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Demographic structure and overlapping generations: A simpler proof with more general conditions

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Abstract

d'Albis (2007) considers a continuous-time general equilibrium overlapping-generations model with age-specific mortality rates. His proof of the existence and uniqueness of the steady-state equilibrium, which can be extended to other overlapping-generations models, relies on the shape of a function that appears in the equation defining the equilibrium. By focusing on the mean age as a function of the stable population growth rate instead of the function used in d'Albis (2007), we provide a simpler proof with more general conditions. We also obtain useful properties about the first and second derivatives of the mean age function that can be applied in future work.

JEL Classification Numbers: E13; E21; J10

Keywords: overlapping-generations model, age-specific mortality, steady-state equilibrium, mean age function

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1. INTRODUCTION

d'Albis (2007) considers an overlapping-generations (OLG) model with lifetime uncertainty and purely life-cycle saving motive. He first obtains sufficient conditions for the existence and uniqueness of the steady-state equilibrium. He then shows that there exists a finite population growth rate maximizing long-run capital per worker, and relates the results to several well-known OLG models, including the two-period OLG model (Diamond, 1965) and the continuous-time OLG model with age-invariant mortality (Blanchard, 1985; Weil, 1989).

The proof of the existence and uniqueness of the steady-state equilibrium in the OLG model with a general survival function is important. It provides the foundation for the analysis of further topics in d'Albis (2007). The proof can also be extended to other general equilibrium OLG models with age-specific mortality, such as the model incorporating technological progress and a retirement phase (Lau, 2009). In this paper we focus on the existence and uniqueness proof in the d'Albis (2007) model. First, we point out a missing condition in his proof, which relies heavily on the convexity property of a function that appears in the equation defining the steady-state equilibrium. Second, by focusing on the mean age as a function of the stable population growth rate instead of the function used in d'Albis (2007), we show that it is possible to provide a simpler proof with more general conditions.

The remaining sections are organized as follows. Section 2 describes briefly the model and discusses the proof in d'Albis (2007), including a condition missing in his proof. Section 3 presents an alternative proof, and obtains two conditions which are more general than those in his paper. Furthermore, by using the first and second derivatives of the mean age function, we show that in some special cases, a key condition holds for all survival functions. Section 4 provides concluding remarks.

2. THE MODEL AND STEADY-STATE EQUILIBRIUM

d'Albis (2007) analyzes a continuous-time OLG model with a general demographic structure and age-specific discount rates. Since either the age-varying or age-invariant discount rate assumption is usually not the major determinant of economic phenomena (except perhaps on issues related to present-biased preferences), many authors working with OLG models use the constant discount rate assumption, which simplifies the notations without much loss of generality.[1] We consider the d'Albis (2007) model with constant discount rate in this paper.

In this section we provide a brief description of the model—first, the demographic and labor supply features of the model economy, then individual cohort's consumption problem, and finally, the firms' production technology. A more detailed description of the model can be found in d'Albis (2007).

The demographic structure of the model is based on the stable population theory. It is assumed that age-specific fertility rates and age-specific mortality rates of different cohorts remain unchanged. On the mortality side, represent the probability that an individual survives to at least age x by the survival function $l(x)$, where $x \in [0, \Omega]$, Ω is the maximum age, $l(0) = 1$ and $l(\Omega) = 0$. The instantaneous mortality rate at age x, $\mu(x)$, is related to $l(x)$ according to $\mu(x) = -\frac{1}{l(x)}\frac{dl(x)}{dx}$. Based on the stable population theory (Lotka, 1939; Coale, 1972; Keyfitz and Caswell, 2005), it can be shown that eventually, the population growth rate is time-invariant. Since labor supply decision is not the major focus in d'Albis (2007), he follows Blanchard (1985) to model it exogenously. Specifically, each individual is assumed to supply one unit of labor inelastically at every age. Therefore, the growth rate of labor supply is the same as the population growth rate.

[1]Moreover, since there is no commonly-agreed choice of an age-specific discount rate schedule, a constant discount rate assumption is usually adopted for simplicity in applied work using the OLG model.

Next, consider individual's consumption decision in the presence of lifetime uncertainty and an actuarially fair financial market (Yaari, 1965). To focus purely on the saving-for-retirement motive, d'Albis (2007) assumes that individuals have no bequest motive. At time t, an individual, who was born at time s (where $s \leq t \leq s+\Omega$) and whose current financial wealth is $A(s,t)$, chooses $\{C(s,v)\}_t^{s+\Omega}$ to maximize

$$\int_t^{s+\Omega} e^{-\rho(v-t)} \frac{l(v-s)}{l(t-s)} \left[\frac{C(s,v)^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} \right] dv, \tag{1}$$

subject to the flow budget constraint

$$\frac{\partial A(s,v)}{\partial v} = [r(v) + \mu(v-s)] A(s,v) + w(v) - C(s,v), \tag{2}$$

where ρ is the discount rate, σ is the intertemporal elasticity of substitution, $C(s,v)$ is consumption of a cohort s individual at time v, $A(s,v)$ is the financial wealth of a cohort s individual at time v, $r(v)$ is the (real) interest rate at v, and $w(v)$ is the (real) wage rate at v. Individuals are born without financial assets or liabilities, and face a terminal condition of non-negative financial wealth.

