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Research Article

On Lyapunov-Type Inequalities for Two-Dimensional Nonlinear Partial Systems

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We establish a new Laypunov-type inequality for two nonlinear systems of partial differential equations and the discrete analogue is also established. As application, boundness of the two-dimensional Emden-Fowler-type equation is proved.

1. Introduction

In a celebrated paper of 1893, Liapunov [1] proved the following well-known inequality: if y is a nontrivial solution of

$$y'' + q(t)y = 0, \tag{1.1}$$

on an interval containing the points a and b $(a < b)$ such that $y(a) = y(b) = 0$, then

$$4 < (b-a)\int_a^b |q(s)|\,ds. \tag{1.2}$$

Since the appearance of Liapunov's fundamental paper [1], considerable attention has been given to various extensions and improvements of the Lyapunov-type inequality from different viewpoints [2–7]. In particular, the Lyapunov-type inequalities for the following nonlinear system of differential equations were given in [8]

$$\begin{aligned} x'(t) &= \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), \\ u'(t) &= -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t). \end{aligned} \tag{1.3}$$

In this paper, we obtain new Lyapunov-type inequalities for the two-dimensional nonlinear system and discrete nonlinear system, respectively.

2. The Lyapunov-Type Integral Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned} \frac{\partial^2 x(s,t)}{\partial s\partial t} &= \alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t), \\ \frac{\partial^2 u(s,t)}{\partial s\partial t} &= -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t) - \alpha_1(s,t)u(s,t). \end{aligned} \tag{2.1}$$

We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (2.1), and furthermore, (2.1) satisfies the following assumptions (i), (ii), and (iii):

(i) $\gamma > 1, \beta > 1$ are real constants;

(ii) $\beta_1(s,t)$, $\beta_2(s,t) : [s_0,\infty) \times [t_0,\infty) \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that $\beta_1(s,t) > 0$ for $(s,t) \in [s_0,\infty) \times [t_0,\infty)$;

(iii) $\alpha_1(s,t) : [s_0,\infty) \times [t_0,\infty) \to \mathbb{R}$ is a continuous function.

Theorem 2.1. *Let the hypotheses (i)–(iii) hold. If the nonlinear system* (2.1) *has a real solution* $(x(s,t), u(s,t))$ *such that* $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ *for* $(s,t) \in [a,b] \times [c,d]$, *and* $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t) + (\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s)$ *and* $x(s,t)$ *is not identically zero on* $[a,b] \times [c,d]$, *where* $a,b,c,d \in \mathbb{R}$ *with* $a < b$, $c < d$, *then*

$$2 \le \int_a^b \int_c^d |\alpha_1(s,t)| dt\, ds + M^{\beta/\alpha-1} \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t) dt\, ds \right)^{1/\alpha}, \tag{2.2}$$

where $(1/\alpha) + (1/\gamma) = 1$, $M = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)|$, *and* $\beta_2^+(s,t) = \max_{\substack{a<s<b \\ c<t<d}} \{\beta_2(s,t), 0\}$ *is the nonnegative part of* $\beta_2(s,t)$.

Proof. Since $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $x(s,t)$ is not identically zero on $[a,b] \times [c,d]$, we can choose $(\tau,\sigma) \in (a,b) \times (c,d)$ such that $|x(\tau,\sigma)| = \max_{\substack{a<s<b \\ c<t<d}} |x(s,t)| > 0$.

Let $M = |x(\tau,\sigma)| > 0$. Integrating the first equation of system (2.1) over t from c to σ and over s from a to τ, respectively, we obtain

$$\int_a^\tau\int_c^\sigma \frac{\partial^2 x(s,t)}{\partial s\partial t}dt\,ds = \int_a^\tau\int_c^\sigma \Big(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\Big)dt\,ds. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}\int_a^\tau\int_c^\sigma \frac{\partial^2 x(s,t)}{\partial s\partial t}dt\,ds &= \int_a^\tau\int_c^\sigma \frac{\partial}{\partial t}\left(\frac{\partial x(s,t)}{\partial s}\right)dt\,ds\\ &= \int_a^\tau\left[\int_c^\sigma \frac{\partial x(s,t)}{\partial s}\bigg|_t dt\right]ds\\ &= \int_a^\tau \frac{\partial x(s,\sigma)}{\partial s}ds - \int_a^\tau \frac{\partial x(s,c)}{\partial s}ds\\ &= x(\tau,\sigma) - x(a,\sigma) - x(\tau,c) + x(a,c)\\ &= x(\tau,\sigma).\end{aligned} \tag{2.4}$$

