

Union Bounds for BER Evaluation and Code Optimization of Space-Time codes in 2-by-2 MIMO systems

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Abstract- In this paper, an exact closed-form formula for the Pair-wise Error Probability (PEP) is derived for two transmit and two receive antennas MIMO systems using the probability density function (PDF) of the modified Euclidean distance. An exact Union Bound formed by this formula, together with the Asymptotic Union Bound, are studied for optimization and bit-error rate (BER) evaluation of Space-Time (S-T) codes. Numerical calculations and Monte Carlo computer simulation have been used to study these two Union Bounds on a 2-by-2 MIMO system using a rotation-based diagonal S-T code (D code) in a block fading channel. Results show that the exact Union Bound is a very tight Bound for BER evaluation while the Asymptotic Union Bound is very accurate for code optimization.

Index Terms- Space-Time codes, Union Bound, MIMO

I. INTRODUCTION

Since the work on the construction of Space-Time (S-T) codes in [1], optimization in terms of minimizing the error probability and BER evaluation on S-T codes in the wireless channels have been the important issues for many researchers. For evaluation of the error probability performances of S-T codes, the Union Bound has been proved to be an accurate means for both Single-Input-Single-Out (SISO) Trellis coded systems [2~3] and Multiple-Input-Multiple-Out (MIMO) [4]. The Union Bound is the result of all possible Pair-wise Error Probabilities (PEPs) in the code-book, therefore having an accurate PEP formula for use in the Union Bound is the key to obtain a more precise evaluation of the error probability. Existing results for exact PEP formulas involving either numerical quadrature [5] or the residue [4] are not suitable for simple and efficient calculations. This paper uses the probability density function (PDF) of the modified Euclidean distance between two codewords in the code-book to derive the closed-form exact PEP formula for a 2-by-2 MIMO system, i.e. two transmit and two receive antennas. An exact Union Bound formed by this exact PEP formula, together with the Asymptotic Union Bound, is studied for optimization and bit-error rate (BER) evaluation of Space-Time (S-T) codes in block fading channels. Results show that the proposed exact Union Bound is very accurate for BER evaluation. However, when it is used to construct a S-T code by optimizing (or minimize) the BER performance, the optimized result is SNR dependent and so is inconvenient to be used in practice. The paper proposes an alternate Asymptotic Union Bound using the asymptotic PEP. With this Asymptotic Union Bound, the optimization results become independent of SNR.

Numerical calculations and Monte Carlo simulations have been used to study the optimization results and BER performances using the two Union bounds in a 2-by-2 rotation-based diagonal S-T coded (D code) MIMO system under block fading channel. Results show that the proposed exact Union Bound is a very tight bound in terms of BER performance, particularly at high SNR, and the Asymptotic Union Bound is very accurate for code optimization.

The remainder of this paper is organized as follows. Section II describes the system model under study. Closed-form formula of the exact PEP is derived in section III. The exact Union Bound and the Asymptotic Union Bound are in section IV. Results and discussions are reported in section V. Section VI is the conclusion.

II. SYSTEM MODEL

A 2-by-2 S-T coded MIMO system is used for the study in this paper. At the transmitter of the system, each quadruple of binary independent information symbols, $s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}$, are coded and then placed in a 2×2 matrix X according to :

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} = \begin{bmatrix} \cos(\theta_1)s_{1,1} - \sin(\theta_1)s_{1,2} & \sin(\theta_2)s_{2,1} + \cos(\theta_2)s_{2,2} \\ \cos(\theta_2)s_{2,1} - \sin(\theta_2)s_{2,2} & \sin(\theta_1)s_{1,1} + \cos(\theta_1)s_{1,2} \end{bmatrix} \quad (1)$$

where the values of the angle pair (θ_1, θ_2) are to be optimized so as to minimize the BER. The four coded elements in X are transmitted in a diagonal [6] way to two antennas, i.e. the code symbol $x_{i,t}$ in X , for $i = 1, 2$ and $t = 1, 2$, is modulated and then transmitted from the i -th transmit antenna in the time interval t . We assume Binary Phase Shift Keying is used here. Each data matrix X carries 4 binary data symbol and takes two time interval to transmit. So the system has a transmission rate of 2 coded symbols per time interval t , same as that of the V-BLAST system [7]. The operation in (1) can be thought of as a rotation, and thus we call the code a rotation-based diagonal space-time code or a D code here. The information symbols in the D code are encoded and transmitted in a block-by-block basis, so the matrix X in (1) could be regarded as a codeword in the code-book of the D code.

