Cosmological dynamics of dark matter Bose-Einstein condensation

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Once the critical temperature of a cosmological boson gas is less than the critical temperature, a Bose-Einstein condensation process can always take place during the cosmic history of the Universe. In the Bose-Einstein condensation model, dark matter can be described as a nonrelativistic, Newtonian gravitational condensate, whose density and pressure are related by a barotropic equation of state, with a barotropic index equal to one. In the present work, we study the Bose-Einstein condensation process in a cosmological context by assuming that this process can be described (at least approximately) as a first-order phase transition. We analyze the evolution of the physical quantities relevant for the physical description of the early universe, namely, the energy density, temperature, and scale factor, before, during, and after the Bose-Einstein condensation (phase transition). We also consider in detail the epoch when the Universe evolved through a mixed condensate—normal dark matter phase—with a monotonically growing Bose-Einstein dark matter component. An important parameter characterizing the Bose-Einstein condensation is the condensate dark matter fraction, whose time evolution describes the time dynamics of the conversion process. The behavior of this parameter during the cosmological condensation process is also analyzed in detail. To study the cosmological dynamics and evolution we use both analytical and numerical methods. The presence of the condensate dark matter and of the Bose-Einstein phase transition could have modified drastically the cosmological evolution of the early universe, as well as the large scale structure formation process.

I. INTRODUCTION

At very low temperatures, particles in a dilute Bose gas can occupy the same quantum ground state, forming a Bose-Einstein condensate (BEC), which appears as a sharp peak over a broader distribution in both coordinates and momentum space. The possibility to obtain quantum degenerate gases by a combination of laser and evaporative cooling techniques has opened several new lines of research at the border of atomic, statistical, and condensed matter physics (for recent reviews on the Bose-Einstein Condensation see [1–3]).

An ideal system for the experimental observation of the Bose-Einstein condensation is a dilute atomic Bose gas confined in a trap and cooled to very low temperatures. BECs were first observed in 1995 in dilute alkali gases such as vapors of rubidium and sodium [4]. In these experiments, atoms were confined in magnetic traps, evaporatively cooled down to a fraction of a microkelvin, left to expand by switching off the magnetic trap, and subsequently imaged with optical methods. A sharp peak in the velocity distribution was observed below a critical temperature, indicating that condensation has occurred, with the alkali atoms condensed in the same ground state. Under the typical confining conditions of experimental settings, BECs are inhomogeneous, and hence condensates

arise as a narrow peak not only in the momentum space but also in the coordinate space [4].

Since in the terrestrial experiments Bose-Einstein condensation is a well-known phenomenon, the possibility that a similar condensation may have occurred during the cosmological evolution of the Universe cannot be excluded *a priori*. In fact, once the critical temperature of the boson gas is less than the critical temperature, BEC can always take place at some moment during the cosmic history of the Universe. Different aspects of the BEC cosmological transition were considered in [5–7]. The critical temperature for the condensation to take place is $T_{cr} < 2\pi\hbar^2 n^{2/3}/mk_B$, where n is the particle number density, m is the particle mass, and k_B is Boltzmann's constant [1–3]. On the other hand, cosmic evolution has the same temperature dependence, since in an adiabatic expansion process the density of a matter dominated universe evolves as $\rho \propto T^{3/2}$ [7]. Therefore, if the boson temperature is equal, for example, to the radiation temperature at a redshift $z = 1000$, the critical temperature for the Bose-Einstein condensation is at present $T_{cr} = 0.0027 K$ [7]. Since the matter temperature T_m varies as $T_m \propto a^{-2}$, where a is the scale factor of the Universe, it follows that during an adiabatic evolution the ratio of the photon temperature T_γ and of the matter temperature evolves as $T_\gamma/T_m \propto a$. Using for the present day energy density of the Universe the value $\rho_{cr} = 9.44 \times 10^{-30}$ g/cm^3, BEC takes place provided that the boson mass satisfies the restriction $m < 1.87$ eV [6]. On the other hand, we expect that the Universe is

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always under critical temperature, if it is at the present time [7].

Despite the important achievements of the Λ cold dark matter (ΛCDM) model [8], at galactic scales ~ 10 kpc, the standard cosmological model meets with severe difficulties in explaining the observed distribution of the invisible matter around the luminous one. In fact, N-body simulations, performed in this scenario, predict that bound halos surrounding galaxies must have very characteristic density profiles, called the Navarro-Frenk-White (NFW) profiles, which feature a well-pronounced central cusp $\rho_{\mathrm{NFW}}(r) = \rho_s/(r/r_s)(1 + r/r_s)^2$, where r_s is a scale radius and ρ_s is a characteristic density [9]. On the observational side, high-resolution rotation curves show, instead, that the actual distribution of dark matter is much shallower than the above, and it presents a nearly constant density core [10]. The core-cusp problem of standard CDM models can be solved by assuming that dark matter is composed of ultra-light scalar particles, with masses of the order of 10^{-22} eV, initially in a cold Bose-Einstein condensate ("fuzzy dark matter") [11]. The wave properties of the dark matter stabilize gravitational collapse, providing halo cores. On the other hand, dark matter models with pressure, satisfying a polytropic equation of state, give an excellent fit to the observed galactic rotational curves [12]. The polytropic equation of state can describe extended theories of dark matter involving self-interaction, nonextensive thermostatistics, or boson condensation (in a classical limit). In such models, the flat-cored mass profiles widely observed in disc galaxies are due to innate dark physics, regardless of any baryonic motion. The fine structure in the observed inner mass distribution of the MilkyWay can be explained only if the infalling dark matter particles, from which such galactic halos formed, had a net overall rotation, causing a tricusp caustic ring of dark matter [13]. For standard, noninteracting CDM models, however, one expects the infall to be irrotational. Bose-Einstein condensate dark matter can form vortices, thus leading to a net overall rotation of the galactic halos. Hence Milky-Way observations may have already detected some specific signatures of the Bose-Einstein condensate dark matter [14].

The possibility that the galactic dark matter is in the form of a (cold) Bose-Einstein condensate was considered in detail in [15]. The density distribution ρ of the static gravitationally bounded single component dark matter Bose-Einstein condensate is given by $\rho(r) = \rho_c \sin kr/kr$, where ρ_c is the central density of the condensate, $\rho_c = \rho(0)$, and k is a constant. At the boundary of the dark matter distribution $\rho(R) = 0$, giving the condition $kR = \pi$, which fixes the radius of the condensate dark matter halo as $R = \pi\sqrt{\hbar^2 l_a/Gm^3}$, where l_a is the s-wave scattering length, and m is the particle mass. The total mass of the condensate dark matter halo M can be obtained as $M = 4\pi^2(\hbar^2 l_a/Gm^3)^{3/2} \rho_c = 4R^3\rho_c/\pi$, giving for the mean value $\langle\rho\rangle$ of the condensate density the expression

$\langle\rho\rangle = 3\rho c/\pi^2$. The mass of the particle in the condensate is given by

$$m = \left(\frac{\pi^2\hbar^2 l_a}{GR^2}\right)^{1/3}$$
$$\approx 6.73 \times 10^{-2} \times [l_a \ (\mathrm{fm})]^{1/3}[R \ (\mathrm{kpc})]^{-2/3} \ \mathrm{eV}. \quad (1)$$

The global cosmological evolution and the evolution of the density contrast in the Bose-Einstein condensate dark matter model, in the framework of a post-Newtonian cosmological approach, were investigated in [16,17]. The global cosmological evolution as well as the evolution of the perturbations of the condensate dark matter shows significant differences with respect to the pressureless dark matter model considered in the framework of standard cosmology. Therefore, the presence of condensate dark matter could have modified drastically the cosmological evolution of the early universe, as well as the large scale structure formation process. The core/cusp problem and the dark halo properties of the dwarf galaxies were recently analyzed in [18]. Different properties of the dark matter condensate have been extensively studied in the physical literature [19].