Hence,

$$x(\tau,\sigma) = \int_a^\tau\int_c^\sigma \Big(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\Big)dt\,ds, \tag{2.5}$$

and similarly, we have

$$x(\tau,\sigma) = \int_\tau^b\int_\sigma^d \Big(\alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t)\Big)dt\,ds. \tag{2.6}$$

Employing the triangle inequality gives

$$|x(\tau,\sigma)| \le \int_a^\tau\int_c^\sigma |\alpha_1(s,t)||x(s,t)|dt\,ds + \int_a^\tau\int_c^\sigma \beta_1(s,t)|u(s,t)|^{\gamma-1}dt\,ds, \tag{2.7}$$

$$|x(\tau,\sigma)| \le \int_\tau^b\int_\sigma^d |\alpha_1(s,t)||x(s,t)|dt\,ds + \int_\tau^b\int_\sigma^d \beta_1(s,t)|u(s,t)|^{\gamma-1}dt\,ds. \tag{2.8}$$

Summing (2.7) and (2.8), we obtain

$$2|x(\tau,\sigma)| \le \int_a^b\int_c^d |\alpha_1(s,t)||x(s,t)|dt\,ds + \int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma-1}dt\,ds. \tag{2.9}$$

By using Hölder inequality on the second integral of the right side of (2.9) with indices α and γ, we have

$$\begin{aligned}
&\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\gamma-1} dt\, ds \\
&\quad = \int_a^b \int_c^d \beta_1(s,t)^{1/\gamma} \beta_1(s,t)^{1/\alpha} |u(s,t)|^{\gamma-1} dt\, ds \\
&\quad \leq \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)} dt\, ds \right)^{1/\alpha} \\
&\quad = \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\gamma} dt\, ds \right)^{1/\alpha},
\end{aligned} \tag{2.10}$$

where $(1/\alpha) + (1/\gamma) = 1$.

Therefore, we obtain from (2.9)

$$\begin{aligned}
2|x(\tau,\sigma)| \leq &\int_a^b \int_c^d |\alpha_1(s,t)||x(s,t)| dt\, ds \\
&+ \left(\int_a^b \int_c^d \beta_1(s,t) dt\, ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_1(s,t)|u(s,t)|^{\gamma} dt\, ds \right)^{1/\alpha}.
\end{aligned} \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial t}(x(s,t)u(s,t)) &= \frac{\partial}{\partial t}\left(\frac{\partial x(s,t)}{\partial s} \cdot u(s,t) + x(s,t) \cdot \frac{\partial u(s,t)}{\partial s} \right) \\
&= \frac{\partial^2 x(s,t)}{\partial s \partial t} \cdot u(s,t) + \frac{\partial x(s,t)}{\partial s} \cdot \frac{\partial u(s,t)}{\partial t} \\
&\quad + \frac{\partial x(s,t)}{\partial t} \cdot \frac{\partial u(s,t)}{\partial s} + x(s,t) \cdot \frac{\partial^2 u(s,t)}{\partial s \partial t}.
\end{aligned} \tag{2.12}$$

Multiplying the first equation of (2.1) by $u(s,t)$ and the second one by $x(s,t)$, adding the result, and noting $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t) + (\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s) = 0$, we have

$$\frac{\partial^2}{\partial s \partial t}[x(s,t)u(s,t)] = \beta_1(s,t)|u(s,t)|^{\gamma} - \beta_2(s,t)|x(s,t)|^{\beta}. \tag{2.13}$$