The signal from the transmitter passes through a Rayleigh

fading channel before reaching the receiver where additive white Gaussian noise is added to the signal at the input. The receiver uses 2 antennas to receive the signal which can be written as:

$$R=HX+Y \quad (2)$$

where H is a 2×2 channel matrix with elements $\{h_{j,i}\}$, for $j = 1, 2$ and $i = 1, 2$, being the channel transfer function from the i -th transmit antenna to the j -th receive antenna. All elements in $\{h_{j,i}\}$ are i.i.d complex Gaussian variables with zero mean and variance one. The Rayleigh fading channel is assumed to be static within a transmission interval for a block of coded symbols. The received signal matrix R in (2) is also a 2×2 matrix with elements $\{r_{j,i}\}$, for $j = 1, 2$ and $i = 1, 2$, being the signal received from the j -th antenna in the i -th transmission interval. Transmission delay is neglected here. Additive white Gaussian noise in the channel is modeled by the 2×2 matrix Y in (2) with all elements being i.i.d complex Gaussian variables with zero-mean and a fixed variance $N_0/2$ for the real and imaginary parts. We assume perfect estimation of the channel matrix H at the receiver. A Maximum-Likelihood detector is then used to make decision on the received data $x_{i,t}$ based on the minimal Euclidean distance :

$$\text{Minimize} \left\{ \sum_{j=1}^2 \sum_{t=1}^2 \left| r_{j,t} - \sum_{i=1}^2 h_{j,i} x_{i,t} \right|^2 \right\} \quad (3)$$

III Closed-Form PEP for S-T codes

3.1 Conditional PEP and modified Euclidean distance:

The PEP, denoted here as $P_e(X \rightarrow \hat{X})$, is defined as the error probability when codeword X is transmitted but is falsely decided as \hat{X} which is any codeword in the codebook other than X . With the system model in (2) and the decision rule described in (3), the conditional PEP on a given channel H , can be written as [8]:

$$P_e(X \rightarrow \hat{X} | H) = Q \left(\sqrt{d^2(X, \hat{X}) \frac{E_s}{2N_0}} \right) \quad (4)$$

where:

$$d^2(X, \hat{X}) = \sum_{j=1}^2 \sum_{t=1}^2 \left| \sum_{i=1}^2 h_{j,i} (x_{i,t} - \hat{x}_{i,t}) \right|^2 \quad (5)$$

is defined as the modified Euclidean distance between two codewords X and \hat{X} . Since $h_{j,i}$ is an i.i.d complex

Gaussian random variable with zero mean and variance one, (5) can be simplified as [1]:

$$d^2(X, \hat{X}) = \sum_{j=1}^2 \sum_{t=1}^2 \lambda_j |\beta_{j,t}|^2 \quad (6)$$

where $\beta_{j,t}$, for $i = 1, 2$ and $j = 1, 2$, is an independent complex Gaussian random variable with variance one, and λ_i , for $i = 1, 2$, is the i -th eigenvalue of the matrix $(X - \hat{X})(X - \hat{X})^*$ with $*$ being the transpose conjugate.

Here we assume both eigenvalues, i.e. λ_1, λ_2 are nonzero. If one of them is zero, then this 2-by-2 MIMO system degenerates to a two-branch diversity system, which can be analyzed in a straightforward manner.

3.2 Closed-Form exact PEP:

Although the exact PEP can be obtained by substituting (6) into (4) and then taking the average of the Q function over all $\beta_{j,t}$, this however involves complicated operations of multiple integrals. So a simpler approach is adopted here by using the modified Euclidean distance $d^2(X, \hat{X})$ as the variable to find the PDF $p_d(x)$ and then integrate the conditional PEP in (4) over $d^2(X, \hat{X})$ to get the closed-form expression.

