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On the Admissible Equilibrium Points of Nonlinear Dynamical Systems Affected by Parametric Uncertainty: Characterization via LMIs

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Abstract— **This paper investigates the set of admissible equilibrium points of nonlinear dynamical systems affected by parametric uncertainty. As it is well-known, determining this set is a difficult problem since one should compute the solutions of a system of nonlinear equations for all the admissible values of the uncertainty, which typically amounts to an infinite number of times. In order to address this problem, this paper proposes a characterization of this set via convex optimization for the case of polynomial nonlinearities and uncertainty constrained in a polytope. Specifically, it is shown that an upper bound of the smallest outer estimate with a freely selectable fixed shape can be obtained by solving a linear matrix inequality (LMI) problem built through the square matrix representation (SMR). Then, a necessary and sufficient condition is provided for establishing the tightness of the found upper bound. The proposed methodology and its benefits are illustrated through several numerical examples.**

I. INTRODUCTION

It is well-known that analyzing and designing a control system often requires the knowledge of the equilibrium points of a nonlinear dynamical system. In fact, this knowledge is exploited in various tasks, such as establishing the stability of the steady states, their domains of attractions, the input-output properties of the system when working in a neighborhood of a steady state of interest, etc. See e.g. [1], [2] and reference therein.

The determination of the equilibrium points amounts to solving a system of nonlinear equations for a given mathematical model of a nonlinear dynamical system. However, mathematical models of real nonlinear systems are almost always affected by uncertainty. This is due, for example, to the fact that the coefficients of the model (such as friction, mass and stiffness in a mechanical systems) cannot be measured exactly. Another reason is that these system often present some variable components, such as potentiometers in electric circuits, in order to allow one to vary some system performance. As a consequence, a family of admissible nonlinear systems has to be considered, and hence the determination of the equilibrium points has to be repeated for all admissible values of the uncertainty.

Clearly, it is generally undesirable, and often even infeasible, repeating the determination of the equilibrium points for all admissible uncertainties. In fact, determining the equilibrium points is a nontrivial problem even in the case of nonlinear systems without uncertainty: this is either due to the computational burden of symbolic tools or to the fact that numerical techniques do not guarantee to find all solutions. Moreover, the set of admissible values of the uncertainty is typically not finite, and considering a finite grid only could easily miss key equilibrium points achievable by the system.

This paper proposes a characterization of the set of admissible equilibrium points via convex optimization. Specifically, nonlinear dynamical systems affected by parametric uncertainty are considered in the case of polynomial nonlinearities and polynomial dependence on the uncertainty constrained in a polytope. Hence, the problem of determining the smallest outer estimate with a freely selectable fixed shape is considered. This estimate is expressed as a sublevel set of a given polynomial, and it is shown that an upper bound of the optimal level can be obtained by solving an eigenvalue problem (EVP), which belongs to the class of convex optimization problems with LMI constraints [3]. This EVP is constructed by adopting the square matrix representation (SMR) and by introducing a suitable expression of parameter-dependent polynomials and a parameter-dependent polynomial multiplier whose degrees allow one to regulate the conservatism of the found upper bound. Moreover, a necessary and sufficient condition for establishing the tightness of the found upper bound is provided. The use of the proposed methodology and its benefits are illustrated through several numerical examples.

The paper is organized as follows. Section II provides the problem formulation and some preliminaries about the representation of polynomials. Section III describes the proposed methodology. Section IV illustrate some numerical examples. Lastly, Section V concludes the paper with some final remarks.

Before proceeding it is useful mentioning that LMI techniques have been proposed for solving systems of polynomial equations, see for instance [4], [5] and references therein. Also, it is worth mentioning that methodologies for the study and design of uncertain nonlinear systems have been proposed in the literature, mainly by assuming that the system has an equilibrium point of interest that does not change with the uncertainty. This is the case, for instance, of [6], [7] that investigate robust stability in uncertain nonlinear systems, and of [8], [9] that consider stability analysis and synthesis in nonlinear switching systems.

II. PRELIMINARIES

A. Problem formulation

The notation used throughout the paper is as follows: $\mathbb{N}, \mathbb{R}$: natural number set (including 0) and real number set; 0_n: origin of $\mathbb{R}^n$; $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$; I_n: $n \times n$ identity matrix; A':

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transpose of the vector/matrix A; $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A; $\mathrm{conv}(\mathcal{S})$: convex hull of the elements in the set $\mathcal{S}$; $\mathrm{vol}(\mathcal{S})$: volume of the set $\mathcal{S}$; $\|x\| = \sqrt{x'x}$ with $x \in \mathbb{R}^n$; $x^y = x_1^{y_1} \cdots x_n^{y_n}$ with $x, y \in \mathbb{R}^n$; s.t.: subject to.