Integrating the left side of (2.13) over t from c to d and over s from a to b, respectively, we get

$$\begin{aligned}
&\int_a^b\int_c^d \frac{\partial^2}{\partial s\partial t}[x(s,t)u(s,t)]dt\,ds \\
&= \int_a^b\int_c^d \frac{\partial}{\partial t}\left[\frac{\partial(x(s,t)u(s,t))}{\partial s}\right]dt\,ds \\
&= \int_a^b\left[\int_c^d \frac{\partial(x(s,t)u(s,t))}{\partial s}\Big|_t\,dt\right]ds \\
&= \int_a^b \frac{\partial(x(s,d)u(s,d))}{\partial s}ds - \int_a^b \frac{\partial(x(s,c)u(s,c))}{\partial s}ds \\
&= x(b,d)u(b,d) - x(a,d)u(a,d) - x(b,c)u(b,c) + x(a,c)u(a,c).
\end{aligned} \tag{2.14}$$

Now integrating both sides of (2.13) over t from c to d and over s from a to b, respectively, and noting $x(a,t) = x(b,t) = 0$, we get

$$\int_a^b\int_c^d \beta_1(s,t)|u(s,t)|^{\gamma}dt\,ds = \int_a^b\int_c^d \beta_2(s,t)|x(s,t)|^{\beta}dt\,ds. \tag{2.15}$$

Substituting equality (2.15) by (2.11), we have

$$\begin{aligned}
2|x(\tau,\sigma)| \le &\int_a^b\int_c^d |\alpha_1(s,t)||x(s,t)|dt\,ds \\
&+ \left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_2(s,t)|x(s,t)|^{\beta}dt\,ds\right)^{1/\alpha}.
\end{aligned} \tag{2.16}$$

Noticing that $M = |x(\tau,\sigma)| = \max_{\substack{a<s<b\\c<t<d}}|x(s,t)| > 0$ and $\beta_2^{+}(s,t) = \max_{\substack{a<s<b\\c<t<d}}\{\beta_2(s,t),0\}$, we obtain

$$2 \le \int_a^b\int_c^d |\alpha_1(s,t)|dt\,ds + M^{\beta/\alpha-1}\left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_2^{+}(s,t)dt\,ds\right)^{1/\alpha}. \tag{2.17}$$

The proof is complete. □

Remark 2.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$, and $\beta_1(s,t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (2.2), and with suitable changes, (2.2) changes to the following result:

$$2 \le \int_a^b |\alpha_1(t)|dt + M^{\beta/\alpha-1}\left(\int_a^b \beta_1(t)dt\right)^{1/\gamma}\left(\int_a^b \beta_2^+(t)dt\right)^{1/\alpha}. \tag{2.18}$$

This is just a new Lyapunov-type inequality which was given by Tiryaki et al. [8].

3. The Lyapunov-Type Discrete Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned}\Delta_1\Delta_2 x(s,t) &= \alpha_1(s,t)x(s+1,t+1) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\\ \Delta_1\Delta_2 u(s,t) &= -\beta_2(s,t)|x(s+1,t+1)|^{\beta-2}x(s+1,t+1) - \alpha_1(s,t)u(s,t),\end{aligned} \tag{3.1}$$

where $s,t \in \mathbb{Z}$, Δ_1 denotes the forward difference operator for s, that is, $\Delta_1 x(s,t) = x(s+1,t) - x(s,t)$, and Δ_2 denotes the forward difference operator for t, that is, $\Delta_2 x(s,t) = x(s,t+1) - x(s,t)$. We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (3.1), and furthermore, (3.1) satisfies the following assumptions (i), (ii), and (iii):

(i) $\gamma > 1, \beta > 1$ are real constants;

(ii) $\beta_1(s,t), \beta_2(s,t)$ are real-valued functions such that $\beta_1(s,t) > 0$ for all $s,t \in \mathbb{Z}$;

(iii) $\alpha_1(s,t)$ is a real-valued function for all $s,t \in \mathbb{Z}$.