It is known that the characteristic function of the sum of independent random variables is the product of characteristic functions of individual independent random variables. Also, once the characteristic function is found, the PDF can be obtained by using the inverse Fourier transform. Since $\beta_{j,t}$ is an i.i.d complex-Gaussian-random variable, thus $d^2(X, \hat{X})$ in (6) is the sum of 4 independent chi-square random variables. An easy way to obtain the PDF for $d^2(X, \hat{X})$ is to find first the characteristic function of individual chi-square random variable using the Fourier transform of its PDF:

$$p_{dk}(j\omega) = \int_{-\infty}^{\infty} p_{dk}(x) e^{-j\omega x} dx, \quad k = 1, 2, 3, 4 \quad (7)$$

The resultant characteristic function $p_d(j\omega)$ for $d^2(X, \hat{X})$ is the products of the four characteristic functions $p_{dk}(j\omega)$. The PDF for $d^2(X, \hat{X})$ then can be obtained by performing the inverse Fourier transform (IFT) on the resultant characteristic function:

$$p_d(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p_d(j\omega) e^{j\omega x} d\omega \quad (8)$$

The PDF of $d^2(X, \hat{X})$ can be obtained in the following two ways, depending on whether the eigenvalues are all equal or not.

Equal eigenvalues case

If the eigenvalues are all equal, then

$$\lambda_1 = \lambda_2 = \lambda \quad (9)$$

In this case, $d^2(X, \hat{X})$ in (6) is the chi-square distributed random variable with 8 degrees of freedom. The PDF for it could be found in many literatures such as in [9] where the characteristic function is also used to derive the PDF

$$p_d(x) = \frac{1}{6\lambda^4} x^3 e^{-\frac{x}{\lambda}} \quad (10)$$

for chi-square random variables.

Unequal eigenvalues case

In this case, the characteristic function is:

$$p_d(j\omega) = \frac{1}{(1 + j\omega\lambda_1)^2 (1 + j\omega\lambda_2)^2} \quad (11)$$

Performing IFT on (11) or substituting (11) into (8) yields:

$$p_d(x) = \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \right)^2 \left[\frac{1}{\lambda_1^2} x e^{-\frac{x}{\lambda_1}} \right] + \left(\frac{\lambda_2}{\lambda_1 - \lambda_2} \right)^2 \left[\frac{1}{\lambda_2^2} x e^{-\frac{x}{\lambda_2}} \right] - \frac{2\lambda_1^2\lambda_2}{(\lambda_1 - \lambda_2)^3} \left[\frac{1}{\lambda_1} e^{-\frac{x}{\lambda_1}} \right] + \frac{2\lambda_2^2\lambda_1}{(\lambda_1 - \lambda_2)^3} \left[\frac{1}{\lambda_2} e^{-\frac{x}{\lambda_2}} \right] \quad (12)$$

Thus (12) and (10) give the distribution for the modified Euclidean distance $d^2(X, \hat{X})$ defined in (3) for the equal and unequal eigenvalues cases, respectively.

Once the PDF for $d^2(X, \hat{X})$ is obtained, the exact PEP can be calculated by integrating the conditional PEP in (4) over $p_d(x)$ as:

$$P_e(X \rightarrow \hat{X}) = \int_0^\infty Q \left(\sqrt{x \frac{E_s}{2N_0}} \right) p_d(x) dx \quad (13)$$