Let us consider the uncertain nonlinear system

$$\left\{ \begin{array}{lcl} \dot{x} & = & f(x,\theta) \\ \theta & \in & \Theta \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $\theta \in \mathbb{R}^q$ is the time-invariant uncertain vector, and $\Theta \subset \mathbb{R}^q$ is the set of admissible values for θ. It is supposed that Θ is a bounded convex polytope, expressed as

$$\Theta = \mathrm{conv}\left(\left\{\theta^{(1)}, \ldots, \theta^{(r)}\right\}\right) \tag{2}$$

where $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathbb{R}^q$ are given vectors, and $\mathrm{conv}(\cdot)$ denotes the convex hull. The function $f(x,\theta) \in \mathbb{R}^n$ is a vector polynomial (i.e., a vector of polynomials) in x and θ, and we denote with d_1 the degree of $f(x,\theta)$ in x for fixed θ and with d_2 the degree in θ for fixed x, i.e.

$$f(x,\theta) = \sum_{\substack{i \in \mathbb{N}^n,\ i_1 + \ldots + i_n \leq d_1 \\ j \in \mathbb{N}^q,\ j_1 + \ldots + j_q \leq d_2}} c_{i,j} x^i \theta^j \tag{3}$$

for some vector coefficients $c_{i,j} \in \mathbb{R}^n$. The set of admissible equilibrium points of (1) is the set of equilibrium points that this system can own for different values of θ in Θ, and is given by

$$\mathcal{E} = \{x \in \mathbb{R}^n :\ f(x,\theta) = 0_n \text{ for some } \theta \in \Theta\}. \tag{4}$$

The problem addressed in this paper consists of determining outer estimates of $\mathcal{E}$ of the form

$$\mathcal{G}(\gamma) = \{x \in \mathbb{R}^n :\ g(x) \leq \gamma\} \tag{5}$$

where $g(x)$ is a given polynomial of degree $2d_g$ and $\gamma \in \mathbb{R}$. In particular, the problem consists of estimating the smallest outer estimate of $\mathcal{E}$ with fixed shape defined by $g(x)$, which is denoted by $\mathcal{G}(\gamma^*)$ where γ^* is the solution of the optimization problem

$$\gamma^* = \inf_{\gamma \geq 0} \gamma \quad \text{s.t.} \quad \mathcal{E} \subseteq \mathcal{G}(\gamma). \tag{6}$$

This problem will be addressed in Section III-A by providing an upper bound of γ^* through a convex optimization problem. Moreover, a necessary and sufficient condition for establishing tightness of the found upper bound will be provided in Section III-B.

B. SMR

Before proceeding we briefly introduce a key tool that will be exploited in the next sections to derive the proposed conditions. For $x \in \mathbb{R}^n$, let $p(x)$ be a polynomial of degree $2d$. Let $x_{pol}^{\{d\}} \in \mathbb{R}^{\sigma(n,d)}$ be a vector containing all monomials of degree less than or equal to d in x, where $\sigma(n,d)$ is the number of such monomials given by

$$\sigma(n,d) = \frac{(n+d)!}{n!d!}. \tag{7}$$

Then, $p(x)$ can be expressed via the square matrix representation (SMR) introduced in [10] as

$$p(x) = x_{pol}^{\{d\}'} \left(P + L(\alpha)\right) x_{pol}^{\{d\}} \tag{8}$$

where $P = P' \in \mathbb{R}^{\sigma(n,d) \times \sigma(n,d)}$ is a symmetric matrix such that $p(x) = x_{pol}^{\{d\}'} P x_{pol}^{\{d\}}$, $L(\alpha) = L(\alpha)' \in \mathbb{R}^{\sigma(n,d) \times \sigma(n,d)}$ is a linear parametrization of the set

$$\mathcal{L} = \left\{L = L' :\ x_{pol}^{\{d\}'} L x_{pol}^{\{d\}} = 0\right\}, \tag{9}$$

and $\alpha \in \mathbb{R}^{\sigma_{\mathcal{L}}}$ is a vector of free parameters, where ω is the dimension of the linear subspace $\mathcal{L}$ given by

$$\sigma_{\mathcal{L}} = \frac{1}{2}\sigma(n,m)(\sigma(n,m)+1) - \sigma(n,2m). \tag{10}$$

The matrices P and $P + L(\alpha)$ are referred to as SMR matrix and complete SMR matrix, respectively, of $p(x)$. The matrix P is also known as Gram matrix of $p(x)$.

Homogeneous polynomials can be represented with a more compact SMR. Specifically, let $h(x)$ be a homogeneous polynomial of degree $2d$, and let $x_{hom}^{\{d\}} \in \mathbb{R}^{\sigma(n-1,d)}$ be a vector containing all monomials of degree d in x. Then, $h(x)$ can be expressed via the SMR as

$$h(x) = x_{hom}^{\{d\}'} \left(H + L(\alpha)\right) x_{hom}^{\{d\}} \tag{11}$$

where H and $L(\alpha)$ are defined analogously to the previous case.