Theorem 3.1. *Let the hypotheses* (*i*)–(*iii*) *hold. Assume* $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ *and* $n_1 < m_1 - 2, n_2 < m_2 - 2$. *If the nonlinear system* (3.1) *has a real solution* $(x(s,t), u(s,t))$ *such that* $x(n_1,t) = x(m_1,t) = x(s,n_2) = x(s,m_2) = 0$ *for all* $(s,t) \in [n_1,m_1] \times [n_2,m_2]$, *and* $\Delta_1 x(s,t+1) \cdot \Delta_2 u(s,t) + \Delta_2 x(s+1,t) \cdot \Delta_1 u(s,t) = 0$ *and* $x(s,t)$ *is not identically zero on* $[n_1,m_1] \times [n_2,m_2]$, *then*

$$2 \le \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)| + M^{\beta/\alpha-1}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^+(s,t)\right)^{1/\alpha}, \tag{3.2}$$

where $(1/\alpha) + (1/\gamma) = 1, M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}|x(s,t)|$, *and* $\beta_2^+(s,t) = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}\{\beta_2(s,t), 0\}$.

Proof. Let $(x(s,t), u(s,t))$ be nontrivial real solution of system (3.1) such that $x(n_1,t) = x(m_1,t) = x(s,n_2) = x(s,m_2) = 0$ and $x(s,t)$ is not identically zero on $[n_1,m_1] \times [n_2,m_2]$.

Then multiplying the first equation of (3.1) by $u(s,t)$ and the second one by $x(s+1,t+1)$, adding the result, and noting $\Delta_1 x(s,t+1)\cdot\Delta_2 u(s,t)+\Delta_2 x(s+1,t)\cdot\Delta_1 u(s,t)=0$, and

$$\begin{aligned}
&\Delta_1\Delta_2[x(s,t)u(s,t)]\\
&\quad= \Delta_2((x(s+1,t)-x(s,t))u(s,t)+x(s+1,t)(u(s+1,t)-u(s,t)))\\
&\quad= \Delta_2((x(s+1,t)-x(s,t))u(s,t))+\Delta_2(x(s+1,t)(u(s+1,t)-u(s,t)))\\
&\quad= (x(s+1,t+1)-x(s,t+1)-(x(s+1,t)-x(s,t)))u(s,t)\\
&\qquad+(x(s+1,t+1)-x(s,t+1))(u(s,t+1)-u(s,t))\\
&\qquad+(x(s+1,t+1)-x(s+1,t))(u(s+1,t)-u(s,t))\\
&\qquad+x(s+1,t+1)(u(s+1,t+1)-u(s,t+1)-(u(s+1,t)-u(s,t)))\\
&\quad= (\Delta_1\Delta_2 x(s,t))u(s,t)+\Delta_1 x(s,t+1)\Delta_2 u(s,t)\\
&\qquad+\Delta_2 x(s+1,t)\Delta_1 u(s,t)+x(s+1,t+1)(\Delta_1\Delta_2 u(s,t)),
\end{aligned} \tag{3.3}$$

we have

$$\Delta_1\Delta_2[x(s,t)u(s,t)]=\beta_1(s,t)|u(s,t)|^{\gamma}-\beta_2(s,t)|x(s+1,t+1)|^{\beta}. \tag{3.4}$$

Summing the left side of (3.4) over t from n_2 to m_2-1 and over s from n_1 to m_1-1, respectively, we have

$$\begin{aligned}
&\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\Delta_1\Delta_2(x(s,t)u(s,t))\\
&\quad=\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}(x(s+1,t+1)u(s+1,t+1)-x(s+1,t)u(s+1,t)\\
&\qquad\qquad-(x(s,t+1)u(s,t+1)-x(s,t)u(s,t)))\\
&\quad=\sum_{s=n_1}^{m_1-1}(x(s+1,m_2)u(s+1,m_2)-x(s,m_2)u(s,m_2)\\
&\qquad\quad-(x(s+1,n_2)u(s+1,n_2)-x(s,n_2)u(s,n_2)))\\
&\quad=x(m_1,m_2)u(m_1,m_2)-x(n_1,m_2)u(n_1,m_2)-x(m_1,n_2)u(m_1,n_2)\\
&\qquad+x(n_1,n_2)u(n_1,n_2).
\end{aligned} \tag{3.5}$$

Summing both sides of (3.4) over t from n_2 to m_2-1 and over s from n_1 to m_1-1, respectively, and noting $x(n_1,t)=x(m_1,t)=0$, we obtain

$$\sum_{s=n_1}^{m_1-1}\sum_{t=\mathrm{m}_1}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}=\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_2(s,t)|x(s+1,t+1)|^{\beta}. \tag{3.6}$$