It is the purpose of the present paper to investigate the cosmological Bose-Einstein condensation process of dark matter. The condensation process is interpreted as a phase transition taking place some time during the cosmic history of the Universe. A Bose-Einstein condensation process of the dark matter in the expanding universe can be described generically as follows. As the normal bosonic dark matter cools below the critical temperature T_{cr} of the condensation, it becomes energetically favorable to form a condensate, in which all particles are in the same quantum state. However, the new phase does not appear instantaneously and the two phases coexist for some time. The transition ends when all normal dark matter has been converted to a condensed state. During the transition phase the global cosmological evolution of the Universe is changed. By assuming that the phase transition is of the first order, we study in detail the evolution of the relevant cosmological parameters (energy density, temperature, scale factor, etc.) of the normal dark matter and Bose-Einstein condensed dark matter phase and the condensation process itself. An important parameter to describe the Bose-Einstein condensation is the condensate dark matter fraction, whose time evolution describes the time dynamics of the conversion process. The behavior of this parameter is also analyzed in detail.

The present paper is organized as follows. In Sec. II, we briefly outline, for self-completeness and self-consistency, the basic properties of the normal and Bose-Einstein condensed dark matter. We also lay down the equations of state and the relevant physical quantities that are analyzed in the remaining sections. The cosmological dynamics of the Bose-Einstein condensation of dark matter is analyzed in Sec. III. We discuss and conclude our results in

Sec. IV. In the present paper we use the centimeter-gram-second system of units.

II. COSMOLOGICAL DARK MATTER IN NORMAL AND BOSE-EINSTEIN CONDENSED STATES

In the present section we outline the relevant physical processes and quantities of the Bose-Einstein Condensation, which will be used in the following sections to study the dynamics of the condensation and some of its cosmological implications.

We assume that the space-time geometry is the flat Friedmann-Robertson-Walker metric, given by

$$ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (2)$$

where a is the scale factor describing the cosmological expansion. For the matter energy-momentum tensor we restrict our analysis to the case of the perfect fluid energy-momentum tensor,

$$T^{\mu\nu} = (\rho c^2 + p)u^\mu u^\nu + pg^{\mu\nu}. \quad (3)$$

The Hubble function $H(a)$ is defined as $H = \dot{a}/a$. As for the matter content of the Universe, we assume that it consists of radiation, with energy density $\rho_{\rm rad}$ and pressure $p_{\rm rad}$, pressureless ($p_b = 0$) baryonic matter, with energy density ρ_b, and dark matter, with energy density ρ_χ, and pressure p_χ, respectively. In the following, we neglect any possible interaction between these components by assuming that the energy of each component is individually conserved. Thus, the gravitational field equations, corresponding to the line element (2) become

$$3\frac{\dot{a}^2}{a^2} = 8\pi G(\rho_b + \rho_{\rm rad} + \rho_\chi) + \Lambda, \quad (4)$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{8\pi G}{c^2}(p_b + p_{\rm rad} + p_\chi) + \Lambda, \quad (5)$$

$$\dot{\rho}_i + 3\left(\rho_i + \frac{p_i}{c^2}\right)\frac{\dot{a}}{a} = 0, i = b, {\rm rad}, \chi. \quad (6)$$

The cosmological evolution of the energy density of the baryonic matter and radiation are given by $\rho_b = \rho_{b,0}/(a/a_0)^3$ and $\rho_{\rm rad} = \rho_{{\rm rad},0}/(a/a_0)^4$, respectively, where $\rho_{b,0}$ and $\rho_{{\rm rad},0}$ are the energy densities of the matter corresponding to the value $a = a_0$ of the scale factor. For the dark matter we consider a general density evolution of the form $\rho_\chi = \rho_{\chi,0}/f(a/a_0)$, where $\rho_{\chi,0}$ is the value of the energy density of the dark matter at $a = a_0$, and $f(a/a_0)$ is an arbitrary function of the scale factor, depending on the particular dark matter model. By introducing the critical density at $a = a_0$ as $\rho_{cr,0} = 3H_0^2/8\pi G$, where $H_0 = H(a_0)$, and the density parameters $\Omega_{i,0} = \rho_{i,0}/\rho_{\rm cr,0}$, $i = b, {\rm rad}, \chi$, we obtain the basic equation describing the dynamics of the cosmological models as

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\frac{\Omega_{b,0}}{(a/a_0)^3} + \frac{\Omega_{{\rm rad},0}}{(a/a_0)^4} + \frac{\Omega_{\chi,0}}{f(a/a_0)} + \Omega_\Lambda\right], \quad (7)$$

where Ω_Λ is the density parameter of the dark energy. The density parameters satisfy the relation $\Omega_{b,0} + \Omega_{{\rm rad},0} + \Omega_{\chi,0} + \Omega_\Lambda = 1$.

A. Normal dark matter in the early universe

We assume that in the early stages of the evolution of the Universe dark matter consisted of bosonic particles of mass m_χ and temperature T, originating in equilibrium and decoupling at a temperature T_D or chemical potential $\mu \gg m_\chi$. By assuming that the dark matter forms an isotropic gas of particles in kinetic equilibrium, the spatial number density is given by

$$n = \frac{g}{h^3}\int 4\pi f(p)p^2 dp, \quad (8)$$

where h is Planck's constant, g is the number of helicity states, and

$$f(p) = \{\exp[(E - \mu)] - 1\}^{-1}, \quad (9)$$

where p is the momentum of the particle and $E = \sqrt{p^2 + m_\chi^2 c^4}$ is the energy. A particle species that decouples in the early universe from the remaining plasma at temperature T_D redshifts its momenta according to $p(t) = p_D a_D/a(t)$, where $a(t)$ is the cosmological scale factor and a_D is the value of the scale factor at the decoupling. The number density of the particles n evolves as $n_\chi \sim a^{-3}(t)$ [20]. The distribution function f at a time t after the decoupling is related to the value of the distribution function at the decoupling by $f(p) = f(pa/a_D)$. The distribution function keeps an equilibrium shape in two regimes. In the extreme-relativistic case, when $E \approx pc$, $T = T_D a_D/a$, and $\mu = \mu_D a_D/a$, respectively, the distribution function is given by $f_{\rm ER}(p) = \{\exp[(pc - \mu)] - 1\}^{-1}$. In the nonrelativistic decoupling case $E - \mu \approx p^2/2m_\chi - \mu_{\rm kin}$, where we have defined $\mu_{\rm kin} \equiv \mu - m_\chi c^2$, the distribution function is $f_{\rm NR}(p) = \{\exp[(p^2/2m_\chi - \mu_{\rm kin})] - 1\}^{-1}$. In the nonrelativistic case μ_{kin} and T evolve as $\mu_{\rm kin} = \mu_{{\rm kin},D}(a_D/a)^2$ and $T = T_D/(a_D/a)^2$, respectively [20].