The SMR is useful because it allows one to investigate positivity of polynomials. Indeed, one can establish whether a polynomial is a sum of squares of polynomials (SOS) by solving a convex optimization problem with linear matrix inequalities (LMIs). Specifically, $p(x)$ (resp., $h(x)$) is SOS if and only if there exists α such that

$$P + L(\alpha) \geq 0 \quad \text{(resp., } H + L(\alpha) \geq 0\text{)} \tag{12}$$

which is an LMI feasibility test since P (resp., H) is constant and $L(\alpha)$ is a linear matrix function, see for instance [10]. LMI feasibility tests can be checked by solving a convex optimization problem, see for instance [3]. See also [11], [12] for details and algorithms about the SMR and SOS polynomials.

In the sequel it will be assumed that the first entry of any vector $x_{pol}^{\{d\}}$ is 1, e.g.

$$x_{pol}^{\{d\}} = (1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^d)'. \tag{13}$$

III. ESTIMATING THE SET OF ADMISSIBLE EQUILIBRIUM POINTS

Here we address the computation of outer estimates with fixed shape. Specifically, Section III-A considers the computation of an upper bound of the smallest outer estimate with fixed shape. Then, Section III-B provides a necessary and sufficient condition for establishing the tightness of the found upper bound.

A. Upper Bound Computation

First of all, let us express a generic θ in the set Θ as

$$\theta = \sum_{i=1}^{r} \phi_i \theta^{(i)} \tag{14}$$

where $\phi = (\phi_1, \ldots, \phi_r)'$ is a vector in the simplex Φ given by

$$\Phi = \left\{ \phi \in \mathbb{R}^r : \ \sum_{i=1}^{r} \phi_i = 1, \ \phi_i \geq 0 \right\}. \tag{15}$$

Let $c_{i,j}$ be the generic vector coefficient of $f(x, \theta)$ in (3) and let us define

$$h(x, \phi) = \sum_{\substack{i \in \mathbb{N}^n, \ i_1 + \ldots + i_n \leq d_1 \\ j \in \mathbb{N}^q, \ j_1 + \ldots + j_r \leq d_2}} c_{i,j} x^i a^j b^{d_2 - j_1 - \ldots - j_r} \tag{16}$$

where

$$a = \sum_{i=1}^{r} \phi_i \theta^{(i)}, \quad b = \sum_{i=1}^{r} \phi_i. \tag{17}$$

It turns out that each entry of the function $h(x, \phi) \in \mathbb{R}^n$ is a homogeneous polynomial in ϕ of degree d_2 for any fixed x and a polynomial in x of degree d_1 for any fixed ϕ. Moreover,

$$\begin{array}{l} \forall \theta \in \Theta \ \exists \phi \in \Phi : \ f(x, \theta) = h(x, \phi) \\ \forall \phi \in \Phi \ \exists \theta \in \Theta : \ h(x, \phi) = f(x, \theta). \end{array} \tag{18}$$

This implies that (1) can be rewritten as

$$\left\{ \begin{array}{lcl} \dot{x} & = & h(x, \phi) \\ \phi & \in & \Phi \end{array} \right. \tag{19}$$

and hence $\mathcal{E}$ is given by

$$\mathcal{E} = \{ x \in \mathbb{R}^n : \ h(x, \phi) = 0_n \text{ for some } \phi \in \Phi \}. \tag{20}$$

Let us introduce the notation

$$\mathrm{sq}(\phi) = \left(\phi_1^2, \ldots, \phi_r^2 \right)' \tag{21}$$

and

$$\Delta(A, b, c) = (b \otimes c)' A (b \otimes c) \tag{22}$$

where b and c are vectors and A is a matrix of suitable dimension. Let us define the integers

$$\begin{array}{lcl} d_3 & = & \left\lceil \dfrac{d_1 + d_x}{2} \right\rceil - d_g \\ d_4 & = & d_2 + d_\phi \end{array} \tag{23}$$

for some $d_x, d_\phi \in \mathbb{N}$ (with d_x such that $d_3 \geq 0$). Let us define the polynomial

$$u(x, \phi) = U \left(x_{pol}^{\{d_x\}} \otimes \phi_{hom}^{\{d_\phi\}} \right) \tag{24}$$

where $U \in \mathbb{R}^{n \times \sigma(n, d_x) \sigma(r-1, d_\phi)}$ is a variable matrix to be determined. Let $H(U) = H(U)'$, $V = V'$ and $W = W'$ be any symmetric matrix functions of suitable dimension satisfying

$$\begin{array}{rcl} u(x, \mathrm{sq}(\phi))' h(x, \mathrm{sq}(\phi)) & = & \Delta \left(H(U), x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}} \right) \\ a(x, \mathrm{sq}(\phi)) g(x) & = & \Delta \left(V, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}} \right) \\ a(x, \mathrm{sq}(\phi)) & = & \Delta \left(W, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}} \right) \end{array} \tag{25}$$

where

$$a(x, \phi) = \left(1 + \|x\|^2 \right)^{d_3} \left(\sum_{i=1}^{r} \phi_i \right)^{d_4}. \tag{26}$$