Noticing that $x(m_1,t)=x(s,m_2)=0$ and $\beta_2^+(s,t)=\max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}\{\beta_2(s,t),0\}$, we have

$$\begin{aligned}\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}&=\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2(s,t)|x(s+1,t+1)|^{\beta}\\&\le\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^+(s,t)|x(s+1,t+1)|^{\beta}.\end{aligned}\tag{3.7}$$

Choose $(\tau,\sigma)\in[n_1+1,m_1-1]\times[n_2+1,m_2-1]$ such that $M=|x(\tau,\sigma)|=\max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}}|x(s,t)|$. Hence $M=|x(\tau,\sigma)|>0$. Summing the first equation of (3.1) over t from n_2 to $\sigma-1$ and over s from n_1 to $\tau-1$, respectively, we obtain

$$\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\Delta_1\Delta_2x(s,t)=\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\alpha_1(s,t)x(s+1,t+1)+\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t).\tag{3.8}$$

Considering the left side of (3.8) and noting $x(n_1,t)=x(s,n_2)=0$ for all $(s,t)\in[n_1,m_1]\times[n_2,m_2]$, we have

$$\begin{aligned}\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\Delta_1\Delta_2x(s,t)&=\sum_{s=n_1}^{\tau-1}\left(\sum_{t=n_2}^{\sigma-1}(x(s+1,t+1)-x(s+1,t)-(x(s,t+1)-x(s,t)))\right)\\&=\sum_{s=n_1}^{\tau-1}(x(s+1,\sigma)-x(s,\sigma)-(x(s+1,n_2)-x(s,n_2)))\\&=x(\tau,\sigma)-x(n_1,\sigma)-x(\tau,n_2)+x(n_1,n_2)\\&=x(\tau,\sigma).\end{aligned}\tag{3.9}$$

Hence,

$$x(\tau,\sigma)=\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\alpha_1(s,t)x(s+1,t+1)+\sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\tag{3.10}$$

and similarly, we have

$$x(\tau,\sigma)=\sum_{s=\tau}^{m_1-2}\sum_{t=\sigma}^{m_2-2}\alpha_1(s,t)x(s+1,t+1)+\sum_{s=\tau}^{m_1-1}\sum_{t=\sigma}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t).\tag{3.11}$$

Employing the triangle inequality gives

$$|x(\tau,\sigma)| \le \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=n_1}^{\tau-1}\sum_{t=n_2}^{\sigma-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}, \tag{3.12}$$

$$|x(\tau,\sigma)| \le \sum_{s=\tau}^{m_1-2}\sum_{t=\sigma}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=\tau}^{m_1-1}\sum_{t=\sigma}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}. \tag{3.13}$$

Summing (3.12) and (3.13), we obtain

$$2|x(\tau,\sigma)| \le \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)| + \sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1}. \tag{3.14}$$

On the other hand, using Hölder inequality on the second sum of the right side of (3.14) with indices α and γ, we have

$$\begin{aligned}\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1} &= \sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)^{1/\gamma}\beta_1(s,t)^{1/\alpha}|u(s,t)|^{\gamma-1}\\ &\le \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\alpha(\gamma-1)}\right)^{1/\alpha}\\ &= \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma}\right)^{1/\alpha},\end{aligned} \tag{3.15}$$

where $(1/\alpha)+(1/\gamma)=1$. Therefore, from (3.7) and (3.10), we obtain

$$\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)|u(s,t)|^{\gamma-1} \le \left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)|x(s+1,t+1)|^{\beta}\right)^{1/\alpha}. \tag{3.16}$$

Substituting (3.16) to (3.14), we have

$$\begin{aligned}2|x(\tau,\sigma)| \le{}& \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)||x(s+1,t+1)|\\ &+\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)|x(s+1,t+1)|^{\beta}\right)^{1/\alpha}.\end{aligned} \tag{3.17}$$