The model is closed by assuming that competitive firms use a neoclassical production technology with capital and labor inputs. Output at time t, $Y(t)$, is determined according to $Y(t) = F(K(t), N(t))$, where $K(t)$ and $N(t)$ represent, respectively, capital input and labor input at time t. Define a variable per worker as the variable divided by N, and denote it in lower case letter. With this transformation, the neoclassical production function in intensive form is given by $y(t) = \frac{Y(t)}{N(t)} = F\left(\frac{K(t)}{N(t)}, 1\right) \equiv f(k(t))$, with $f'(k) > 0$, $f''(k) < 0$, $\lim_{k\to 0} f'(k) = \infty$, and $\lim_{k\to\infty} f'(k) = 0$. Finally, it is assumed that capital depreciates at a constant rate δ (≥ 0), and

$$0 < \delta + n < \infty, \tag{3}$$

where n is the constant growth rate of the stable population. The assumption of a positive value of $\delta + n$ in (3) is empirically more relevant than a negative value of

$\delta + n$.[2] The same assumption is used in d'Albis (2007), and a similar assumption incorporating the rate of technological progress is used in Lau (2009).

It is useful to define

$$H(u) = \int_0^{\Omega} e^{-ux} l(x)\, dx \tag{4}$$

for $u \in (-\infty, \infty)$. Also define

$$J(u) = \frac{H(u)\, H(-\sigma u + n + \sigma\rho)}{H((1-\sigma)u + \sigma\rho)\, H(n)}. \tag{5}$$

Note that $H(u)$ is positive and decreasing in u, since $l(x)$ is positive for $x \in (0, \Omega)$. As a result, $J(u)$ is also positive.

Denote a variable at the steady-state equilibrium with a $*$. As in Blanchard (1985) and d'Albis (2007), the equilibrium condition is obtained by equating asset holdings with capital stock.[3] Using steps similar to those in d'Albis (2007) or Lau (2009), it can be shown that the steady-state equilibrium of this economy is defined by

$$\phi(k^*) = k^*, \tag{6}$$

where k^* is the steady-state value of capital per worker, and function $\phi(k)$ is defined

[2] Since δ is non-negative, (3) is always satisfied for a stable population with a positive growth rate. If one takes δ to be positive (say, at the commonly used value of 0.05, as in Barro et al., 1995), then (3) is satisfied even for an economy with a shrinking population, as long as the rate of decline is relatively mild.

[3] This condition corresponds to the concept of "balanced" equilibrium according to the terminology used in Gale (1973), who considers pure exchange economies, and in Willis (1988), who considers productive economies.

as[4]

$$\phi(k) = \frac{f(k) - kf'(k)}{f'(k) - (\delta + n)} \left[J(f'(k) - \delta) - 1 \right]. \tag{7}$$

d'Albis (2007) provides sufficient conditions for the existence and uniqueness of the steady-state equilibrium in his Proposition 3, and the conditions are more general than those in Bommier and Lee (2003). These results represent an improvement. However, there is a missing condition in his proof of Lemma 8, which affects the subsequent existence and uniqueness proof. The missing condition (for the constant discount rate version of his model) is

$$\frac{d}{du}\left[\frac{J'(u)}{J(u)}\right] = -\sigma^2 h'(-\sigma u + n + \sigma\rho) + (1-\sigma)^2 h'((1-\sigma)u + \sigma\rho) - h'(u) > 0 \tag{8}$$

for all $u \in (-\delta, \infty)$, where $h'(.)$ is the derivative of the mean age function $h(.)$ defined in (9) in the next section.[5]

The reason for the missing condition (8) is as follows. The proof of d'Albis (2007) relies heavily on the shape of function $J(u)$. Specifically, his existence proof is based on the assumed convexity of $J(u)$, and his uniqueness proof relies on the assumption that $\frac{J'(u)}{J(u)}$ is increasing in u. According to the proof of Lemma 8 in d'Albis

[4]It is well known that the competitively determined interest rate and wage rate are given by $r = f'(k) - \delta$ and $w = f(k) - kf'(k)$, respectively. Thus, (7) can also be written as $\phi(k) = \frac{w}{r-n}[J(r) - 1]$ or $f(k) - wJ(r) = (\delta + n)k$, where r and w depend on k implicitly. From (17) of d'Albis (2007) or (21) of Lau (2009), we know that $wJ(r)$ is the steady-state consumption per worker. Under assumption (3), aggregate saving is positive (i.e., $f(k) - wJ(r) > 0$) at the steady-state equilibrium.

[5]Since we assume a constant discount rate while d'Albis (2007) allows for age-specific discount rates, our definition of function $H(u)$, as in (4), is slightly different from that in (6) of his paper. As a result, the term $(1-\sigma)^2 h'((1-\sigma)u)$ in (61) of d'Albis (2007) is modified to $(1-\sigma)^2 h'((1-\sigma)u + \sigma\rho)$ in (8) of this paper, and so on. Note that if we assume age-specific discount rates, (8) should be replaced by:

$$-\sigma^2 h'(n - \sigma u) + (1-\sigma)^2 h'((1-\sigma)u) - g'(u) > 0, \tag{8a}$$

where $g(u)$ and $h(u)$ are defined in (59) and (60), respectively, of d'Albis (2007).