Lastly, let $N(\alpha)$ be any linear parametrization of the linear subspace

$$\mathcal{N} = \left\{ N = N' : \ \Delta \left(N, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}} \right) = 0 \right\} \tag{27}$$

where α is a free vector of suitable dimension, and let us define the notation

$$\mathrm{jrp}(\phi) = \frac{\left(\sqrt{\phi_1}, \ldots, \sqrt{\phi_r} \right)'}{\sum_{i=1}^{r} \sqrt{\phi_i}}. \tag{28}$$

The following result provides an upper bound of γ^* in (6) via a convex optimization problem.

Theorem 1: Let $g(x) \in \mathcal{P}_{n, 2d_g}$ be given. Define the optimization problem

$$\gamma^\# = \inf_{\gamma, U, \alpha} \gamma \ \text{ s.t. } \ H(U) - V + \gamma W + N(\alpha) > 0. \tag{29}$$

Then, $\gamma^\# \geq \gamma^*$.

Proof. Let us suppose that the LMI in (29) is fulfilled for some $\bar{\gamma}, \bar{U}, \bar{\alpha}$. Let us consider any $\bar{x}$ in $\mathcal{E}$, and let $\bar{\phi}$ be a vector of admissible uncertain parameters corresponding to $\bar{x}$, i.e. such that

$$\left\{ \begin{array}{l} h(\bar{x}, \bar{\phi}) = 0_n \\ \bar{\phi} \in \Phi. \end{array} \right.$$

Since $\bar{\phi}_i \geq 0$ for all $i = 1, \ldots, r$, we can define the vector

$$\bar{\psi} = \mathrm{jrp}(\bar{\phi}).$$

Let us pre- and post- multiply the LMI by $\left(\bar{x}_{pol}^{\{d_3\}} \otimes \bar{\psi}_{hom}^{\{d_4\}} \right)'$ and $\bar{x}_{pol}^{\{d_3\}} \otimes \bar{\psi}_{hom}^{\{d_4\}}$, respectively. It follows that

$$\begin{array}{rcl} 0 & < & \Delta \left(H(\bar{U}) - V + \bar{\gamma} W + N(\bar{\alpha}), \bar{x}_{pol}^{\{d_3\}}, \bar{\psi}_{hom}^{\{d_4\}} \right) \\ & = & \bar{u}(\bar{x}, \mathrm{sq}(\bar{\psi}))' h(\bar{x}, \mathrm{sq}(\bar{\psi})) - a(\bar{x}, \mathrm{sq}(\bar{\psi})) g(\bar{x}) \\ & & + \bar{\gamma} a(\bar{x}, \mathrm{sq}(\bar{\psi})) \end{array}$$

where $\bar{u}(\bar{x}, \mathrm{sq}(\bar{\psi}))$ is given by (24) for $U = \bar{U}$, and where it has been taken into account that

$$\Delta \left(N(\bar{\alpha}), \bar{x}_{pol}^{\{d_3\}}, \bar{\psi}_{hom}^{\{d_4\}} \right) = 0.$$

Let us observe that

$$\mathrm{sq}(\bar{\psi}) = c \bar{\phi}$$

where

$$c = \left(\sum_{i=1}^{r} \sqrt{\bar{\phi}_i} \right)^{-2}.$$

Since $h(x, \phi)$ is a homogeneous polynomial of degree d_2 in ϕ, it follows that

$$h(\bar{x}, c\bar{\phi}) = c^{d_2} h(\bar{x}, \bar{\phi}).$$

Hence, from $h(\bar{x}, \bar{\phi}) = 0_n$ one has that

$$0 < -a(\bar{x}, c\bar{\phi}) g(\bar{x}) + \bar{\gamma} a(\bar{x}, c\bar{\phi}).$$

Finally, let us observe that $a(\bar{x}, c\bar{\phi}) > 0$ since

$$a(\bar{x}, c\bar{\phi}) = c^{d_2}\left(1 + \|\bar{x}\|^2\right)^{d_3}$$

and $c > 0$, which implies that $g(\bar{x}) < \bar{\gamma}$. Hence, $\mathcal{E} \subseteq \mathcal{G}(\bar{\gamma})$, and therefore $\gamma^{\#} \geq \gamma^*$. □

Theorem 1 provides the upper bound $\gamma^{\#}$ of γ^* in (6), and hence the outer estimate $\mathcal{G}(\gamma^{\#})$ of $\mathcal{E}$. This upper bound is obtained by solving the optimization problem (29), which is an EVP and belongs to the class of convex optimization problems with LMI constraints [3]. This EVP is constructed by introducing a suitable expression of parameter-dependent polynomials via the function $\Delta(\cdot, x_{pol}^{\{d_3\}}, \phi_{hom}^{\{d_4\}})$ and the parameter-dependent polynomial multiplier $u(x, \phi)$. The conservatism of the upper bound $\gamma^{\#}$ decreases by increasing the degrees d_x and d_ϕ of $u(x, \phi)$.