Noticing that $M = |x(\tau,\sigma)| = \max_{\substack{n_1+1<s<m_1-1\\ n_2+1<t<m_2-1}} |x(s,t)| > 0$, we get

$$2 \leq \sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}|\alpha_1(s,t)| + M^{\beta/\alpha-1}\left(\sum_{s=n_1}^{m_1-1}\sum_{t=n_2}^{m_2-1}\beta_1(s,t)\right)^{1/\gamma}\left(\sum_{s=n_1}^{m_1-2}\sum_{t=n_2}^{m_2-2}\beta_2^{+}(s,t)\right)^{1/\alpha}. \tag{3.18}$$

This completes the proof. □

Remark 3.2. Let $x(s,t), u(s,t), \alpha_1(s,t)$, and $\beta_1(s,t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (3.2) and with suitable changes, (3.2) changes to the following result:

$$2 \leq \sum_{t=n}^{m-2}|\alpha_1(t)| + M^{\beta/\alpha-1}\left(\sum_{t=n}^{m-1}\beta_1(t)\right)^{1/\gamma}\left(\sum_{t=n}^{m-2}\beta_2^{+}(t)\right)^{1/\alpha}. \tag{3.19}$$

This is just a new Lyapunov-type inequality which was given by Ünal et al. [2].

4. An application

Two-dimensional Emden-Fowler-type equation

$$\frac{\partial}{\partial s\partial t}\left(r(s,t)\left|\frac{\partial x(x,t)}{\partial s\partial t}\right|^{\alpha-2}\frac{\partial x(x,t)}{\partial s\partial t}\right) + q(s,t)|x(s,t)|^{\beta-2}x(s,t) = 0, \tag{4.1}$$

where $\alpha > 1$ is a constant, $r(s,t)$ and $q(s,t)$ are real functions, and $r(s,t) > 0$ for all $(s,t) \in \mathbb{R}\times\mathbb{R}$.

Consider the following special case of system (2.1), which is an equivalent system for the two-dimensional Emden-Fowler-type equation (4.1)

$$\begin{aligned}\frac{\partial x^2(s,t)}{\partial s\partial t} &= \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t),\\ \frac{\partial u^2(s,t)}{\partial s\partial t} &= -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t),\end{aligned} \tag{4.2}$$

where $\beta_1(s,t) = r(s,t)^{1-\gamma}$ and $\beta_2(s,t) = q(s,t)$.

Obviously Theorem 2.1 for the two-dimensional nonlinear system (2.1) with $\alpha_1(s,t) \equiv 0$ is satisfied for system (4.2). Therefore, we have

$$2 \leq M^{\beta/\alpha-1}\left(\int_a^b\int_c^d \beta_1(s,t)dt\,ds\right)^{1/\gamma}\left(\int_a^b\int_c^d \beta_2^{+}(s,t)dt\,ds\right)^{1/\alpha}. \tag{4.3}$$

A nontrivial solution $(x(s,t),u(s,t))$ of system (4.2) defined on $[s_0,\infty)\times[t_0,\infty)$ is said to be *proper* if and only if

$$\sup\{|x(s,t)|+|u(s,t)| : a\le s<\infty, c\le t<\infty\}>0, \tag{4.4}$$

for any $a\ge s_0, c\ge t_0$.A proper solution $(x(s,t),u(s,t))$ of system (4.2) is called *weakly oscillatory* if and only if at least one component has a sequence of zeros tending to $+\infty$.

Theorem 4.1. *If $|x(\tau,\sigma)| = \max\{|x(s,t)| : a<s<b, c<t<d\}$, where $a>s_0$, $c>t_0$ and $s_0,t_0,a,b,c,d\in\mathbb{R}$, $u(\tau,t)$ is bounded on $[t_0,\infty)$ and $u(s,\sigma)$ is bounded on $[s_0,\infty)$,*

$$\int^{\infty}\int^{\infty}\beta_1(s,t)dt\,ds<\infty, \qquad \int^{\infty}\int^{\infty}\left|\beta_2(s,t)\right|dt\,ds<\infty, \tag{4.5}$$

then every weakly oscillatory proper solution of (4.2) *is bounded on $I=[s_0,\infty)\times[t_0,\infty)$.*