Substitute (10) into (13) and use the closed-form integral provided in [9], the exact PEP for the equal eigenvalues case can be obtained in closed form as:

$$P_e(X \rightarrow \hat{X}) = [0.5(1-u)]^4 \sum_{k=0}^3 \binom{3+k}{k} (0.5(1+u))^k \quad (14)$$

$$u = \sqrt{\frac{\frac{E_s}{4N_0}}{1 + \lambda \frac{E_s}{4N_0}}}$$

Similarly, substitute (12) into (13), the exact PEP for the unequal eigenvalues case can be obtained in closed form as:

$$P_e(X \rightarrow \hat{X}) = \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \right)^2 \left[\frac{1}{4} (1-u_1)^2 (u_1+2) \right] + \left(\frac{\lambda_2}{\lambda_1 - \lambda_2} \right)^2 \left[\frac{1}{4} (1-u_2)^2 (u_2+2) \right] - \frac{2\lambda_1^2\lambda_2}{(\lambda_1 - \lambda_2)^3} [0.5(1-u_1)] + \frac{2\lambda_2^2\lambda_1}{(\lambda_1 - \lambda_2)^3} [0.5(1-u_2)] \quad (15)$$

$$u_1 = \sqrt{\frac{\lambda_1 \frac{E_s}{4N_0}}{1 + \lambda_1 \frac{E_s}{4N_0}}}, u_2 = \sqrt{\frac{\lambda_2 \frac{E_s}{4N_0}}{1 + \lambda_2 \frac{E_s}{4N_0}}}$$

Therefore (14) and (15) give the closed-form exact PEPs of 2-by-2 MIMO systems for the equal and unequal eigenvalues cases, respectively. Although this paper uses the D code as a study, the derivation of the exact PEP formulas in (14) and (15) does not depend on any special S-T coding scheme. So (14) and (15) are applicable to any general 2-by-2 S-T codes.

3.3 Chernoff bound PEP and Asymptotic PEP:

Applying the Chernoff Bound to the Q function :

$$Q(x) \leq e^{-\frac{x^2}{2}} \quad (16)$$

yields the well known Chernoff bound PEP [1]. Substituting (16) & (6) into (4) and then averaging the result over all $\beta_{i,j}$ produces:

$$P_e(X \rightarrow \hat{X}) \leq \left(\frac{1}{\left(\left(1 + \lambda_1 \frac{E_s}{4N_0} \right) \left(1 + \lambda_2 \frac{E_s}{4N_0} \right) \right)} \right)^2 \quad (17)$$

If SNR is high enough, then (17) can be further upper-bounded by:

$$P_e(X \rightarrow \hat{X}) \leq (\lambda_1 \lambda_2)^{-2} \left(\frac{E_s}{4N_0} \right)^{-4} \quad (18)$$

It can be seen that, as the SNR increases, (18) will be tighter and closer to the Chernoff bound in (17), so we denote the bound in (18) here as the Asymptotic PEP.

IV Exact Union Bound and Asymptotic Union Bound

4.1 Union Bound

The Union bound is defined as [10]:

$$P_U = \sum_X \sum_{\hat{X}|\hat{X} \neq X} r(X, \hat{X}) P_e(X \rightarrow \hat{X}) P(X) \quad (19)$$

where $P(X)$ is the priori probability of X being transmitted and $r(X, \hat{X})$ is the error rate from X to \hat{X} which could be the Bit Error Rate (BER), the Symbol Error Rate (SER) or the Block Error Rate (BLER). When all codewords in X are equally likely to be transmitted, (19) is simplified to:

$$P_U = \frac{1}{C} \sum_X \sum_{\hat{X}|\hat{X} \neq X} r(X, \hat{X}) P_e(X \rightarrow \hat{X}) \quad (20)$$

Here, C is the size of the code-book.

Substituting the Asymptotic Bound given by (18) into (20) gives the Asymptotic Union Bound; while substituting the exact PEP in (14) or (15) into (20) yields the exact Union Bound.

V Results and Discussions

Optimization results

Numerical calculations have been performed on the exact Union Bound, i.e., by substituting (1), (14) and (15) into (20). At each SNR tested, the angle pairs were varied by a step size of 0.001 rad from zero to π so as to minimize the BER for the D code. The optimum angle pair, leading to the minimum BER, for the SNRs tested are shown Table I. It can be seen that the optimization angle pair varies with SNR; thus the results are not convenient to use.