B. Establishing Tightness

In Section III-A we have shown how an upper bound $\gamma^{\#}$ of γ^* can be computed by solving an EVP. A question that naturally arises concerns the tightness of the found upper bound: is $\gamma^{\#} = \gamma^*$?

The following result provides an answer to this question, by proposing a necessary and sufficient condition for establishing whether the found upper bound $\gamma^{\#}$ is tight.

Theorem 2: Let $U^{\#}, \alpha^{\#}$ be the optimal values of U, α in (29), and define

$$J = H(U^{\#}) - V + \gamma^{\#} W + N(\alpha^{\#}). \tag{30}$$

Then, $\gamma^{\#} = \gamma^*$ if and only if there exist $x \in \mathbb{R}^n$ and $\psi \in \mathbb{R}_0^r$ such that

$$\left\{ \begin{array}{l} \left(x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}\right) \in \ker(J) \\ h(x, \mathrm{prj}(\psi)) = 0_n \\ g(x) = \gamma^{\#} \end{array} \right. \tag{31}$$

where

$$\mathrm{prj}(\psi) = \frac{\mathrm{sq}(\psi)}{\|\psi\|^2}. \tag{32}$$

Proof. "$\Rightarrow$" Let us suppose that $\gamma^{\#} = \gamma^*$. Let $x^* \in \mathcal{E}$ be a tangent point between $\mathcal{E}$ and $\mathcal{G}(\gamma^*)$, and let $\phi^* \in \Phi$ be a vector of admissible uncertain parameters corresponding to x^*, i.e.

$$\left\{ \begin{array}{l} g(x^*) = \gamma^* \\ h(x^*, \phi^*) = 0_n \\ \phi^* \in \Phi. \end{array} \right.$$

Since $\phi_i^* \geq 0$ for all $i = 1, \ldots, r$, we can define the vector $\psi^* = \mathrm{jrp}(\phi^*)$. Let us observe that $\mathrm{sq}(\psi^*) = c\phi^*$ where $c > 0$. Due to the fact that $h(x, \phi)$ is a homogeneous polynomial of degree d_2 in ϕ, one has that

$$\begin{array}{rcl} h(x^*, \mathrm{sq}(\psi^*)) & = & c^{d_2} h(x^*, \phi^*) \\ & = & 0. \end{array}$$

Let us observe that $J \geq 0$ since J is the left-hand side of the LMI in (29) evaluated for the optimal values of the EVP.

Let us pre- and post- multiply J by $\left(x^{*\{d_3\}}_{pol} \otimes \psi^{*\{d_4\}}_{hom}\right)'$ and $x^{*\{d_3\}}_{pol} \otimes \psi^{*\{d_4\}}_{hom}$, respectively. It follows that

$$\begin{array}{rcl} 0 & \leq & \Delta\left(J, x^{*\{d_3\}}_{pol}, \psi^{*\{d_4\}}_{hom}\right) \\ & = & u^{\#}(x^*, \mathrm{sq}(\psi^*))' h(x^*, \mathrm{sq}(\psi^*)) \\ & & -a(x^*, \mathrm{sq}(\psi^*)) g(x^*) + \gamma^* a(x^*, \mathrm{sq}(\psi^*)) \\ & = & 0 \end{array}$$

since $g(x^*) = \gamma^*$ and $h(x^*, \mathrm{sq}(\psi^*)) = 0$, where $u^{\#}(x^*, \mathrm{sq}(\psi^*))$ is given by (24) for $U = U^{\#}$. Since $J \geq 0$, this implies that the vector $x^{*\{d_3\}}_{pol} \otimes \psi^{*\{d_4\}}_{hom}$ must belong to the null space of J. Moreover, one has that

$$\begin{array}{rcl} \mathrm{prj}(\psi^*) & = & \mathrm{prj}(\mathrm{jrp}(\phi^*)) \\ & = & \phi^* \end{array}$$

and hence (31) holds.

"$\Leftarrow$" Let us suppose that (31) holds for some $x \in \mathbb{R}^n$ and $\psi \in \mathbb{R}_0^r$. Let us observe that

$$\mathrm{prj}(\psi) \in \Phi$$

which means that x is an admissible equilibrium point of the system, i.e. $x \in \mathcal{E}$. Moreover, x satisfies $g(x) = \gamma^{\#}$, i.e. x lies on the boundary of $\mathcal{G}(\gamma^{\#})$. This implies that

$$\gamma^{\#} \leq \gamma^*,$$

and since $\gamma^{\#}$ is an upper bound of γ^* from Theorem 1, one finally has that $\gamma^{\#} = \gamma^*$. □

As explained in Theorem 2, a sufficient and necessary condition for the found upper bound $\gamma^{\#}$ to be tight is the existence of vectors $x \in \mathbb{R}^n$ and $\psi \in \mathbb{R}_0^r$ satisfying (31). From the first condition in (31), it follows that these vectors have to satisfy

$$x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}} = J_1 y \tag{33}$$

where J_1 is a matrix whose columns form a base of the null space of J, and y is a vector of suitable dimension.