(2007), a positive value of $\frac{\partial}{\partial\sigma}\left[(1-\sigma)^2 h'((1-\sigma)u+\sigma\rho)\right]$ for all $\sigma \in (0,1]$ and $\lim_{\sigma\to 1}(1-\sigma)^2 h'((1-\sigma)u+\sigma\rho)=0$ lead to $\frac{d}{du}\left[\frac{J'(u)}{J(u)}\right]>0$ and thus also the convexity of $J(u)$.[6] However, these two properties about $(1-\sigma)^2 h'((1-\sigma)u+\sigma\rho)$ imply that this term is negative unless $\sigma=1$. Thus, even though the other two terms in the middle expression of (8) are positive because of (12), $\frac{d}{du}\left[\frac{J'(u)}{J(u)}\right]$ is not necessarily positive. We conclude that, contrary to the claim in d'Albis (2007), condition (8) is not always satisfied for $\sigma \in (0,1)$. In order to ensure that $\frac{J'(u)}{J(u)}$ is increasing in u and $J''(u)>0$ for all $u \in (-\delta,\infty)$, Proposition 3 of d'Albis (2007) requires condition (8) explicitly.

3. A SIMPLER PROOF WITH MORE GENERAL CONDITIONS

In the previous section, we point out that the correct proof of the existence and uniqueness of the steady-state equilibrium in d'Albis (2007) requires condition (8) when the discount rate is constant. In this section, we provide an alternative proof which relaxes this condition and another condition in d'Albis (2007). Moreover, the proof covers the model with $\sigma>1$, which is not considered in d'Albis (2007).[7]

[6]Note that (11) implies that

$$J''(u)=\left[-\sigma^2 h'(-\sigma u+n+\sigma\rho)+(1-\sigma)^2 h'((1-\sigma)u+\sigma\rho)-h'(u)\right]J(u)$$
$$+\left[\sigma h(-\sigma u+n+\sigma\rho)+(1-\sigma)h((1-\sigma)u+\sigma\rho)-h(u)\right]^2 J(u).$$

Comparing this equation with (8), it can be concluded that the condition "$\frac{d}{du}\left[\frac{J'(u)}{J(u)}\right]>0$", which is required for the proof of uniqueness of the steady-state equilibrium in d'Albis (2007), is stronger than the condition "$J''(u)>0$", which is required for his proof of existence of the steady-state equilibrium.

[7]Even though σ is specified to be smaller than or equal to 1 in most applied studies (as in Barro et al., 1995), we also consider the $\sigma>1$ case for the sake of completeness. In particular, our analysis suggests that the properties about the first and second derivatives of $h(u)$ affect the sufficient conditions for the OLG models with $\sigma\le 1$ and those with $\sigma>1$ differently, as elaborated

Our analysis makes use of the property of a function which expresses the mean age of the stable population when the population growth rate u changes, with the age-specific mortality rates given by $l(x)$. This mean age function is given by

$$h(u) = \frac{-H'(u)}{H(u)} = \frac{\int_0^\Omega xe^{-ux}l(x)\,dx}{\int_0^\Omega e^{-ux}l(x)\,dx} = \int_0^\Omega xq(x,u)\,dx, \tag{9}$$

where

$$q(x,u) = \frac{e^{-ux}l(x)}{\int_0^\Omega e^{-ux}l(x)\,dx} \tag{10}$$

is the density function of individuals aged x in the stable population with survival function $l(x)$ and population growth rate u.[8] The mean age function is related to the first derivative of $J(.)$ according to

$$J'(u) = [\sigma h(-\sigma u + n + \sigma\rho) + (1-\sigma)h((1-\sigma)u + \sigma\rho) - h(u)]\,J(u). \tag{11}$$

It can be shown that the first derivative of $h(u)$ is the negative of the variance of individual's age of the population (when population size grows at a constant rate u), and the second derivative of $h(u)$ is equal to the third central moment of age. These properties, which will be used in subsequent analysis, are summarized in the following lemma.

Lemma 1 *The first and second derivatives of function $h(u)$ are given by*

$$h'(u) = -\int_0^\Omega [x - h(u)]^2\,q(x,u)\,dx < 0, \tag{12}$$

and

$$h''(u) = \int_0^\Omega [x - h(u)]^3\,q(x,u)\,dx. \tag{13}$$

In particular, $h''(u) > 0$ for $u \geq 0$.

in the paragraph following Proposition 3.

[8] See, for example, (2.1) and (2.3) in Coale (1972), or Keyfitz and Caswell (2005, p. 104).

Proof. See Appendix A.[9]

The negative slope of $h(u)$ and its convexity for positive values of u are illustrated in Figure 1, using the life table information for the USA (men and women combined) in 2005.[10]

We now state, in the following proposition, an alternative set of sufficient conditions under which the steady-state equilibrium exists and is unique.

