THE PROBABILITIES OF ABSOLUTE RUIN IN THE RENEWAL RISK MODEL WITH CONSTANT FORCE OF INTEREST

DIMITRIOS G. KONSTANTINIDES, KAI W. NG, AND QIHE TANG

Abstract. In this paper we consider the probabilities of finite- and infinite-time absolute ruin in the renewal risk model with constant premium rate and constant force of interest. In the particular case of compound Poisson model, explicit asymptotic expressions for the finite- and infinite-time absolute ruin probabilities are given. For the general renewal risk model, we present an asymptotic expression for the infinite-time absolute ruin probability. Conditional distributions of Poisson processes and probabilistic techniques regarding randomly weighted sums are employed in the course of this study.

Keywords: Absolute ruin; Asymptotics; Constant force of interest; Convolution equivalence; Heavy tails; Renewal risk model

1. Introduction

In this paper we address the probabilities of finite- and infinite-time absolute ruin in the renewal risk model with constant premium rate and constant force of interest. In this model, the claim sizes, X_k, $k = 1, 2, \ldots$, form a sequence of independent, identically distributed (i.i.d.), nonnegative random variables with generic random variable X and common distribution $F = 1 - \overline{F}$. The inter-occurrence times θ_k, $k = 1, 2, \ldots$, form another sequence of i.i.d. positive random variables with generic random variable θ. We assume that the sequences $\{\theta, \theta_k, k = 1, 2, \ldots\}$ and $\{X, X_k, k = 1, 2, \ldots\}$ are mutually independent. The occurrence times of the successive claims, $T_n = \sum_{k=1}^{n} \theta_k$, $n = 1, 2, \ldots$, constitute a renewal counting process

$$N_t = \#\{n = 1, 2, \ldots : T_n \leq t\}, \qquad t \geq 0.$$

Therefore, the compound renewal process, $C_t = \sum_{k=1}^{N_t} X_k$, represents aggregate claims up to time $t \geq 0$, with $C_t = 0$ when $N_t = 0$. Let $x \geq 0$ be the initial surplus of the insurance company, let $c > 0$ be the constant premium rate, and let $\delta > 0$ be the constant force of interest so that after time t a capital x becomes $x\mathrm{e}^{\delta t}$. Then the total surplus up to time t,

Date: December 20, 2009.

Department of Statistics and Actuarial - Financial Mathematics, University of the Aegean, Karlovassi, GR-83 200 Samos, Greece; E-mail: konstant@aegean.gr.

Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong; E-mail: kaing@hku.hk.

Department of Statistics and Actuarial Science, University of Iowa, 241 Schaeffer Hall, Iowa City, IA 52242, USA; E-mail: qtang@stat.uiowa.edu.

denoted as $W_\delta(t)$, is given by

$$W_\delta(t) = x\mathrm{e}^{\delta t} + c\int_0^t \mathrm{e}^{\delta(t-y)}\mathrm{d}y - \int_0^t \mathrm{e}^{\delta(t-y)}\mathrm{d}C_y, \qquad t \geq 0. \tag{1.1}$$

If the inter-occurrence time θ is exponentially distributed with mean $1/\lambda$, or, equivalently, $\{N_t, t \geq 0\}$ is a Poisson process with intensity λ, then the model above reduces to the compound Poisson risk model, also called the classical risk model.

In the actuarial literature, the probability of infinite-time ruin is defined to be the probability that the surplus falls below zero. This probability has been extensively investigated in the compound Poisson model with constant force of interest; see, e.g. [23], [24], [1], [19], [16], [20], and [25].

As commented by Embrechts and Schmidli [10], the boundary zero here plays an unrealistic role. They used the alternative boundary $-c/\delta$. Whenever the surplus process hits this boundary, the company will not be able to repay its debts. Motivated by the work of [10], we define the probability of infinite-time absolute ruin as

$$\psi(x,\infty) = \Pr\left(\inf_{t\geq 0} W_\delta(t) < -\frac{c}{\delta}\middle| W_\delta(0) = x\right), \qquad x \geq 0, \tag{1.2}$$

and define the probability of finite-time absolute ruin as

$$\psi(x,t) = \Pr\left(\inf_{0\leq s\leq t} W_\delta(s) < -\frac{c}{\delta}\middle| W_\delta(0) = x\right), \qquad x \geq 0,\ t \geq 0. \tag{1.3}$$

Compared with the study on the ruin probabilities in the ordinary sense, the absolute ruin probabilities have received less attention than it deserves. In the compound Poisson model and for the general case with possibly different forces of interest for invested and borrowed money, using the technique of piecewise deterministic Markov processes and martingales, [10, Th. 1] showed an equality as an estimate for the infinite-time absolute ruin probability. This estimate involves a function that can be explicitly expressed in certain cases such as that of exponential claims, but it is not easy in general.