A way to verify the existence of vectors $x \in \mathbb{R}^n$ and $\psi \in \mathbb{R}_0^r$ fulfilling (33) for some y and determine them is as follows. First, let us observe that the vector $x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}}$ has the structure

$$x_{pol}^{\{d_3\}} \otimes \psi_{hom}^{\{d_4\}} = \begin{pmatrix} \psi_{hom}^{\{d_4\}} \\ x_1 \psi_{hom}^{\{d_4\}} \\ \vdots \\ x_n^{d_3} \psi_{hom}^{\{d_4\}} \end{pmatrix}. \tag{34}$$

Let $J_1^{(0)}, \ldots, J_1^{(\sigma(n,d_3))}$ be the sub-matrices of J_1, with $J_1^{(i)}$ containing the rows of J_1 from the $(i-1)\sigma(r-1, d_4)+1$-th row to the $i\sigma(r-1, d_4)$-th row. It follows from (33) that

$$\psi_{hom}^{\{d_4\}} = J_1^{(0)} y. \tag{35}$$

The vectors ψ and y satisfying (35) can be found with the approach proposed in [4], [12]. Once ψ and y have been determined, the vector x fulfilling (33) can directly be read from (34) according to

$$x_i \psi_{hom}^{\{d_4\}} = J_1^{(i)} y. \tag{36}$$

IV. EXAMPLES

Here we present some examples of the proposed methodology. The problem (29) is solved by using the toolbox SeDuMi for Matlab on a standard computer. The parameter-dependent polynomial multiplier $u(x, \phi)$ is built as in (24) with $d_x = 1$ and $d_\phi = 0$.

A. Example 1

Let us consider the uncertain nonlinear system

$$\begin{cases} \dot{x}_1 &= (1+3\theta)x_1^2 + 2x_1x_2 + (4-3\theta)x_2^2 + x_2 - 2 \\ \dot{x}_2 &= x_1^2 + (2-4\theta)x_1x_2 + 2x_2^2 - 2x_1 - 2 \end{cases}$$

where θ is the uncertain time-invariant parameter satisfying

$$\theta \in [0, 1].$$

This system can be written in the form of (19) with $\phi = (\phi_1, \phi_2)'$, $\theta = \phi_1$ and

$$\begin{array}{rcl} h_1(x,\phi) &=& (4\phi_1+\phi_2)x_1^2 + (\phi_1+4\phi_2)x_2^2 \\ && +(\phi_1+\phi_2)(2x_1x_2+x_2-2) \\ h_2(x,\phi) &=& (\phi_1+\phi_2)(x_1^2+x_2^2-2x_1-2) \\ && +(2\phi_2-2\phi_1)x_1x_2. \end{array}$$

Let us select the shape function $g(x) = \|x\|^2$. From Theorem 1 we find that an upper bound of γ^* is given by $\gamma^\# = 2.147$ (the computational time is 1.6 seconds). Figure 1a shows the boundary of the found estimate $\mathcal{G}(\gamma^\#)$. This figure also shows the equilibrium points computed for 101 values of θ equally distributed in $[0, 1]$.

In order to establish whether the found upper bound $\gamma^\#$ is tight, we use Theorem 2, in particular (31) holds with $\psi^\# = (0.774, 0.634)'$ and $x^\# = (0.925, -1.136)'$. This implies that $\gamma^\#$ is tight, i.e. $\gamma^\# = \gamma^*$. Moreover, from Theorem 2 we have that $x^\#$ is an equilibrium point of the system achieved for the uncertain parameter

$$\phi^\# = \mathrm{prj}(\psi^\#) = (0.599, 0.401)'.$$

Figure 1b shows the equilibrium points for $\phi^\#$.

It is worth observing that the equilibrium points shown in Figure 1a for 101 values of θ are unable to determine the outer estimate of $\mathcal{E}$: in fact, none of these lies on the boundary of $\mathcal{G}(\gamma^\#)$. Such an extreme point is obtained for $\phi^\#$ found via Theorem 2 and is shown in Figure 1b.