The kinetic energy-momentum tensor T_ν^μ associated to the frozen distribution of dark matter is given by

$$T_\nu^\mu = \frac{g}{h^3}\int \frac{p^\mu p_\nu}{p^0} f(p)d^3p. \quad (10)$$

The energy density ϵ of the system is defined as

$$\epsilon = \frac{g}{3h^3}\int E f(p)d^3p, \quad (11)$$

while the pressure of a system with an isotropic distribution of momenta is given by

$$P = \frac{g}{3h^3} \int p v f(p) d^3 p = \frac{g}{3h^3} \int \frac{c^2 p^2}{E} f(p) d^3 p, \quad (12)$$

where the velocity v is related to the momentum by $v = pc^2/E$ [21]. In the nonrelativistic regime, when $E \approx m_\chi c^2$ and $p \approx m_\chi v_\chi$, the density ρ_χ of the dark matter is given by $\rho_\chi = m_\chi n_\chi$, while its pressure P_χ can be obtained as [20]

$$P_\chi = \frac{g}{3h^3} \int \frac{p^2 c^2}{E} f(p) d^3 p \approx 4\pi \frac{g}{3h^3} \int \frac{p^4}{m_\chi} dp, \quad (13)$$

giving

$$P_\chi = \rho_\chi c^2 \sigma^2, \quad (14)$$

where $\sigma^2 = \langle \vec{v}_\chi^2 \rangle / 3c^2$, and $\langle \vec{v}_\chi^2 \rangle$ is the average squared velocity of the particle. σ is the one-dimensional velocity dispersion. The cosmological dynamics of the dark matter density is described by the equation

$$\dot{\rho}_\chi + 3\rho_\chi (1 + \sigma^2) \frac{\dot{a}}{a} = 0, \quad (15)$$

with the general solution given by

$$\rho_\chi = \frac{\rho_{\chi,0}}{(a/a_0)^{3(1+\sigma^2)}}, \quad (16)$$

where $\rho_{\chi,0}$ is the density of the dark matter at $a = a_0$. The dynamics of the Universe with a normal dark matter component is described by the equation

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\frac{\Omega_{b,0}}{(a/a_0)^3} + \frac{\Omega_{\text{rad},0}}{(a/a_0)^4} + \frac{\Omega_{\chi,0}}{(a/a_0)^{3(1+\sigma^2)}} + \Omega_\Lambda \right]. \quad (17)$$

In the following we consider that $a = a_0$ is the present day scale factor. Therefore, for the Hubble constant we adopt the value $H_0 = 70$ km/s/Mpc $= 2.27 \times 10^{-18}$ s^{-1}, giving for the critical density a value of $\rho_{\text{cr},0} = 9.24 \times 10^{-30}$ g/cm^3. The present day dark matter density parameter is $\Omega_{\chi,0} \approx 0.228$, $\Omega_{b,0} = 0.045$, $\Omega_{\text{rad},0} = 8.24 \times 10^{-5}$, and $\Omega_\Lambda = 0.73$, respectively [22]. We also introduce the Hubble time, defined as $t_H = 1/H_0 = 4.39 \times 10^{17}$ s. Since dark matter is nonrelativistic, the global cosmological evolution of the Universe is very little influenced by the variation of the numerical values of σ^2.

B. Bose-Einstein condensed dark matter

At very low temperatures, all particles in a dilute Bose gas condense to the same quantum ground state, forming a BEC. Particles become correlated with each other when their wavelengths overlap, that is, the thermal wavelength λ_T is greater than the mean interparticles' distance l. This happens at a temperature $T_{\text{cr}} \approx 2\pi \times \hbar^2 \rho^{2/3}/m^{5/3} k_B$, where m is the mass of the particle in the condensate, ρ is the density, and k_B is Boltzmann's constant [1]. A coherent state develops when the particle density is enough high, or the temperature is sufficiently low. We assume that

the dark matter halos are composed of a strongly coupled dilute Bose-Einstein condensate at absolute zero. Hence, almost all the dark matter particles are in the condensate. In a dilute and cold gas, only binary collisions at low energy are relevant, and these collisions are characterized by a single parameter, the s-wave scattering length l_a, independently of the details of the two-body potential. Therefore, one can replace the interaction potential with an effective interaction $V_I(\vec{r}' - \vec{r}) = \lambda \delta(\vec{r}' - \vec{r})$, where the coupling constant λ is related to the scattering length l_a through $\lambda = 4\pi\hbar^2 l_a/m_\chi$ [1]. The ground state properties of the dark matter are described by the mean-field Gross-Pitaevskii (GP) equation. The GP equation for the dark matter halos can be derived from the GP energy functional,

$$E[\psi] = \int \left[\frac{\hbar^2}{2m_\chi} |\nabla \psi(\vec{r})|^2 + \frac{U_0}{2} |\psi(\vec{r})|^4 \right] d\vec{r}$$
$$- \frac{1}{2} G m_\chi^2 \iint \frac{|\psi(\vec{r})|^2 |\psi(\vec{r}')|^2}{|\vec{r} - \vec{r}'|} d\vec{r} d\vec{r}', \quad (18)$$

where $\psi(\vec{r})$ is the wave function of the condensate, and $U_0 = 4\pi\hbar^2 l_a/m_\chi$ [1]. The first term in the energy functional is the quantum pressure, the second is the interaction energy, and the third is the gravitational potential energy. The mass density of the condensate dark matter is defined as

$$\rho_\chi(\vec{r}) = m_\chi |\psi(\vec{r})|^2 = m_\chi \rho(\vec{r}, t), \quad (19)$$

and the normalization condition is $N = \int |\psi(\vec{r})|^2 d\vec{r}$, where N is the total number of dark matter particles. The variational procedure $\delta E[\psi] - \mu \delta \int |\psi(\vec{r})|^2 d\vec{r} = 0$ gives the GP equation as

$$-\frac{\hbar^2}{2m_\chi} \nabla^2 \psi(\vec{r}) + m_\chi V(\vec{r}) \psi(\vec{r}) + U_0 |\psi(\vec{r})|^2 \psi(\vec{r})$$
$$= \mu \psi(\vec{r}), \quad (20)$$

where μ is the chemical potential, and the gravitational potential V satisfies the Poisson equation $\nabla^2 V = 4\pi G \rho$. In the time-dependent case the generalized Gross-Pitaevskii equation describing a gravitationally trapped rotating Bose-Einstein condensate is given by