Absolute ruin was initially considered in [12] and further included in the book [13]. It is to be noted that in [8] martingale methods in the context of absolute ruin were applied and in [9] the influence of the force of interest on the negative surplus through several examples was described.

Recently, most of the works on absolute ruin have been concentrated on the compound Poisson risk model. In [2] the Gerber-Shiu discounted penalty function was used to study the relation between the asymptotic expressions for the infinite-time absolute ruin probability and the ordinary infinite-time ruin probability. In [14] the compound Poisson risk model enriched with an independent Brownian motion was considered and their analysis was based on the jump diffusion model. There are calculations in three special examples with the corresponding numerical applications. In [30] an asymptotic formula for the infinite-time absolute ruin probability with different forces of interest for invested and borrowed money was established. In [29] the multilayer model with different premium rates on different layers of the surplus process and different forces of interest for invested and borrowed money in the frame of the compound Poisson risk model was examined.

This paper aims at asymptotic estimates for the absolute ruin probabilities defined in (1.2) and (1.3) as the initial surplus x increases for the case where the claim sizes follow a distribution from the class $\mathcal{S}(\gamma)$ for $\gamma \geq 0$. In doing so, we mainly apply some standard probabilistic arguments in [15], [7], [21], [26], [27], and [28].

The rest of this paper consists of three sections. As the starting point of the present research, we establish in Section 2 a proposition which presents a simple structure of the probability of absolute ruin as being the tail probability of a randomly weighted sum of non-negative random variables, for both cases of finite- and infinite-time. In Section 3 three main results are shown, two providing explicit asymptotic estimates for the finite- and infinite-time absolute ruin probabilities in the compound Poisson model and one providing a general asymptotic estimate for the infinite-time absolute ruin probability in the general renewal model. Section 4 proves the main results after a series of lemmas.

2. A treatment on the probabilities of absolute ruin

At occurrence time $T_n = \sum_{k=1}^{n} \theta_k$, we observe the value $W_\delta(T_n)$ which represents the surplus immediately after paying the nth claim, $n = 1, 2, \ldots$. By virtue of (1.1), we can see that this sequence satisfies the recurrence equation

$$W_\delta(0) = x, \qquad W_\delta(T_n) = W_\delta(T_{n-1})\mathrm{e}^{\delta\theta_n} + \frac{c}{\delta}\left(\mathrm{e}^{\delta\theta_n} - 1\right) - X_n, \qquad n = 1, 2, \ldots. \tag{2.1}$$

Consider another sequence

$$V_n = W_\delta(T_n) + \frac{c}{\delta}, \qquad n = 0, 1, \ldots.$$

It follows that

$$V_0 = x + \frac{c}{\delta}, \qquad V_n = V_{n-1}\mathrm{e}^{\delta\theta_n} - X_n, \qquad n = 1, 2, \ldots,$$

and hence that

$$V_n = \left(x + \frac{c}{\delta}\right)\prod_{k=1}^{n} \mathrm{e}^{\delta\theta_k} - \sum_{k=1}^{n} X_k \prod_{i=k+1}^{n} \mathrm{e}^{\delta\theta_i}, \qquad n = 1, 2, \ldots.$$

Since absolute ruin can happen only at the time of a claim occurrence, we rewrite the infinite-time absolute ruin probability in (1.2) as

$$\psi(x, \infty) = \Pr\left(\inf_{n\geq 1} W_\delta(T_n) < -\frac{c}{\delta}\middle| W_\delta(0) = x\right).$$