Proposition 2 *For the continuous-time overlapping-generations model with age-specific mortality rates, a constant discount rate and assumption (3), the steady-state equilibrium (with $k^* > 0$) exists and is unique, if*

$$f(k) - kf'(k) + k^2 f''(k) \geq 0 \tag{14}$$

for all $k \in (0, \infty)$,

$$\lim_{k \to 0} \left[-kf''(k)\right] > 0, \tag{15}$$

[9]Some of the results in Lemma 1 are similar to those in the literature. For example, as in d'Albis (2007), one can apply the Cauchy-Schwarz inequality to show that $[H'(u)]^2 < H(u) H''(u)$. Therefore, $h'(u) = \frac{[H'(u)]^2 - H(u)H''(u)}{[H(u)]^2} < 0$. However, we think that it is useful to summarize the first two derivatives of $h(u)$ in a separate lemma. In particular, our proof of (12) has the additional benefit of relating $h'(u)$ to the variance of age. As a result, we can relate $h''(u)$ to the third central moment of age, and obtain the convexity property of $h(u)$ when $u \geq 0$. Note also that according to the infinite series for the mean age in (5.3.3) of Keyfitz and Caswell (2005), which is based on (48) in Lotka (1939), the derivatives of $h(u)$ are related to the various moments of age. However, the moments used in Lotka (1939) and Keyfitz and Caswell (2005) refer to those of the stationary population (with $u = 0$), whereas the results in Lemma 1 are more general as the moments refer to the those of the stable population (with an arbitrary u).

[10]Since most OLG models assume that the adult stage starts at age 20, we calculate the survival probability based on adult age, which is defined as actual age minus 20, according to $l(x) = \frac{l_{actual}(x+20)}{l_{actual}(20)}$, where $l_{actual}(.)$ is the survival probability based on actual age. Similar shape of $h(u)$ is obtained if $l(x)$ is the survival function based on actual age.

and

$$\frac{J'(u)}{J(u)} - \frac{J'(n)}{J(n)} \equiv [\sigma h(-\sigma u + n + \sigma\rho) + (1-\sigma) h((1-\sigma)u + \sigma\rho) - h(u)] \\ - [h((1-\sigma)n + \sigma\rho) - h(n)] \tag{16}$$

≤ 0 *for all* $u \in (-\delta, n)$ *but* ≥ 0 *for all* $u \in (n, \infty)$.

Proof.

We only consider positive k (capital per worker), since it must be positive for a meaningful economy. It is helpful to establish three preliminary results. First, given the concavity of the production function $f(k)$ for all $k > 0$, we have

$$f(k) - kf'(k) > 0. \tag{17}$$

That is, real wage is positive for all $k > 0$. Second, under assumption (3) and $f''(k) < 0$, it is easy to see that $f'(k) - (\delta + n)$ is positive when k is close to zero, and it changes sign (to negative) only once after k reaches the golden rule level $(k = k_{gr})$, which is defined by

$$f'(k_{gr}) = \delta + n. \tag{18}$$

Note that k_{gr} in (18) is well-defined under assumption (3). Third, we conclude from (6), (7) and (17) that if k^* exists, it will satisfy[11]

$$J(f'(k^*) - \delta) \begin{cases} > 1 & \text{if } k^* < k_{gr} \\ = 1 & \text{if } k^* = k_{gr} \\ < 1 & \text{if } k^* > k_{gr} \end{cases}. \tag{19}$$

(A) Existence

[11] Using steps similar to those in footnote 14, it can further be shown that $J(f'(k^*) - \delta) = 1$ if

$$k^* = k_{gr} = \lim_{k \to k_{gr}} \phi(k) = [f(k_{gr}) - k_{gr}(\delta + n)][h((1-\sigma)n + \sigma\rho) - h(n)]. \tag{19a}$$

It is easy to see from (7) and (17) that $\phi(k^*) = k^*$ is equivalent to $\lambda(k^*) = 0$, where

$$\lambda(k) = \frac{J(f'(k) - \delta) - 1}{f'(k) - (\delta + n)} - \frac{k}{f(k) - kf'(k)}. \tag{20}$$

When k tends to 0, we have $f'(k)$ tends to ∞, $f(k) - kf'(k)$ tends to 0, and $J(f'(k) - \delta)$ tends to ∞.[12] Using the l'Hôpital's rule and the results in footnote 12, it is shown that $\lim_{k\to 0} \frac{J(f'(k)-\delta)-1}{f'(k)-(\delta+n)} = \lim_{k\to 0} J'(f'(k) - \delta) = +\infty$. Using the l'Hôpital's rule and (15), we show that $\lim_{k\to 0} \frac{k}{f(k)-kf'(k)} = \lim_{k\to 0} \left[\frac{1}{-kf''(k)}\right]$ is finite. Combining these results leads to

$$\lim_{k\to 0} \lambda(k) = +\infty. \tag{21}$$

When k tends to ∞, we have $f(k)$ tends to ∞, and $f'(k)$ tends to 0. Therefore, $\frac{J(f'(k)-\delta)-1}{f'(k)-(\delta+n)}$ tends to $\frac{J(-\delta)-1}{-(\delta+n)}$, which is finite. On the other hand, it can be shown that $\lim_{k\to\infty} \frac{k}{f(k)-kf'(k)} = +\infty$.[13] Therefore,

$$\lim_{k\to\infty} \lambda(k) = \frac{J(-\delta) - 1}{-(\delta + n)} - \lim_{k\to\infty} \frac{k}{f(k) - kf'(k)} = -\infty. \tag{22}$$

Since $\lambda(k)$ is continuous for $0 < k < \infty$,[14] one concludes from (21) and (22) that $\lambda(k^*) = 0$ exists for $0 < k^* < \infty$. Equivalently, $\phi(k^*) = k^*$ exists for $0 < k^* < \infty$.

