$

for all $(m, n) \in I_M \times J$. Now, define a function $\Theta(m, n)$ by the right-hand side of (2.30). Clearly, $\Theta(m, n)$ is positive and nondecreasing in each variable, with $\Theta(m_0, n) = W(a(M, n)) > 0$. Thus, (2.30) is equivalent to

$$\Xi(m,n) \le W^{-1}(\Theta(m,n)) \quad \forall\ (m,n) \in I_M \times J_{N_2}, \tag{2.31}$$

where N_2 is defined by (2.25). Similar to the process from (2.14) to (2.18), we obtain

$$\begin{aligned} \Theta(m,n) &\le \Phi_1^{-1}\left(\Phi_1(\Theta(m_0,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_i(s,t)+g_i(s,t)]\right) \\ &= \Phi_1^{-1}\left(\Phi_1(W(a(M,n))) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_i(s,t)+g_i(s,t)]\right) \end{aligned} \tag{2.32}$$

for all $(m,n) \in I_M \times J_{N_2}$. From (2.28), (2.31), and (2.32), we conclude that

$$\begin{aligned} u(m,n) &\le \psi^{-1}(\Xi(m,n)) \le \psi^{-1}\left(W^{-1}(\Theta(m,n))\right) \\ &\le \psi^{-1}\left[W^{-1}\left(\Phi_1^{-1}(\Phi_1(W(a(M,n))) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_i(s,t)+g_i(s,t)])\right)\right] \end{aligned} \tag{2.33}$$

for all $(m,n) \in I_M \times J_{N_2}$. Let $m = M$, from (2.33), we get

$$u(M,n) \le \psi^{-1}\left[W^{-1}\left(\Phi_1^{-1}(\Phi_1(W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1}[f_i(s,t)+g_i(s,t)])\right)\right]. \tag{2.34}$$

Since $M \in I_{M_2}$ is arbitrary, from inequality (2.34), we obtain the required inequality in (2.22).

(*ii*) When $\phi_1(u) \le \phi_2(\log u)$, similar to the process from (2.27) to (2.30), from inequality (1.2), we have

$$W(\Xi(m,n)) \le W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[f_i(s,t)+g_i(s,t)\right]\varphi_2(\psi^{-1}(\Xi(s,t))) \tag{2.35}$$

for all $(m,n) \in I_M \times J, M \in I_{M_3}$, where M_3 is defined in (2.26). Similar to the process from (2.30) to (2.34), we obtain

$$u(M,n) \le \psi^{-1}\left[W^{-1}\left(\Phi_2^{-1}(\Phi_2(W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{n-1}[f_i(s,t)+g_i(s,t)])\right)\right]. \tag{2.36}$$

Since $M \in I_{M_3}$ is arbitrary, from inequality (2.36), we obtain the required inequality in (2.23).

Theorem 3. *Suppose* (H_1 - H_5) *hold and that* L, $M \in C(\mathbb{R}_+^3, \mathbb{R}_+)$*satisfy*

$$0 \le L(s,t,u) - L(s,t,v) \le M(s,t,v)(u-v) \tag{2.37}$$

for s, t, u, $v \in \mathbb{R}_+$*with* $u > v \ge 0$. *If* $u(m, n)$ *is a nonnegative function on* $I \times J$ *satisfying* (1.3) *then we have*

$$u(m,n) \le \psi^{-1}\left[W^{-1}(\Phi_3^{-1}(E(m,n)))\right] \tag{2.38}$$

for all $(m,n) \in I_{M_4} \times J_{N_4}$, *where* W *is defined by* (2.2),

$$\Phi_3(r) := \int_1^r \frac{\mathrm{d}s}{\psi^{-1}(W^{-1}(s))} \quad \textit{for } r > 0; \quad \Phi_3(0) := \lim_{r\to 0^+}\Phi_3(r), \tag{2.39}$$

$$E(m,n) := \Phi_3(F(m,n)) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_i(s,t)M(s,t,0)+g_i(s,t)],$$

$$F(m,n) := W(a(m,n)) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_i(s,t)L(s,t,0),$$

and $(M_4, N_4) \in I \times J$ *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R} := \{(m,n) \in I \times J : E(m,n) \in Dom(\Phi_3^{-1}), \Phi_3^{-1}(E(m,n)) \in Dom(W^{-1})\}. \quad (2.40)$$

Proof. From inequality (1.3), we have

$$\psi(u(m,n)) \le a(M,n) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} w(u(\alpha_i(s),\beta_i(t)))\left[f_i(s,t)L\left(s,t,u(\alpha_i(s),\beta_i(t))\right) + g_i(s,t)u(\alpha_i(s),\beta_i(t))\right] \quad (2.41)$$

for all $(m, n) \in I_M \times J$, where $m_0 \le M \le M_4$ is chosen arbitrarily. Let $P(m, n)$ denote the right-hand side of (2.41), which is a positive and nondecreasing function in each variable, with $P(m_0, n) = a(M, n)$. Similar to the process in the proof of Theorem 2 from (2.27) to (2.30), we obtain

$$W(P(m,n)) \le W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[f_i(s,t)L\left(s,t,\psi^{-1}(P(s,t))\right) + g_i(s,t)\psi^{-1}(P(s,t))\right] \quad (2.42)$$