Proof. Let $(x(s,t),u(s,t))$ be any nontrivial weakly oscillatory proper solution of nonlinear system (4.2) on $I=[s_0,\infty)\times[t_0,\infty)$ such that $x(s,t)$ has a sequence of zeros tending to $+\infty$. Suppose to the contrary that $\limsup|x(s,t)|=\infty$; then given any positive number M_0, we can find positive numbers S_0 and T_0 such that $|x(s,t)|>M_0$ for all $s>S_0, t>T_0$. Since $x(s,t)$ is an oscillatory solution, there exist $(a,b)\times(c,d)\in\mathbb{R}\times\mathbb{R}$ with $a>S_0, c>T_0$ such that $x(a,t)=x(b,t)=x(s,c)=x(s,d)=0$ and $|x(s,t)|>0$ on $(a,b)\times(c,d)$. Choose (τ,σ) in $(a,b)\times(c,d)$ such that $M=|x(\tau,\sigma)|=\max\{|x(s,t)| : a<s<b, c<t<d\}>M_0$; in view of (4.5), we can choose S_0 and T_0 large enough such that for every $a\ge S_0$, $c\ge T_0$,

$$\int_a^{\infty}\int_c^{\infty}\beta_1(s,t)dt\,ds<M^{-(\beta-\alpha)/(\alpha-1)}, \qquad \int_a^{\infty}\int_c^{\infty}\left|\beta_2(s,t)\right|dt\,ds<1. \tag{4.6}$$

Taking αth power of both sides of (4.3) and combining (4.6), we obtain

$$\begin{aligned}
2^{\alpha} &\le M^{\beta-\alpha}\left(\int_a^b\int_c^d\beta_1(s,t)dt\,ds\right)^{\alpha-1}\left(\int_a^b\int_c^d\beta_2^{+}(s,t)dt\,ds\right)\\
&\le M^{\beta-\alpha}\left(\int_a^{\infty}\int_c^{\infty}\beta_1(s,t)dt\,ds\right)^{\alpha-1}\left(\int_a^{\infty}\int_c^{\infty}\left|\beta_2(s,t)\right|dt\,ds\right)\\
&< M^{\beta-\alpha}M^{-\beta+\alpha}=1,
\end{aligned} \tag{4.7}$$

where $\alpha>1$ and $\beta_2^{+}(s,t)\le|\beta_2(s,t)|$.

This contradiction shows that $|x(s,t)|$ is bounded on $I=[s_0,\infty)\times[t_0,\infty)$. Therefore, there exists a positive constant K such that $|x(s,t)|\le K$ for all $(s,t)\in I$.

On the other hand, integrating the second equation of system (4.2) over t from σ to t and over s from σ to s, respectively, we obtain

$$u(s,t)-u(\tau,t)-u(s,\sigma)+u(\tau,\sigma)=\int_{\sigma}^{s}\int_{\tau}^{t}-\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t)dt\,ds. \tag{4.8}$$

Notice that $u(\tau,t)$ is bounded on $[t_0,\infty)$, $u(s,\sigma)$ is bounded on $[s_0,\infty)$, and in view of triangle inequality, we have

$$\begin{aligned} |u(s,t)| &\leq |u(\tau,t)+u(s,\sigma)-C| + \int_\sigma^s \int_\tau^t \left|\beta_2(s,t)\right| |x(s,t)|^{\beta-1} dt\, ds \\ &\leq |u(\tau,t)+u(s,\sigma)-C| + K^{\beta-1} \int_\sigma^\infty \int_\tau^\infty \left|\beta_2(s,t)\right| dt\, ds, \end{aligned} \tag{4.9}$$

where $C = u(\tau,\sigma)$ is a constant.

Equation (4.9) implies that $|u(s,t)|$ is bounded on $I = [s_0,\infty) \times [t_0,\infty)$ since $\int_\tau^\infty \int_\sigma^\infty |\beta_2(s,t)| dt\, ds < \infty$. It follows from

$$\limsup\{|x(s,t)| + |u(s,t)|\} \leq \limsup|x(s,t)| + \limsup|u(s,t)| \tag{4.10}$$

that $\limsup\{|x(s,t)| + |u(s,t)|\}$ is bounden on $I = [s_0,\infty) \times [t_0,\infty)$.

This completes the proof. □

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