[12] It is easy to show from (9) that $\lim_{u\to\infty} h(u) = 0$ and $\lim_{u\to-\infty} h(u) = \Omega$. (See Figure 1 also.) Substituting these results into (11) leads to $\lim_{u\to\infty} \frac{J'(u)}{J(u)} = \sigma\Omega$ if $\sigma < 1$ or $= \Omega$ if $\sigma \geq 1$. As a result, $\lim_{u\to\infty} \frac{J'(u)}{J(u)}$ is strictly positive, and thus, $\lim_{u\to\infty} J(u) = +\infty$ and $\lim_{u\to\infty} J'(u) = +\infty$.

[13] Since $\frac{f(k)-kf'(k)}{k} = \frac{f(k)}{k} - f'(k)$ and $\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f'(k) = 0$, we obtain $\lim_{k\to\infty} \frac{f(k)-kf'(k)}{k} = 0$. Because $f(k) - kf'(k)$ is positive when k is positive, we conclude that $\lim_{k\to\infty} \frac{k}{f(k)-kf'(k)} = +\infty$, instead of $-\infty$.

[14] Since $J(f'(k) - \delta) - 1$ and $f'(k) - (\delta + n)$ are continuous in $0 < k < \infty$, discontinuity of $\frac{J(f'(k)-\delta)-1}{f'(k)-(\delta+n)}$ could only occur at $k = k_{gr}$, as $f'(k_{gr}) - (\delta + n) = 0$. However, it can be shown that this ratio is finite at $k = k_{gr}$, as follows. First, it is easy to show from (5) that $J(f'(k_{gr}) - \delta) = J(n) = 1$. Second, we apply the l'Hôpital's rule and use (11) to conclude that

$$\lim_{k\to k_{gr}} \frac{J(f'(k) - \delta) - 1}{f'(k) - (\delta + n)} = \lim_{k\to k_{gr}} J'(f'(k) - \delta) = J'(n) = h((1 - \sigma)n + \sigma\rho) - h(n).$$

Thus, $\lim_{k\to k_{gr}} \lambda(k) = [h((1 - \sigma)n + \sigma\rho) - h(n)] - \frac{k_{gr}}{f(k_{gr})-k_{gr}(\delta+n)}$, which is finite.

(B) Uniqueness

From (7), we have

$$\phi'(k) = -f''(k)\,\phi(k)\left[\frac{1}{f'(k)-(\delta+n)} + \frac{k}{f(k)-kf'(k)} - \frac{J'(f'(k)-\delta)}{J(f'(k)-\delta)-1}\right].$$

Computing $\phi'(k)$ at the steady-state equilibrium (with $\phi(k^*)=k^*$) and simplifying yields

$$\phi'(k^*) = -f''(k^*)\,k^*\frac{J(f'(k^*)-\delta)}{[J(f'(k^*)-\delta)-1]}\left[\frac{k^*}{f(k^*)-k^*f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\right]. \tag{23}$$

We prove uniqueness of the steady-state equilibrium k^* by showing that $\phi'(k^*)\le 0$. Our proof proceeds as follows. First, (14) implies that

$$\frac{d}{dk}\left[\frac{k}{f(k)-kf'(k)}\right] = \frac{f(k)-kf'(k)+k^2f''(k)}{[f(k)-kf'(k)]^2}\ge 0. \tag{24}$$

Second, we show that $\frac{k_{gr}}{f(k_{gr})-k_{gr}f'(k_{gr})} - \frac{J'(f'(k_{gr})-\delta)}{J(f'(k_{gr})-\delta)} = 0$, and obtain $\lim_{k^*\to k_{gr}}\phi'(k^*)\le 0$. Third, we show that when $k^*<k_{gr}$, $\frac{k^*}{f(k^*)-k^*f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\le 0$. Fourth, we show that when $k^*>k_{gr}$, $\frac{k^*}{f(k^*)-k^*f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\ge 0$. Combining these results with (19), we conclude that $\phi'(k^*)\le 0$ for all $0<k^*<\infty$.

When $k^*=k_{gr}$, we use (11), (18) and (19a) to obtain

$$\frac{k_{gr}}{f(k_{gr})-k_{gr}f'(k_{gr})} - \frac{J'(f'(k_{gr})-\delta)}{J(f'(k_{gr})-\delta)} = [h((1-\sigma)n+\sigma\rho)-h(n)] - J'(n) = 0. \tag{25}$$

However, the term $J(f'(k_{gr})-\delta)-1$ in the denominator of the right-hand side of (23) is also zero. We can use (6), (7) and (23) to obtain

$$\lim_{k^*\to k_{gr}}\phi'(k^*) = -f''(k_{gr})\lim_{k^*\to k_{gr}}\frac{k^*}{[J(f'(k^*)-\delta)-1]}\left[\frac{k^*}{f(k^*)-k^*f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\right]$$

$$= -f''(k_{gr})[f(k_{gr})-k_{gr}f'(k_{gr})]\lim_{k^*\to k_{gr}}\frac{\left[\frac{k^*}{f(k^*)-k^*f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\right]}{f'(k^*)-(\delta+n)}.$$