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2m_\chi} \nabla^2 + m_\chi V(\vec{r}) + U_0 |\psi(\vec{r}, t)|^2 \right]$$
$$\times \psi(\vec{r}, t). \quad (21)$$

The physical properties of a Bose-Einstein condensate described by the generalized Gross-Pitaevskii equation given by Eq. (21) can be understood much more easily by using the so-called Madelung representation of the wave function [1–3], which consists in writing ψ in the form

$$\psi(\vec{r}, t) = \sqrt{\rho(\vec{r}, t)} \exp\left[\frac{i}{\hbar} S(\vec{r}, t) \right], \quad (22)$$

where the function $S(\vec{r}, t)$ has the dimensions of an action. By substituting the above expression of $\psi(\vec{r}, t)$ into Eq. (21), it decouples into a system of two differential equations for the real functions ρ_χ and $\vec{v}$, given by

$$\frac{\partial \rho_\chi}{\partial t} + \nabla \cdot (\rho_\chi \vec{v}) = 0, \tag{23}$$

$$\rho_\chi \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] = -\nabla P_\chi \left(\frac{\rho_\chi}{m_\chi} \right) - \rho_\chi \nabla \left(\frac{V}{m_\chi} \right) - \nabla V_Q, \tag{24}$$

where we have introduced the quantum potential $V_Q = -(\hbar^2/2m_\chi)\nabla^2 \sqrt{\rho_\chi}/\sqrt{\rho_\chi}$, and the velocity of the quantum fluid $\vec{v} = \nabla S/m_\chi$, respectively. The effective pressure of the condensate is given by

$$P_\chi \left(\frac{\rho_\chi}{m_\chi} \right) = u_0 \rho_\chi^2, \tag{25}$$

where

$$u_0 = \frac{2\pi \hbar^2 l_a}{m_\chi^3}. \tag{26}$$

The Bose-Einstein gravitational condensate can be described as a gas whose density and pressure are related by a polytropic equation of state, with index $n = 1$ [15]. When the number of particles in the gravitationally bounded Bose-Einstein condensate becomes large enough, the quantum pressure term makes a significant contribution only near the boundary of the condensate. Thus, the quantum stress term in the equation of motion of the condensate can be neglected. This is the Thomas-Fermi approximation, which has been extensively used for the study of the Bose-Einstein condensates [1–3]. As the number of particles in the condensate becomes infinite, the Thomas-Fermi approximation becomes exact. This approximation also corresponds to the classical limit of the theory. From its definition it follows that the velocity field is irrotational, satisfying the condition $\nabla \times \vec{v} = 0$.

The cosmological evolution of the energy density of the Bose-Einstein condensate is determined by the equation

$$\dot{\rho}_\chi + 3\rho_\chi \left(1 + \frac{u_0}{c^2} \rho_\chi \right) \frac{\dot{a}}{a} = 0, \tag{27}$$

with the general solution given by

$$\rho_\chi = \frac{C_\chi}{(a/a_0)^3 - (u_0/c^2)C_\chi}, \tag{28}$$

where C_χ is an arbitrary constant of integration. By using the condition that $\rho_\chi = \rho_{\chi,0}$ for $a = a_0$, we obtain the density of the condensate in the form

$$\rho_\chi = \frac{c^2}{u_0} \frac{\rho_{0\chi}}{(a/a_0)^3 - \rho_{0\chi}}, \tag{29}$$

where we have denoted

$$\rho_{0\chi} = \frac{\rho_{\chi,0} u_0/c^2}{1 + \rho_{\chi,0} u_0/c^2} = \frac{\rho_{cr,0} \Omega_{\chi,0} u_0/c^2}{1 + \rho_{cr,0} \Omega_{\chi,0} u_0/c^2}. \tag{30}$$

III. COSMOLOGICAL DYNAMICS OF THE BOSE-EINSTEIN CONDENSATION

The order of the phase transition of the Bose-Einstein condensation in an interacting Bose systems, i.e, the passage from the normal to the condensed phase where all particles occupy a single-particle state, has been intensively discussed in the recent physical literature. According to some results the Bose-Einstein condensation shows a spontaneous $U(1)$ gauge symmetry breaking, with the condensate fraction N_0/N playing the role of the order parameter [23]. That would suggest a second-order phase transition. However, by analyzing the dependence of the chemical potential μ on the temperature T and particle density ρ in the framework of several theoretical models, describing the thermodynamical transition from the normal to the BEC phase in weakly interacting Bose gases, it was shown that none of them predicts a second-order phase transition, as required by symmetry-breaking general considerations [24]. These results would imply that the Bose-Einstein condensation represents a first-order phase transition [24,25]. On the other hand, by definition, first-order phase transitions in thermodynamics feature a genuine mathematical singularity. Whether finite systems in nature can literally exhibit such a behavior is a long-standing controversial question in physics [26]. However, a rigorous investigation into the thermodynamic instability of an ideal Bose gas, confined in a cubic box, has shown that a system consisting of a finite number of particles can exhibit a discontinuous phase transition that features a genuine mathematical singularity, provided that not the volume, but the pressure is kept constant [25]. This result was obtained without assuming a thermodynamic limit or a continuous approximation. Hence, it seems currently that the best possible description of the intrinsic dynamics of the Bose-Einstein condensation can be obtained within the framework of a first-order phase transition.

A. Cosmological parameters at the condensation point

Generally, the chemical potential μ of a physical system is a function of the temperature and particle density $n = N/V$, $\mu = \mu(n, T)$, where N is the total particle number, and V is the volume. The extensivity property of the Helmholtz free energy $F = F(N, V, T)$ allows us to either write $F = Vf(n, T)$ or $F = N\tilde{f}(v, T)$, with $v = V/N = n^{-1}$, such that $f = n\tilde{f}$. Hence, both forms carry the same physical information. From the Helmholtz free energy we can obtain the chemical potential as $\mu(n, T) = (\partial f/\partial n)_T$ and the pressure as $p(n, T) = -(\partial \tilde{f}/\partial v)_T$. Therefore, $\mu = \mu(n, T)$ and $p = p(v, T)$ carry the same information.

The laws of thermodynamics require that both the chemical potential and the pressure are single valued, that is, for any given values of n and T, or v and T, there must only exist a single value of μ or p, respectively [24].