B. Example 2

Let us consider the uncertain nonlinear system

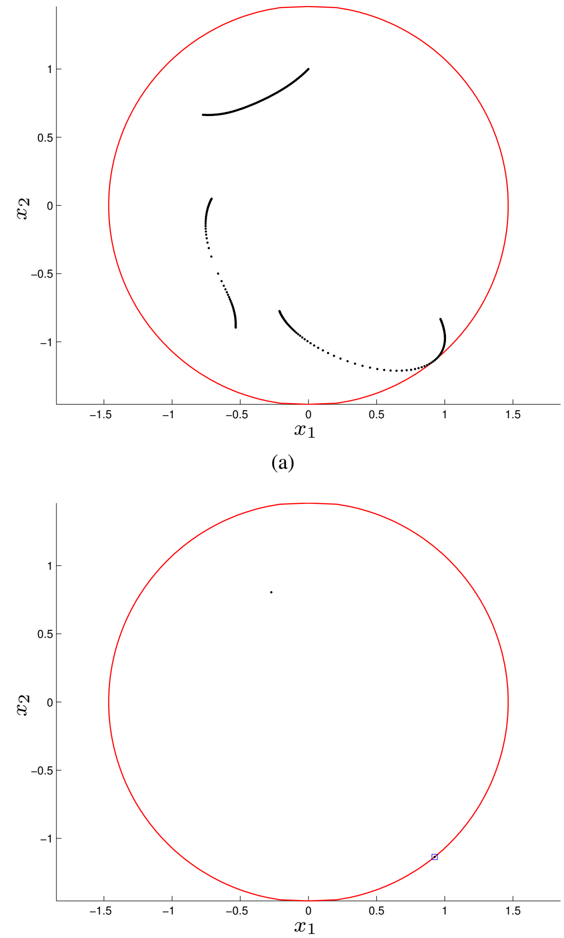
$$\begin{cases} \dot{x}_1 &= x_1^2 + (3.5+2.5\theta_2)x_2^2 + (1+3\theta_1)x_2 - 3 \\ \dot{x}_2 &= (3-2\theta_1)x_1^2 + (1+3\theta_2)x_1x_2 + 2x_2^2 - 2x_2 - 8 \end{cases}$$

where $\theta = (\theta_1, \theta_2)' \in \mathbb{R}^2$ is the uncertain time-invariant parameter satisfying

$$\theta \in [-1, 1]^2.$$

This system can be written as in (19) with $\phi = (\phi_1, \ldots, \phi_4)'$ and

$$\theta = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \phi.$$



(a)

(b)

Fig. 1. Example 1. (a) Boundary of the estimate $\mathcal{G}(\gamma^\#)$ for $g(x) = \|x\|^2$ (red disc) and equilibrium points for 101 values of θ equally distributed in $[0, 1]$ (black dots). (b) Equilibrium points for the uncertain parameter $\phi^\#$ found with Theorem 2 (the blue square is $x^\#$).

Let us select the shape function $g(x) = \|x\|^2$. We find that an upper bound of γ^* is given by $\gamma^\# = 7.918$ (the computational time is 3.4 seconds). Figure 2a shows the boundary of $\mathcal{G}(\gamma^\#)$ and the equilibrium points computed for 289 values of θ equally distributed in $[0, 1]^2$.

In order to establish whether the found upper bound $\gamma^\#$ is tight, we use Theorem 2, in particular (31) holds with $\psi^\# = (0.993, 0.000, 0.117, 0.000)'$ and $x^\# = (0.751, 2.712)'$. This implies that $\gamma^\#$ is tight, i.e. $\gamma^\# = \gamma^*$. Moreover, from Theorem 2 we have that $x^\#$ is an equilibrium point of the system achieved for the uncertain parameter

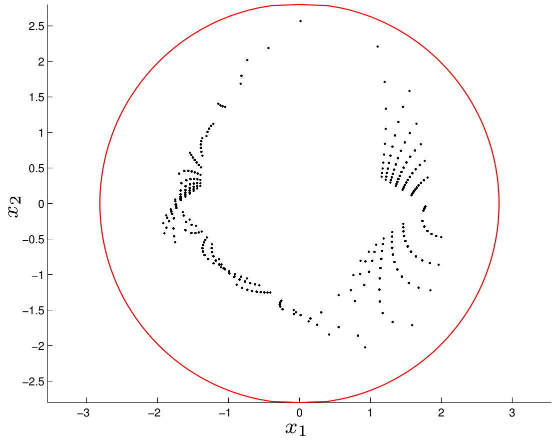
$$\phi^\# = \mathrm{prj}(\psi^\#) = (0.986, 0.000, 0.014, 0.000)'.$$

Figure 2b shows the equilibrium points for $\phi^\#$.