Using the l'Hôpital's rule, we obtain

$$\lim_{k^* \to k_{gr}} \frac{\left[\frac{k^*}{f(k^*)-k^* f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\right]}{f'(k^*) - (\delta+n)}$$

$$= \frac{\lim_{k^* \to k_{gr}} \frac{d}{dk^*}\left[\frac{k^*}{f(k^*)-k^* f'(k^*)}\right] - \lim_{k^* \to k_{gr}} \frac{d}{dk^*}\left[\frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\right]}{f''(k_{gr})}. \tag{26}$$

We know that $\lim_{k^* \to k_{gr}} \frac{d}{dk^*}\left[\frac{k^*}{f(k^*)-k^* f'(k^*)}\right] \geq 0$, because of (24). Condition (16) implies that $\lim_{u \to n} \frac{d}{du}\left[\frac{J'(u)}{J(u)}\right] \geq 0$,[15] which, through $f''(k) < 0$, further implies that $\lim_{k^* \to k_{gr}} \frac{d}{dk^*}\left[\frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)}\right] \leq 0$. As a result, the limit in (26) is non-positive, and $\lim_{k^* \to k_{gr}} \phi'(k^*) \leq 0$.

When $k^* < k_{gr}$ (i.e., $r^* > n$), (24) implies that $\frac{k^*}{f(k^*)-k^* f'(k^*)} \leq \frac{k_{gr}}{f(k_{gr})-k_{gr} f'(k_{gr})}$. From (16), we obtain $\frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)} \geq \frac{J'(f'(k_{gr})-\delta)}{J(f'(k_{gr})-\delta)}$. Therefore, together with (25), we have $\frac{k^*}{f(k^*)-k^* f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)} \leq \frac{k_{gr}}{f(k_{gr})-k_{gr} f'(k_{gr})} - \frac{J'(f'(k_{gr})-\delta)}{J(f'(k_{gr})-\delta)} = 0$. Combining with (19) and (23), we obtain $\phi'(k^*) \leq 0$ for all $k^* < k_{gr}$.

Similarly, when $k^* > k_{gr}$ (i.e., $r^* < n$), we use (16), (24) and (25) to obtain $\frac{k^*}{f(k^*)-k^* f'(k^*)} - \frac{J'(f'(k^*)-\delta)}{J(f'(k^*)-\delta)} \geq \frac{k_{gr}}{f(k_{gr})-k_{gr} f'(k_{gr})} - \frac{J'(f'(k_{gr})-\delta)}{J(f'(k_{gr})-\delta)} = 0$. Combining with (19) and (23), we obtain $\phi'(k^*) \leq 0$ for all $k^* > k_{gr}$. ■

The difference in our proof of Proposition 2 and that of Proposition 3 in d'Albis (2007) is as follows. For the existence proof, we use only the boundary conditions of $J(u)$, and do not require any assumption regarding the convexity of $J(u)$. In this process, we relax condition (20) of d'Albis (2007) to (15) of this paper.[16] For the uniqueness proof, we only compare the values of $\frac{J'(u)}{J(u)}$ with $\frac{J'(n)}{J(n)}$, instead of relying

[15] There are two ways to see that (16) is more general than (8). The first way to see is that $\frac{J'(u)}{J(u)} \leq \frac{J'(n)}{J(n)}$ for all $u \in (-\delta, n)$ and $\frac{J'(u)}{J(u)} \geq \frac{J'(n)}{J(n)}$ for all $u \in (n, \infty)$ according to (16), but $\frac{J'(u)}{J(u)}$ is increasing in u for all $u \in (-\delta, \infty)$ according to (8). Another way to see, which is useful for the proof here, is that $\frac{d}{du}\left[\frac{J'(u)}{J(u)}\right] \geq 0$ holds for $u = n$ according to (16), but it holds for all $u \in (-\delta, \infty)$ according to (8). Note also that we have performed computational check, which confirms that condition (8) is violated but (16) is satisfied under some situations.

[16] With the help of (70) of d'Albis (2007), it can be shown that condition (20) of his paper is based

on the stronger assumption that $\frac{J'(u)}{J(u)}$ is upward sloping. As a result, it is possible to relax condition (8) to (16).

Another advantage of the approach used in this paper is that we are able to extend the proof to cover the case when $\sigma > 1$. In this case, we further show that condition (16) is always satisfied for $\sigma \geq 1$, and thus only (14) and (15) are required in the proof.[17] This is given in the following Proposition.

Proposition 3 *For the continuous-time overlapping-generations model with age-specific mortality rates, a constant discount rate and assumption (3), the steady-state equilibrium (with $k^* > 0$) exists and is unique when $\sigma \geq 1$, if (14) and (15) are satisfied.*

Proof. See Appendix B.

In the proof of Proposition 3, it is observed that the property $h'(.) < 0$ in (12) is used to show that (16) is always satisfied when $\sigma \geq 1$. However, the negative slope of the mean age function is not enough to deliver further results when $\sigma < 1$. We have tried various ways to analyze the model with $\sigma < 1$, and found that condition (16) is required in general. Whether condition (16) holds or not depends on the survival function $l(x)$, among other factors. On some occasions, it will be useful to obtain the conditions which hold for all survival functions. We have found some special cases that the property about the second derivative of $h(.)$ helps to obtain such results. As the method may also be relevant for other OLG models in future work, we present this result in the following proposition.

on a comparison of $\lim_{k\to 0} \frac{k}{f(k)-kf'(k)}$ and $\lim_{k\to 0} \frac{J'(f'(k)-\delta)}{J(f'(k)-\delta)}$. We are able to relax that condition to (15) of this paper because in our existence proof, we only need to consider $\lim_{k\to 0} \frac{k}{f(k)-kf'(k)}$ but not the other limit.