Therefore, a first thermodynamic condition that must be satisfied during the cosmological Bose-Einstein condensation process is the continuity of the pressure at the transition point. With the use of Eqs. (14) and (25) the continuity of the pressure uniquely fixes the critical transition density ρ_χ^{cr} from the normal dark matter state to the Bose-Einstein condensed state as

$$\rho_\chi^{cr} = \frac{c^2\sigma^2}{u_0} = \frac{c^2\sigma^2 m_\chi^3}{2\pi\hbar^2 l_a}. \tag{31}$$

The numerical value of the transition density depends on three unknown parameters, the dark matter particle mass, the scattering length, and the dark matter particles velocity dispersion, respectively. By assuming a typical mass of the dark matter particle of the order of 1 eV (1 eV $= 1.78 \times 10^{-33}$ g), a typical scattering length of the order of 10^{-10} cm, and a mean velocity square of the order of $\langle \vec{v}^2 \rangle = 81 \times 10^{14}$ cm^2/s^2, the critical transition density can be written as

$$\rho_\chi^{cr} = 3.868 \times 10^{-21} \left(\frac{\sigma^2}{3 \times 10^{-6}}\right) \times \left(\frac{m_\chi}{10^{-33} \text{ g}}\right)^3$$
$$\times \left(\frac{l_a}{10^{-10} \text{ cm}}\right)^{-1} \text{ g/cm}^3. \tag{32}$$

The critical temperature at the moment of Bose-Einstein condensate transition is given by [1–3]

$$T_{cr} \approx \frac{2\pi\hbar^2}{\zeta(3/2)^{2/3} m_\chi^{5/3} k_B} (\rho_\chi^{cr})^{2/3}$$
$$= \frac{(2\pi\hbar^2)^{1/3} c^{4/3}}{\zeta(3/2)^{2/3} k_B} \frac{(\sigma^2)^{2/3} m_\chi^{1/3}}{l_a^{2/3}}, \tag{33}$$

where $\zeta(3/2)$ is the Riemann zeta function, or

$$T_{cr} \approx 6.57 \times 10^3 \times \left(\frac{m_\chi}{10^{-33} \text{ g}}\right)^{1/3} \times \left(\frac{\sigma^2}{3 \times 10^{-6}}\right)^{2/3}$$
$$\times \left(\frac{l_a}{10^{-10} \text{ cm}}\right)^{-2/3} \text{ K}. \tag{34}$$

The critical pressure of the dark matter fluid at the condensation moment can be obtained as

$$P_{cr} = 1.04 \times 10^{-5} \left(\frac{\sigma^2}{3 \times 10^{-6}}\right)^2 \times \left(\frac{m_\chi}{10^{-33} \text{ g}}\right)^3$$
$$\times \left(\frac{l_a}{10^{-10} \text{ cm}}\right)^{-1} \text{ dyne/cm}^2. \tag{35}$$

The critical value a_{cr} of the scale factor of the Universe at the moment of the condensation can be obtained from Eq. (31) as

$$a_{cr}/a_0 = \left(\frac{\rho_{\chi,0} u_0}{c^2 \sigma^2}\right)^{1/3(1+\sigma^2)}$$
$$= \left(\frac{2\pi\hbar^2 l_a \rho_{cr,0} \Omega_{\chi,0}}{c^2 \sigma^2 m_\chi^3}\right)^{1/3(1+\sigma^2)}, \tag{36}$$

giving for the critical redshift of the transition the expression

$$1 + z_{cr} = \left(\frac{2\pi\hbar^2 l_a \rho_{cr,0} \Omega_{\chi,0}}{c^2 \sigma^2 m_\chi^3}\right)^{-1/3(1+\sigma^2)}. \tag{37}$$

By using the adopted numerical values of the constants we obtain for the critical scale factor and for the critical redshift the values

$$a_{cr}/a_0 = 8.17 \times 10^{-4} \times \left(\frac{m_\chi}{10^{-33} \text{ g}}\right)^{-(1+\sigma^2)}$$
$$\times \left(\frac{\sigma^2}{3 \times 10^{-6}}\right)^{-1/3(1+\sigma^2)} \times \left(\frac{l_a}{10^{-10} \text{ cm}}\right)^{1/3(1+\sigma^2)}, \tag{38}$$

and

$$1 + z_{cr} = 1.22 \times 10^3 \times \left(\frac{m_\chi}{10^{-33} \text{ g}}\right)^{(1+\sigma^2)}$$
$$\times \left(\frac{\sigma^2}{3 \times 10^{-6}}\right)^{1/3(1+\sigma^2)} \times \left(\frac{l_a}{10^{-10} \text{ cm}}\right)^{-1/3(1+\sigma^2)}, \tag{39}$$

respectively.

B. Cosmological evolution during the Bose-Einstein condensation phase

During a first-order phase transition, the temperature and the pressure are constants, $T = T_{cr}$ and $P = P_{cr}$, respectively. The entropy $S = sa^3$ and the enthalpy $W = (\rho + p)a^3$ are also conserved quantities. After the beginning of the phase transition the density of the dark matter $\rho_\chi(t)$ decreases from $\rho_\chi^{cr}(T_{cr}) \equiv \rho_\chi^{nor}$ (when all the dark matter is in a normal, noncondensed form) to $\rho_\chi(T_{cr}) \equiv \rho_\chi^{BEC}$, corresponding to the full conversion of dark matter into a condensed state. It is convenient to replace $\rho_\chi(t)$ by $h(t)$, the volume fraction of matter in the Bose-Einstein condensed phase, which is defined as

$$h(t) = \frac{\rho_\chi(t) - \rho_\chi^{nor}}{\rho_\chi^{BEC} - \rho_\chi^{nor}}. \tag{40}$$

Therefore, the evolution of the dark matter energy density during the transition process is given by

$$\rho_\chi(t) = \rho_\chi^{BEC} h(t) + \rho_\chi^{nor}[1 - h(t)] = \rho_\chi^{nor}[1 + n_\chi h(t)], \tag{41}$$

where we have denoted

$$n_\chi = \frac{\rho_\chi^{\mathrm{BEC}} - \rho_\chi^{\mathrm{nor}}}{\rho_\chi^{\mathrm{nor}}}. \qquad (42)$$

At the beginning of the Bose-Einstein condensation process $h(t_{\mathrm{cr}}) = 0$, where t_{cr} is the time corresponding to the beginning of the phase transition, and $\rho_\chi(t_{\mathrm{cr}}) \equiv \rho_\chi^{\mathrm{nor}}$. At the end of the condensation $h(t_{\mathrm{BEC}}) = 1$, where t_{BEC} is the time at which the phase transition ends, corresponding to $\rho_\chi(t_{\mathrm{BEC}}) \equiv \rho_\chi^{\mathrm{BEC}}$. For $t > t_{\mathrm{BEC}}$ the Universe enters in the Bose-Einstein condensed dark matter phase.