for all $(m, n) \in I_M \times J$. From inequality (2.37) and (2.42), we get

$$W(P(m,n)) \le W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_i(s,t)L(s,t,0) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[f_i(s,t)M(s,t,0)+g_i(s,t)\right]\psi^{-1}(P(s,t))$$

for all $(m, n) \in I_M \times J$. Similar to the process in the proof of Theorem 2 from (2.30) to (2.34), we obtain

$$u(m,n) \le \psi^{-1}\left[W^{-1}\left(\Phi_3^{-1}\left(\Phi_3\left(W(a(M,n)) + b(M,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_i(s,t)L(s,t,0)\right) + b(m,n)\sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[f_i(s,t)M(s,t,0)+g_i(s,t)]\right)\right)\right]. \quad (2.43)$$

Since $M \in I_{M_4}$ is arbitrary, where M_4 is defined in (2.40), from inequality (2.43), we obtain the required inequality in (2.38).

3 Applications to BVP

In this section, we use our result to study certain properties of the solutions of the following boundary value problem (BVP):

$$\begin{cases} \Delta_2\left(\Delta_1(\psi(z(m,n)))\right) = F\left(m,n,z(\alpha_1(m),\beta_1(n)), z(\alpha_2(m),\beta_2(n)),\ldots,z(\alpha_k(m),\beta_k(n))\right), \\ z(m,n_0) = a_1(m),\ z(m_0,n) = a_2(n), z(m_0,n_0) = a_1(m_0) = a_2(n_0) = 0 \end{cases} \quad (3.1)$$

for $m \in I$, $n \in J$, where $m_0, n_0, m_1, n_1 \in \mathbb{R}_+$ are constants, $I := [m_0, m_1] \cap \mathbb{N}_+$, $J := [n_0, n_1] \cap \mathbb{N}_+$, $F : I \times J \times \mathbb{R}^k \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing on $\mathbb{R}_+$ with $\psi(0) = 0$, $|\psi(r)| = \psi(|r|)$, and $\psi(t) \to \infty$ as $t \to \infty$; functions $\alpha_i: I \to I$ and $\beta_i: J \to J$ are nondecreasing such that $\alpha_i(m) \le m$ and $\beta_i(n) \le n$, $i = 1, 2, \ldots, k$; $|a_1| : I \to \mathbb{R}_+$, $|a_2| : J \to \mathbb{R}_+$ are both nondecreasing.

We give an upper bound estimate for solutions of BVP (3.1).

Corollary 1. *Consider BVP* (3.1) *and suppose that F satisfies*

$$|F(m, n, u_1, u_2, \ldots, u_k)| \le \sum_{i=1}^{k} w(|u_i|)[f_i(m, n)\varphi(|u_i|) + g_i(m, n)], \quad (m, n) \in I \times J, \tag{3.2}$$

where $f_i, g_i: I \times J \to \mathbb{R}_+$ and $w, \phi \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing with $w(u) > 0$, $\phi(u) > 0$ for $u > 0$. Then, all solutions $z(m, n)$ of BVP (3.1) *satisfy*

$$|z(m, n)| \le \psi^{-1}\left(W^{-1}\left(\Phi^{-1}\left(A(m, n)\right)\right)\right), \tag{3.3}$$

for all $(m, n) \in I_{M_1} \times J_{N_1}$, where

$$A(m, n) := \Phi\left(W(\psi(|a_1(m)|) + \psi(|a_2(n)|)) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} g_i(s, t)\right) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(s, t) \tag{3.4}$$

for all $(m, n) \in I_{M_1} \times J_{N_1}$, with W, W^{-1}, Φ, Φ^{-1} and M_1, N_1 as given in Theorem 1.

Proof. BVP (3.1) is equivalent to

$$\begin{aligned} \psi(z(m, n)) &= \psi(a_1(m)) + \psi(a_2(n)) \\ &\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F\left(s, t, z(\alpha_1(s), \beta_1(t)), z(\alpha_2(s), \beta_2(t)), \ldots, z(\alpha_k(s), \beta_k(t))\right) \end{aligned} \tag{3.5}$$

By (3.2) and (3.5), we get

$$\begin{aligned} &\psi(|z(m, n)|) \\ &\quad \le \psi(|a_1(m)|) + \psi(|a_2(n)|) \\ &\qquad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \left|F\left(s, t, z(\alpha_1(s), \beta_1(t)), z(\alpha_2(s), \beta_2(t)), \ldots, z(\alpha_k(s), \beta_k(t))\right)\right| \\ &\quad \le \psi(|a_1(m)|) + \psi(|a_2(n)|) \\ &\qquad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\sum_{i=1}^{k} w\left(|z(\alpha_i(s), \beta_i(t))|\right)\left[f_i(s, t)\varphi(|z(\alpha_i(s), \beta_i(t))|) + g_i(s, t)\right]. \end{aligned} \tag{3.6}$$

Clearly, inequality (3.6) is in the form of (1.1). Thus the estimate (3.3) of the solution $z(m, n)$ follows immediately from Theorem 1.

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All the authors have contributed in all the paper part.

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