[17] Note that the remaining conditions—(14) and (15)—are related to the production function only, and they are satisfied for a Cobb-Douglas production function. Note also that condition (14) is the same as $s(k) \leq \varepsilon(k)$ in (21) of d'Albis (2007), where $s(k) = kf'(k)/f(k)$ is the share of capital in output, and $\varepsilon(k) = -f'(k)[1-s(k)]/[kf''(k)]$ is the elasticity of substitution between capital and labor.

Proposition 4 *For the continuous-time overlapping-generations model with age-specific mortality rates, a constant discount rate and assumption (3), the steady-state equilibrium (with $k^* > 0$) exists and is unique when $0 < \sigma < 1$, if*

$$0 \leq n \leq \rho, \tag{27}$$

$$0 \leq \delta \leq \left(\frac{\sigma}{1-\sigma}\right) n, \tag{28}$$

and (14) and (15) are satisfied.

Proof. See Appendix C.

4. CONCLUDING REMARKS

d'Albis (2007) provides sufficient conditions for the existence and uniqueness of the steady-state equilibrium in an OLG model with lifetime uncertainty, and the conditions are more general than those in Bommier and Lee (2003). His proof relies heavily on the convexity property of function $J(.)$, which appears in the equation defining the equilibrium. In this paper we first point out a missing condition in his proof. We then provide a simpler proof which does not rely on any convexity assumption about $J(.)$, and obtain sufficient conditions which are more general than those in d'Albis (2007). Comparing our method with that used by d'Albis (2007), it is observed that he focuses on function $J(.)$ whereas we focus on the mean age function $h(.)$, which is simpler than $J(.)$. Moreover, we are able to relate the first and second derivatives of $h(.)$ to the various moments of individual's age in the population, and thus to sign these derivatives. On the other hand, the property of $J(.)$, which is a convoluted function of $H(.)$ according to (5), is harder to obtain. Focusing on the property of the mean age function $h(.)$, rather than $J(.)$, is likely to be more fruitful in the analysis of continuous-time OLG models with age-specific mortality rates.

To facilitate comparison with d'Albis (2007), we use the same model in this paper, except that we simplify the discount rates to be age-invariant. The proof is, however, not only applicable in this model, but also in an OLG model with technological progress (Lau, 2009). One of us is currently working on an OLG model consisting of more realistic demographic features of childhood, adulthood and retirement years, and preliminary analysis suggests that the results in this paper can further be extended to that model.[18] The various analyses lead us to believe that the proof in d'Albis (2007), and the additional results reported in this paper, will be useful for other OLG models (such as those with richer demographic and labor supply features, with policy elements such as social security, with other growth-generating elements such as human capital, and so on) as well.

APPENDIX A: PROOF OF LEMMA 1

Partially differentiating (10) with respect to u and simplifying, we obtain

$$\frac{\partial q(x,u)}{\partial u} = -\left[x - h(u)\right] q(x,u). \tag{A1}$$

Differentiating $h(u) = \int_0^\Omega xq(x,u)\,dx$ and using (A1), we have

$$h'(u) = \int_0^\Omega x \frac{\partial q(x,u)}{\partial u} dx = -\int_0^\Omega x\left[x - h(u)\right] q(x,u)\,dx. \tag{A2}$$

Since $h(u)$ is the mean of x when the population growth rate is u, we have

$$\int_0^\Omega \left[x - h(u)\right] q(x,u)\,dx = 0. \tag{A3}$$

[18] Many OLG models—including discrete-time models such as Diamond (1965) and Abel (2003), and continuous-time models such as Blanchard (1985), d'Albis (2007), and Lau (2009)—only assume one stage (working) or two stages (working and retirement) of the life cycle. As mentioned in Bommier and Lee (2003, p. 136), "Two age group models are not capable of representing the most basic feature of the human economic life cycle: that it begins and ends with periods of dependency, separated by a long intermediate period of consuming less than is produced." In studying the consequences of fertility and mortality changes, an OLG model consisting of childhood, working and retirement stages is likely to be better than a two-stage model.

Therefore, we obtain $\int_0^\Omega x\left[x-h\left(u\right)\right]q\left(x,u\right)dx=\int_0^\Omega\left[x-h\left(u\right)\right]^2q\left(x,u\right)dx$. We conclude that $h'\left(u\right)$ in (A2) can be expressed as (12) and is negative.

Differentiating (12) and applying (A1), we obtain

$$h''\left(u\right)=2h'\left(u\right)\int_0^\Omega\left[x-h\left(u\right)\right]q\left(x,u\right)dx+\int_0^\Omega\left[x-h\left(u\right)\right]^3q\left(x,u\right)dx.$$

Using (A3), we have (13).