From Eq. (6) we obtain

$$\frac{\dot{a}}{a} = -\frac{1}{3} \frac{(\rho_\chi^{\mathrm{BEC}} - \rho_\chi^{\mathrm{nor}})\dot{h}}{\rho_\chi^{\mathrm{nor}} + P_{\mathrm{cr}}/c^2 + (\rho_\chi^{\mathrm{BEC}} - \rho_\chi^{\mathrm{nor}})h}$$
$$= -\frac{1}{3} \frac{r\dot{h}}{1 + rh}, \qquad (43)$$

where we have denoted

$$r = \frac{\rho_\chi^{\mathrm{BEC}} - \rho_\chi^{\mathrm{nor}}}{\rho_\chi^{\mathrm{nor}} + P_{\mathrm{cr}}/c^2} = \frac{n_\chi}{1 + P_{\mathrm{cr}}/\rho_\chi^{\mathrm{nor}}c^2}. \qquad (44)$$

Since $\rho_\chi^{\mathrm{BEC}} < \rho_\chi^{\mathrm{nor}}$, generally $r < 0$, $r \in (-1, 0)$, and $n_\chi < 0$, respectively. Equation (43) immediately leads to

the expression of the scale factor of the Universe during the Bose-Einstein condensation phase as

$$a(t) = a_{\mathrm{cr}}[1 + rh(t)]^{-1/3}, \qquad t \in (t_{\mathrm{cr}}, t_{\mathrm{BEC}}), \qquad (45)$$

where we have used the initial condition $h(t_{\mathrm{cr}}) = 0$ and we have denoted $a_{\mathrm{cr}} = a(t_{\mathrm{cr}})$. At the end of the phase transition the scale factor of the Universe has the value

$$a_{\mathrm{BEC}} = a(t_{\mathrm{BEC}}) = a_{\mathrm{cr}}(1 + r)^{-1/3}. \qquad (46)$$

The Bose-Einstein condensation modifies the overall expansion rate of the Universe. Consequently, during the phase transition, the pressureless baryonic matter and the radiation density evolve according to

$$\rho_b = \frac{\rho_{b,0}}{(a_{\mathrm{cr}}/a_0)^3}[1 + rh(t)], \qquad t \in (t_{\mathrm{cr}}, t_{\mathrm{BEC}}), \qquad (47)$$

and

$$\rho_{\mathrm{rad}} = \frac{\rho_{\mathrm{rad},0}}{(a_{\mathrm{cr}}/a_0)^4}[1 + rh(t)]^{4/3}, \qquad t \in (t_{\mathrm{cr}}, t_{\mathrm{BEC}}), \qquad (48)$$

respectively. The evolution of the fraction of the condensed matter $h(t)$ during the Bose-Einstein condensation process is described by the equation

$$\frac{dh}{d\tau} = -3\left(\frac{1}{r} + h\right) \times \sqrt{\frac{\Omega_{b,0}}{(a_{\mathrm{cr}}/a_0)^3}(1 + rh) + \frac{\Omega_{\mathrm{rad},0}}{(a_{\mathrm{cr}}/a_0)^4}(1 + rh)^{4/3} + \Omega_{\chi,\mathrm{nor}}(1 + n_\chi h) + \Omega_\Lambda}, \qquad (49)$$

where we have introduced a dimensionless time variable $\tau = H_0 t$, and we have denoted $\Omega_{\chi,\mathrm{nor}} = \rho_\chi^{\mathrm{nor}}/\rho_{\mathrm{cr},0}$, respectively. Since $P_{\mathrm{cr}}/\rho_\chi^{\mathrm{nor}}c^2 = \sigma^2 \ll 1$, from Eq. (44) it follows that $r \approx n_\chi$, an approximation we adopt in the following. If the energy density contribution of the radiation to the total energy density of the Universe can be neglected, Eq. (49) can be integrated exactly to give

$$h(t) = \frac{\Omega_\Lambda^2}{r\Omega_{\mathrm{tr}}} \times \left\{\frac{1 + \Omega_{\mathrm{cond}} \times \exp[-(3\Omega_\Lambda/t_H)(t - t_{\mathrm{cr}})]}{1 - \Omega_{\mathrm{cond}} \times \exp[-(3\Omega_\Lambda/t_H)(t - t_{\mathrm{cr}})]}\right\}^2$$
$$- \frac{\Omega_\Lambda + \Omega_{\mathrm{tr}}}{r\Omega_{\mathrm{tr}}}, \qquad (50)$$

where we have denoted

$$\Omega_{\mathrm{tr}} = \frac{\Omega_{b,0}}{(a_{\mathrm{cr}}/a_0)^3} + \Omega_{\chi,\mathrm{nor}}, \qquad (51)$$

and

$$\Omega_{\mathrm{cond}} = \frac{\sqrt{\Omega_{\mathrm{tr}} + \Omega_\Lambda} - \Omega_\Lambda}{\sqrt{\Omega_{\mathrm{tr}} + \Omega_\Lambda} + \Omega_\Lambda}, \qquad (52)$$

respectively.

The time interval necessary to convert the entire existing normal dark matter to the Bose-Einstein condensed phased is given by

$$\Delta t_{\mathrm{cond}}$$
$$= t_{\mathrm{BEC}} - t_{\mathrm{cr}}$$
$$= \frac{1}{3\Omega_\Lambda} \ln \frac{[\sqrt{\Omega_{\mathrm{tr}} + \Omega_\Lambda} - \Omega_\Lambda][\sqrt{\Omega_{\mathrm{tr}}(1 + r) + \Omega_\Lambda} + \Omega_\Lambda]}{[\sqrt{\Omega_{\mathrm{tr}} + \Omega_\Lambda} + \Omega_\Lambda][\sqrt{\Omega_{\mathrm{tr}}(1 + r) + \Omega_\Lambda} - \Omega_\Lambda]}$$
$$\times t_H. \qquad (53)$$

If the dark energy density can also be neglected, the volume fraction of matter $h(t)$ in the Bose-Einstein condensed phase evolves as

$$h(t) = \frac{1}{r}\left[1 + \frac{2}{3}H_0\sqrt{\Omega_{\mathrm{tr}}}(t - t_{\mathrm{cr}})\right]^{-2} - \frac{1}{r}. \qquad (54)$$

In this approximation the conversion of the entire dark matter of the Uuniverse to the BEC phase is given by

$$\Delta t_{\mathrm{cond}} = \frac{2}{3}[(1 + r)^{-1/2} - 1]\frac{t_H}{\sqrt{\Omega_{\mathrm{tr}}}}. \qquad (55)$$

By assuming the standard values $l_a = 10^{-10}$ cm, $m = 1$ eV, and $\sigma^2 = 3 \times 10^{-6}$ we obtain $\Omega_{\mathrm{tr}} = 5.02 \times 10^8$, giving for the total condensation time an approximate value of

$$\Delta t_{\mathrm{cond}} = 1.31 \times [(1 + r)^{-1/2} - 1] \times 10^{13} \text{ s}. \qquad (56)$$

The time evolution of $h(t)$, as given by Eq. (49) is represented, for several values of r, in Fig. 1.

C. The post-condensation phase

At $t = t_{BEC}$ and $a_{BEC} = a_{cr}(1 + r)^{-1/3}$, all the dark matter in the Universe has been converted to a Bose-Einstein condensate phase. At this moment the cosmological density of the dark matter is given by

$$\rho_\chi^{BEC} = \frac{c^2}{u_0} \frac{\rho_{0\chi}(1 + r)}{(a_{cr}/a_0)^3 - \rho_{0\chi}(1 + r)}, \tag{57}$$

where the constant $\rho_{0\chi}$ can be represented as

$$\rho_{0\chi} = \frac{1.63 \times 10^{-15} \times (l_a/10^{-10} \text{ cm})(m/10^{-33} \text{ g})^{-3}}{1 + 1.63 \times 10^{-15} \times (l_a/10^{-10} \text{ cm})(m/10^{-33} \text{ g})^{-3}}. \tag{58}$$

The condition of the positivity of the density imposes the constraint $(a_{cr}/a_0) > \rho_{0\chi}^{1/3}(1 + r)^{1/3}$ on the model parameters.