Exact Union Bound optimization for BER		
SNR (dB)	θ_1 (rad)	θ_2 (rad)
9	1.112	1.112
10	2.674	2.719
11	1.063	0.349
12	1.878	0.52
13	0.279	2.616
14	1.83	2.613
15	2.101	2.896
16	2.906	2.102
17	0.228	0.532
18	2.103	2.919
19	2.103	1.789

Table I: Optimum angle pair using exact Union Bound

Numerical calculations has also been performed on the Asymptotic Union Bound, i.e., by substituting (1) and (18) into (20). Here the optimum angle pair is SNR independent, as described previously. Results have shown that the optimum angle pair (1.368 rad, 0.531 rad) achieves the minimum BER.

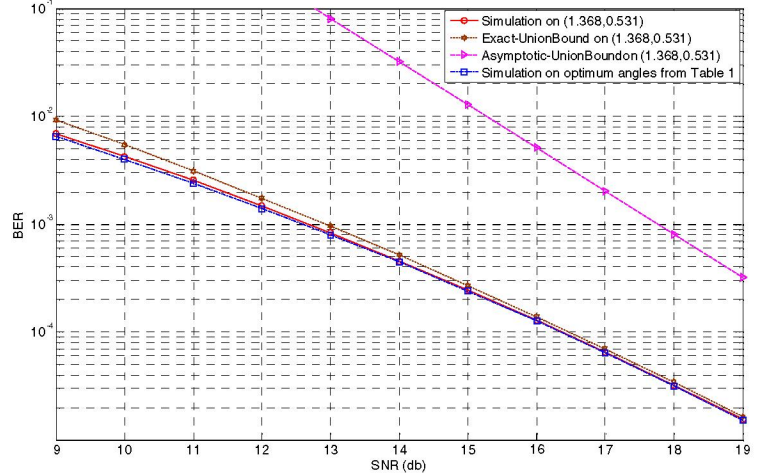


Fig 1 BER performances of D code using simulation and different Union Bounds

BER results

Then Monte Carlo simulation has been used to evaluate the BER performance of the system using 2 receive antennas and the optimum angle pair (1.368, 0.531) obtained from the Asymptotic Union Bound. Results are shown in Fig 1. With the use of the same optimum angle pair (1.368, 0.531), the numerical calculation results on the BER performances using the exact Union Bound and the Asymptotic Union Bound are also shown in the same figure for comparison. These results indicate that the Asymptotic Union Bound is a loose bound and has a difference of about 5 dB at $BER = 10^{-3}$ from the simulation result. The exact Union Bound is very close to the simulation result, particularly at high SNRs, so it is a very tight bound for the true BERs. As a result, as far as BER evaluation is concerned, the exact Union Bound using the proposed exact PEP formula is much more accurate.

Since the optimum angle pair obtained from using the exact Union Bound varies with the SNR used, Monte Carlo simulation has been used to evaluate the BER performances of the system at different SNRs but with the corresponding optimum angle pair shown in Table I. The results are also shown in Fig 1. It can be seen that the two BER performances, obtained from using the angle pair (1.368, 0.531) and the corresponding optimum angle pairs for different SNRs, are nearly identical. Thus, although the Asymptotic Union Bound is not accurate for BER performance evaluation, it is quite an

accurate method to find the optimum angles.

VI CONCLUSION

The paper has derived the closed-form exact PEP formula which is subsequently proposed to be used in the exact Union Bound for BER evaluation of an S-T coded 2-by-2 MIMO system in a blocking fading channel. Numerical analyses and Monte Carlo computer simulation have been used to study the proposed exact Union Bound and the Asymptotic Union Bound on BER evaluation and code optimization. Results have shown that the proposed exact Union Bound is a very tight bound in terms of BER performance, but the optimized angle pair obtained varies with SNR. Results have also shown that the BER performances obtained using the Asymptotic Union Bound is only a loose bound; however, the angle pair obtained using this bound is independent of SNR and can be used in the exact Union Bound for accurate BER evaluation, so the Asymptotic Union Bound is good for optimization of the angle pairs.

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