The numerator on the right-hand side of (10) is decreasing in x when $u\geq 0$, since e^{-ux} is non-increasing in x when $u\geq 0$ and $l\left(x\right)$ is decreasing in x. The term in the denominator, $\int_0^\Omega e^{-ux}l\left(x\right)dx$, is independent of x. Therefore, $\frac{\partial q(x,u)}{\partial x}<0$. It can be shown that a random variable x with density function monotonically decreasing in x is positively skewed. Therefore, when $u\geq 0$, the third central moment of x is positive, and

$$h''\left(u\right)>0 \tag{A4}$$

according to (13). ■

APPENDIX B: PROOF OF PROPOSITION 3

When $u<n$, we use $h'\left(.\right)<0$ in (12) to obtain $h\left(-\sigma u+n+\sigma\rho\right)<h\left(-\sigma u+u+\sigma\rho\right)$. Thus, when $\sigma\geq 1$, we have

$$\frac{J'\left(u\right)}{J\left(u\right)}<h\left(\left(1-\sigma\right)u+\sigma\rho\right)-h\left(u\right)<h\left(\left(1-\sigma\right)n+\sigma\rho\right)-h\left(n\right)=\frac{J'\left(n\right)}{J\left(n\right)}.$$

Similar, when $u>n$, we use $h'\left(.\right)<0$ and $\sigma\geq 1$ to obtain

$$\frac{J'\left(u\right)}{J\left(u\right)}>h\left(\left(1-\sigma\right)u+\sigma\rho\right)-h\left(u\right)>h\left(\left(1-\sigma\right)n+\sigma\rho\right)-h\left(n\right)=\frac{J'\left(n\right)}{J\left(n\right)}.$$

Thus, (16) is always satisfied when $\sigma\geq 1$. Applying Proposition 2, we prove Proposition 3. ■

APPENDIX C: PROOF OF PROPOSITION 4

In the following analysis, $0 < \sigma < 1$. Conditions (27) and (28) imply that $-\delta \leq 0 \leq n \leq \rho$. Since $u \in (-\delta, \infty)$, there are only four distinct cases to consider. They are: $u > \rho$ (case 1), $n < u \leq \rho$ (case 2), $0 < u \leq n$ (case 3) and $-\delta < u \leq 0$ (case 4).

For case 1 (with $-\delta \leq 0 \leq n \leq \rho < u$), we first use (11), (12), $u > n$ and $u > \rho$ to obtain

$$\frac{J'(u)}{J(u)} > h((1-\sigma)u + \sigma\rho) - h(u) > 0.$$

Second, we use (12) and $\rho \geq n$ to obtain $h((1-\sigma)n + \sigma\rho) \leq h(n)$ and thus, $\frac{J'(n)}{J(n)} \leq 0$. Combining these results, we have $\frac{J'(u)}{J(u)} > \frac{J'(n)}{J(n)}$.

For case 2 (with $-\delta \leq 0 \leq n < u \leq \rho$), we first use (11), (12) and $u > n$ to obtain

$$\frac{J'(u)}{J(u)} > h((1-\sigma)u + \sigma\rho) - h(u) > h(u - \sigma n + \sigma\rho) - h(u).$$

Second, since $u - \sigma n + \sigma\rho$, u, $n - \sigma n + \sigma\rho$, and n are all non-negative in case 2, we can use the convexity result ($h''(u) > 0$ for $u \geq 0$) in (A4) and $u > n$ to conclude that

$$h(u - \sigma n + \sigma\rho) - h(u) > h(n - \sigma n + \sigma\rho) - h(n).$$

Combining these results, we have $\frac{J'(u)}{J(u)} > \frac{J'(n)}{J(n)}$.

The proof for case 3 (with $-\delta \leq 0 < u \leq n \leq \rho$) is similar to that for case 2, except that the various inequalities are reversed since $0 < u \leq n$.

For case 4 (with $-\delta < u \leq 0 \leq n \leq \rho$), we first use (12) to obtain

$$h(u) \geq h(0). \tag{A5}$$

Second, since $u \leq 0 \leq n$, we have

$$\sigma h(-\sigma u + n + \sigma\rho) + (1-\sigma) h((1-\sigma)u + \sigma\rho) \leq h((1-\sigma)u + \sigma\rho). \tag{A6}$$

Third, (28) implies that

$$n - 0 \geq [(1-\sigma)n + \sigma\rho] - [(1-\sigma)u + \sigma\rho]$$

for $u \in (-\delta, 0]$. Using (A4), we obtain[19]

$$h(0) - h(n) \geq h((1-\sigma)u + \sigma\rho) - h((1-\sigma)n + \sigma\rho). \tag{A7}$$

Combining (A5) to (A7), we obtain that for $u \in (-\delta, 0]$,

$$h(u) - h(n) \geq \sigma h(-\sigma u + n + \sigma\rho) + (1-\sigma)h((1-\sigma)u + \sigma\rho) - h((1-\sigma)n + \sigma\rho),$$

or equivalently, $\frac{J'(u)}{J(u)} - \frac{J'(n)}{J(n)} \leq 0$. This completes the proof of case 4.

Condition (16) is always satisfied, because $\frac{J'(u)}{J(u)} > \frac{J'(n)}{J(n)}$ in cases 1 and 2, and $\frac{J'(u)}{J(u)} \leq \frac{J'(n)}{J(n)}$ in cases 3 and 4. Applying Proposition 2, we prove Proposition 4. ■

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[19] For $u \in (-\delta, 0]$, it can be shown that $(1-\sigma)u + \sigma\rho \geq 0$. However, $(1-\sigma)u + \sigma\rho$ may be greater or smaller than n. In either case, we can apply (A4) to show that (A7) holds.

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Figure 1 : The mean age function, h(u)