The equation determining the time evolution of the scale factor of the dark matter in the Bose-Einstein condensate phase is given by

$$\frac{d(a/a_0)}{dt} = H_0\sqrt{\Omega_{BE}} \frac{a/a_0}{\sqrt{(a/a_0)^3 - \rho_{0\chi}}}, \tag{59}$$

where we have denoted

$$\Omega_{BE} = \frac{\Omega_{\chi,0}}{1 + \rho_{\chi,0}u_0/c^2}, \tag{60}$$

and can be integrated exactly to give

$$\sqrt{\Omega_{BE}}H_0(t - C) = \frac{2}{3}\sqrt{\left(\frac{a}{a_0}\right)^3 - \rho_{0\chi}}$$
$$- \frac{2}{3}\sqrt{\rho_{0\chi}} \times \arctan\left[\sqrt{\frac{(a/a_0)^3 - \rho_{0\chi}}{\rho_{0\chi}}}\right], \tag{61}$$

where C is an arbitrary constant of integration, which can be determined from the condition $a = a_{BEC}$ at $t = t_{BEC}$. Thus, we obtain

$$C = t_{BEC} - \frac{2}{3\sqrt{\Omega_{BE}}H_0}\sqrt{\left(\frac{a_{BEC}}{a_0}\right)^3 + \rho_{0\chi}}$$
$$- \frac{2}{3} \times \sqrt{\frac{\rho_{0\chi}}{\Omega_{BE}H_0^2}}\arctan\left[\sqrt{\frac{(a_{BEC}/a_0)^3 - \rho_{0\chi}}{\rho_{0\chi}}}\right]. \tag{62}$$

In the case of a universe filled with dark energy, radiation, baryonic matter with negligible pressure, and Bose-Einstein condensed dark matter, respectively, the time evolution of the scale factor is given by the differential equation

$$\frac{1}{(a/a_0)}\frac{d(a/a_0)}{dt}$$
$$= H_0\sqrt{\frac{\Omega_{b,0}}{(a/a_0)^3} + \frac{\Omega_{rad,0}}{(a/a_0)^4} + \frac{\Omega_{BE}}{(a/a_0)^3 - \rho_{0\chi}} + \Omega_\Lambda},$$
$$t \geq t_{BEC}, \tag{63}$$

which must be integrated with the initial condition $a(t_{BEC}) = a(0) = a_{BEC}$. The time evolutions of the scale factors for universes containing BEC dark matter are represented, for different values of the BEC parameter $\rho_{0\chi}$, in Fig. 2.

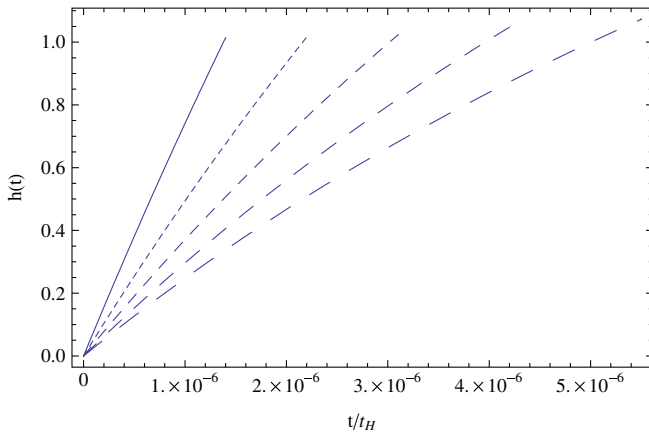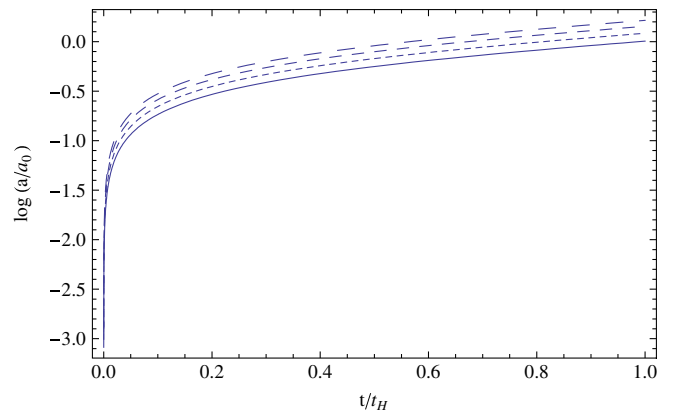

FIG. 1 (color online). Time evolution of the condensed dark matter fraction $h(t)$ for different values of r: $r = -0.10$ (solid curve), $r = -0.15$ (dotted curve), $r = -0.20$ (dashed curve), $r = -0.25$ (long dashed curve), and $r = -0.3$ (ultra-long dashed curve), respectively.

FIG. 2 (color online). Time evolution (in a logarithmic scale) of the scale factor of the Universe in the post–Bose-Einstein condensation phase, for $a_{BEC} = a(0) = 8.1 \times 10^{-4}$ ($z \approx 1200$), and for different values of $\rho_{0\chi}$: $\rho_{0\chi} = 10^{-11}$ (solid curve), $\rho_{0\chi} = 5 \times 10^{-11}$ (dotted curve), $\rho_{0\chi} = 10^{-10}$ (dashed curve), and $\rho_{0\chi} = 5 \times 10^{-10}$ (long dashed curve), respectively.

The presence of the condensate dark matter changes the global cosmological dynamics of the Universe in the post-condensation phase, and the magnitude of the changes increases with the increase of the BEC parameter $\rho_{0\chi}$.

IV. DISCUSSIONS AND FINAL REMARKS

The Bose-Einstein condensation process has been extensively studied (especially for the case of scalar fields) in the physical literature. However, many important questions have yet to be answered. For example, one would like to know more precisely if the condensation process of the dark matter takes place (almost) instantaneously, or if it evolves during a long time interval. Many details of the thermodynamics of the Bose-Einstein condensation are not yet conclusively understood. Even the order of the phase transition (crossover?) is still a matter of debate. And, of course, from a cosmological point of view, one would like to know more precisely in what cosmological epoch the condensation did occur, what were the values of the cosmological parameters at that time, how long the transition lasted, and what were its implications on the global evolution of the Universe. In the present paper we have tried to give some preliminary and qualitative answers to these questions.