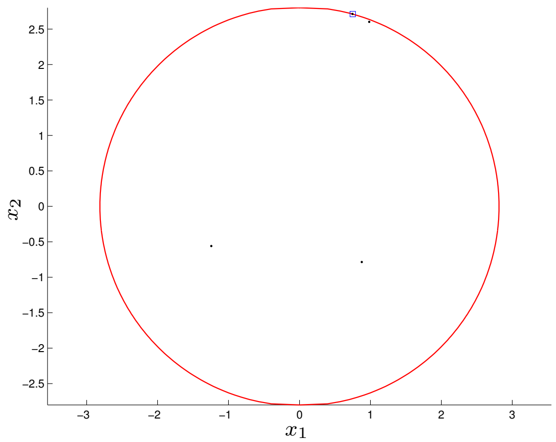
C. Example 3

Let us consider the uncertain nonlinear system

$$\begin{cases} \dot{x}_1 &= 1 + x_1^2 + x_1^3 + x_2^3 + (3\theta-4)x_3^3 \\ \dot{x}_2 &= 1 + x_2^2 - x_1^3 + (\theta+1)x_2^3 + x_3^3 \\ \dot{x}_3 &= 1 + x_3^2 + (2\theta+1)x_1^3 - x_2^3 + x_3^3 \end{cases}$$



(a)



(b)

Fig. 2. Example 2. (a) Boundary of the estimate $\mathcal{G}(\gamma^{\#})$ (red disc) and equilibrium points for 289 values of θ equally distributed in $[0,1]^2$ (black dots). (b) Equilibrium points for the uncertain parameter $\phi^{\#}$ found with Theorem 2 (the blue square is $x^{\#}$).

where θ is the uncertain time-invariant parameter satisfying

$$\theta \in [0,1].$$

This system can be written in the form of (19) with $\phi = (\phi_1, \phi_2)'$, $\theta = \phi_1$, and $h(x,\phi)$ given by (16).

Let us select the shape function $g(x) = \|x\|^2$. We find that an upper bound of γ^* is given by $\gamma^{\#} = 15.294$.

In order to establish whether the found upper bound $\gamma^{\#}$ is tight, we use Theorem 2, in particular (31) holds with $\psi^{\#} = (0.000, 1.000)'$ and $x^{\#} = (-2.417, -2.501, -1.789)'$. This implies that $\gamma^{\#}$ is tight, i.e. $\gamma^{\#} = \gamma^*$. Moreover, from Theorem 2 we have that $x^{\#}$ is an equilibrium point of the system achieved for the uncertain parameter

$$\phi^{\#} = \mathrm{prj}(\psi^{\#}) = (0.000, 1.000)'.$$

It is worth observing that determining the equilibrium point of this system for a fixed value of θ takes more than 13 minutes with symbolic functions of Matlab (which means that considering a hundred values of θ as in Examples 1 and 2 would take more than 21 hours). Instead, the computational time of $\gamma^{\#}$ is 3.0 seconds.

V. CONCLUSION

The problem of characterizing the set of admissible equilibrium points of nonlinear dynamical systems affected by parametric uncertainty has been addressed for the case of polynomial nonlinearities and uncertainty constrained in a polytope. Specifically, it has been shown that an upper bound of the smallest outer estimate with a freely selectable fixed shape can be obtained by solving an LMI problem. Then, a necessary and sufficient condition has been provided for establishing the tightness of the found upper bound. Future work will be devoted to extend the proposed methodology to the case of estimates with variable shape.

The benefit of the proposed methodology is twofold. First, determining outer estimates of the set of admissible equilibrium points would require to repeat the determination of the equilibrium points an infinite number of times while this paper provides outer estimates by solving a convex optimization problem. Second, the determination of the equilibrium points can hardly be done even for fixed values of the uncertainty (either due to the computational burden of symbolic tools or due to the fact that numerical techniques do not guarantee to find all solutions), while the computational burden of the proposed techniques is indeed reasonable.

VI. ACKNOWLEDGMENTS

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REFERENCES

[1] D. D. Šiljak, *Nonlinear Systems: Parametric Analysis and Design*. John Wiley & Sons, 1969.

[2] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 2001.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[4] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "An LMI-based approach for characterizing the solution set of polynomial systems," in *IEEE Conf. on Decision and Control*, Sydney, Australia, 2000, pp. 1501–1506.

[5] G. Chesi, "Solving polynomial systems via LMI: worst and best representation matrices," *Int. Journal of Applied Mathematics and Statistics*, vol. 14, no. 9, pp. 47–59, 2009.

[6] ——, "Estimating the domain of attraction for uncertain polynomial systems," *Automatica*, vol. 40, no. 11, pp. 1981–1986, 2004.

[7] U. Topcu and A. Packard, "Local stability analysis for uncertain nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 54, no. 5, pp. 1042–1047, 2009.

[8] G. Chesi, "On the minimum stable commutation time for switching nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 54, no. 6, pp. 1284–1289, 2009.

[9] J. Xu, L. Xie, and Y. Wang, "Simultaneous stabilization and robust control of polynomial nonlinear systems using sos techniques," *IEEE Trans. on Automatic Control*, vol. 54, no. 8, pp. 1892–1897, 2009.

[10] G. Chesi, A. Tesi, A. Vicino, and R. Genesio, "On convexification of some minimum distance problems," in *European Control Conf.*, Karlsruhe, Germany, 1999.

[11] G. Chesi, "On the gap between positive polynomials and SOS of polynomials," *IEEE Trans. on Automatic Control*, vol. 52, no. 6, pp. 1066–1072, 2007.

[12] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems*. Springer, 2009.