Once the Bose-Einstein condensation occurs, dark matter becomes a mixture of two phases, the normal and the condensed phase, respectively. From a general thermodynamic point of view the coexistence of these two phases requires the continuity of the pressure of the two phases at the beginning of the condensation (phase transition). This thermodynamical condition uniquely fixes the value of the dark matter density at the condensation moment, as well as the values of all the other thermodynamical parameters (temperature and pressure). The concrete numerical values of the condensation parameters depend on the mean square velocity of the normal dark matter σ^2, and of the scattering length l_a and mass m_χ of the dark matter particle. Since the values of σ^2, l_a, and m_χ are not known for the dark matter particles, or are very uncertain (like m_χ and σ, for example), it is difficult to predict the exact cosmological moment of the Bose-Einstein condensation and the corresponding values of the cosmological parameters. However, by adopting some "standard" numerical values, one can obtain a qualitative picture of the transition. Thus, the mass of the dark matter particle was assumed to be of the order of 1 eV [7,27], while the mean velocity of the nonrelativistic dark matter particles was taken to be of the order of 900 km/s, which is somewhat larger than the values inferred from the study of the galactic dark matter halos [20,27]. The numerical value of the scattering length l_a for dark matter particles is very uncertain. As can be inferred from present experiments in ultracold gases, the scattering length is a quantity that determines the thermodynamic state of the gas. Terrestrial experiments on Bose-Einstein atomic condensates give a value of

10^{-7} cm [1–3]. For simplicity we have adopted for this parameter a value of $l_a = 10^{-10}$ cm.

The general analysis of the condensation process shows a distinct three-phase cosmological history of the Universe. In the first phase, the Universe evolved according to the standard Λ cold dark matter (ΛCDM) model. The condensation process of the dark matter did start when the temperature of the bosonic gas did fall below the critical value, and the equality of the pressure allowed the two phases to coexist simultaneously. With the adopted numerical values of the dark matter parameters the condensation began at a redshift of around $z = 1200$. The condensation process took place gradually, the normal and condensed phases coexisting for around 10^6 yr. To describe this transition period we have introduced the fraction $h(t)$ of the condensed dark matter, and studied its time evolution. After the transition phase the Universe entered in the condensed dark matter phase. Depending on the moment of its occurrence in the cosmological history, the presence of the condensate dark matter and of the Bose-Einstein phase transition could have modified drastically the cosmological evolution of the early universe, as well as the large scale structure formation process. Of course, even small changes in the numerical values of the parameter set (σ^2, l_a, m_χ) could lead to significant changes in the values of the critical redshift or duration of the condensation phase.

Immediately after the phase transition the presence of a Bose-Einstein condensate may drastically change the cosmological dynamics of the Universe. As one can see from Eq. (63), if for some time interval (a/a_0) is very close to $\rho_{0\chi}^{1/3}$, the cosmological dynamic of the Universe is determined by the condensed dark matter. In this case the condensed dark matter energy density $\rho_\chi^{\text{BEC}} = \Omega_{\text{BE}}/[(a/a_0)^3 - \rho_{0\chi}]$ is very large, and dominates all the other cosmological energy terms. When this condition is fulfilled, the expansion of the Universe enters in an accelerated phase. For the time interval for which ρ_χ^{BE} can be approximated as a constant, $\rho_\chi^{\text{BE}} \approx$ constant, the expansion of the Universe can be described as a de Sitter one. The presence of an accelerating expansion period after the phase transition can also be seen from the analysis of the deceleration parameter $q = -a\ddot{a}/\dot{a}^2 = d(1/H)/dt - 1$ of the Universe. For $(a/a_0) \rightarrow \rho_{0\chi}^{1/3}$, the scale factor of the Universe with Bose-Einstein condensate dark matter, given by Eq. (62), can be approximated as

$$\frac{a}{a_0} \approx \left[\frac{9}{4} \Omega_{\text{BE}} H_0^2 (t - C)^2 + \rho_{0\chi} \right]^{1/3}. \quad (64)$$

During this phase the deceleration parameter q is given by

$$q \approx \frac{1}{2} - \frac{2\rho_{0\chi}}{3\Omega_{\text{BE}} H_0^2 (t - C)^2}. \quad (65)$$

During the time interval given by $t < C + \sqrt{(4/3)\rho_{0\chi}/\Omega_{\text{BE}} H_0^2}$ the deceleration parameter satisfies

the condition $q < 0$. However, in the limit of large times, the expansion of the Universe decelerates, with the deceleration parameter given by $q \approx 1/2$.

Therefore, the Bose-Einstein condensation of the dark matter may also provide the explanation for the observed recent acceleration of the Universe, which could be just a transient phenomenon related to the recent entering of the Universe in the post–Bose-Einstein condensation phase. Hence BEC could also provide an alternative, physically realistic solution to the dark energy problem (for reviews of the dark energy problem see [8]). However, such a model would require that the cosmological BEC phase transition took place very recently, at $z \approx 1$–2, on a global cosmic scale. Currently there is no astrophysical evidence for such a scenario.

A central problem in the theory of the Bose-Einstein condensation is the nature of the dark matter particle, as well as its physical properties. The core/cusp problem can be solved if the dark matter is composed of ultralight scalar particles with mass $m = 10^{-22}$ eV, initially in a (cold) Bose-Einstein condensate [11]. The wave properties of the dark matter stabilize gravitational collapse, providing halo cores and sharply suppressing small-scale linear power. Ultralight dark matter particles with masses of the order of $m \sim 10^{-23}$ eV and with Compton wavelengths of the order of galactic scales have temperatures of about 0.9 K, a temperature substantially lower than the temperature of CMB neutrinos. Therefore, such a mass would guarantee that big bang nucleosynthesis remains unaffected. In addition, the temperature is consistent with WMAP observations without fine-tuning [28]. It was shown that axions with mass $m \approx 6 \times 10^{-6}$ eV 10^{12} GeV$/f$ also form a BEC, where $f \gtrsim 10^9$ GeV [14]. This would imply an axion mass of the order of $m \approx 10^{-3}$ eV. The axions were postulated shortly after the standard model of elementary particles was established to explain why the strong interactions

conserve the discrete symmetries P and CP (for more details see [14] and references therein). A cosmological bound on the mass of the condensate particle can be obtained as $m < 2.696(g_d/g)(T_d/T_{\mathrm{cr}})^3$ eV [27], where g is the number of internal degrees of freedom of the particle before decoupling, g_d is the number of internal degrees of freedom of the particle at the decoupling, and T_d is the decoupling temperature. In the Bose condensed case $T_d/T_c < 1$, and it follows that the BEC particle should be light, unless it decouples very early on, at high temperature and with a large g_d. Therefore, depending on the relation between the critical and the decoupling temperatures, in order for a BEC light relic to act as cold dark matter, the decoupling scale must be higher than the electroweak scale [27]. On the other hand, from Eq. (1) for $l_a \approx 1$ fm and $R \approx 10$ kpc, it follows that the typical mass of the condensate particle is of the order of $m \approx 14$ meV. For $l_a \approx 10^6$ fm, corresponding to the values of a observed in terrestrial laboratory experiments, $m \approx 1.44$ eV. These values are consistent with the limit $m < 1.87$ eV obtained for the mass of the condensate particle from cosmological considerations [6].

A better understanding of the numerical values of the Bose-Einstein condensation parameters (scattering length and dark matter particle mass), would be very helpful in obtaining accurate cosmological conclusions in the framework of the BEC model. Such an advance may also provide a powerful method for observationally testing on a cosmological scale the theoretical predictions of the Bose-Einstein condensation model and the possible existence of the condensed dark matter.

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