With $Y_k = \mathrm{e}^{-\delta\theta_k}$ for $k = 1, 2, \ldots$, we further rewrite this probability as

$$\begin{aligned} \psi(x,\infty) &= \Pr\left(\inf_{n\geq 1} V_n < 0 \middle| V_0 = x + \frac{c}{\delta}\right) \\ &= \Pr\left(\inf_{n\geq 1}\left(\left(x+\frac{c}{\delta}\right) - \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i\right) < 0\right) \\ &= \Pr\left(\sum_{k=1}^{\infty} X_k \prod_{i=1}^{k} Y_i > x + \frac{c}{\delta}\right). \end{aligned} \tag{2.2}$$

Relation (2.2) can be interpreted easily. Each term $X_k \prod_{i=1}^{k} Y_i$ is the effective claim-size, as discounted by the constant force of interest or as inflation-adjusted, viewing from starting point. The sum $\sum_{k=1}^{\infty} X_k \prod_{i=1}^{k} Y_i$ denotes the total of all discounted future claims while the threshold for absolute ruin, $x + c/\delta$, denotes the initial surplus plus the total discounted premium.

Similarly, for the finite-time absolute ruin probability defined in (1.3), we have

$$\begin{aligned} \psi(x,t) &= \Pr\left(\inf_{1\leq n\leq N_t} W_\delta(T_n) < -\frac{c}{\delta} \middle| W_\delta(0) = x\right) \\ &= \Pr\left(\inf_{1\leq n\leq N_t} V_n < 0 \middle| V_0 = x + \frac{c}{\delta}\right) \\ &= \Pr\left(\sum_{k=1}^{N_t} X_k \prod_{i=1}^{k} Y_i > x + \frac{c}{\delta}\right). \end{aligned} \tag{2.3}$$

We therefore record the following proposition:

Proposition 2.1. *Consider the renewal risk model with constant force of interest $\delta > 0$. The absolute ruin probabilities in* (1.2) *and* (1.3) *can be expressed as in* (2.2) *and* (2.3)*, respectively.*

Proposition 2.1, which forms the foundation of our study, rewrites the absolute ruin probabilities as the tail probabilities of randomly weighted sums. This gives rise to the opportunity of applying some techniques well developed in the study of randomly weighted sums. We also remark that relations (2.2) and (2.3) hold most generally, since in deriving them neither independence nor i.i.d. assumption is used. However, in developing (2.2) and (2.3) our assumption of the same force of interest on invested and borrowed money is essential. Therefore, Proposition 2.1 cannot handle the case of varying force of interest.

3. Main results

Here and henceforth, all limit relationships are for $x \to \infty$ unless stated otherwise and the symbol $\sim$ means that the quotient of both sides tends to 1. Clearly, for two positive functions $f(\cdot)$ and $g(\cdot)$, the relation $f(x) \sim g(x)$ amounts to the conjunction of the relations

$\limsup f(x)/g(x) \leq 1$ and $\liminf f(x)/g(x) \geq 1$, which are denoted as $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$, respectively. For two distributions F_1 and F_2 on $[0,\infty)$, denote by $F_1 * F_2$ their convolution; that is, for every $x \geq 0$,

$$F_1 * F_2(x) = \int_{0-}^{x} F_1(x-y) F_2(\mathrm{d}y).$$

Furthermore, we write $F^{1*} = F$ and $F^{n*} = F^{(n-1)*} * F$ for every $n = 2, 3, \ldots$.

A distribution F on $[0,\infty)$ is said to belong to the class $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$ if

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = \mathrm{e}^{\gamma y} \tag{3.1}$$

for every real number y and the limit

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2\int_{0-}^{\infty} \mathrm{e}^{\gamma y} F(\mathrm{d}y) \tag{3.2}$$

exists and is finite. A larger class, $\mathcal{L}(\gamma)$, is defined by relation (3.1) alone. For the well-known subexponential class $\mathcal{S}(0)$, when $\gamma = 0$, the right-hand side of (3.2) becomes 2. For two distributions, $F_1 \in \mathcal{L}(\gamma)$ and $F_2 \in \mathcal{L}(\gamma)$, satisfying $0 < \liminf \overline{F_1}(x)/\overline{F_2}(x) \leq \limsup \overline{F_1}(x)/\overline{F_2}(x) < \infty$, it is known that $F_1 \in \mathcal{S}(\gamma)$ if and only if $F_2 \in \mathcal{S}(\gamma)$; see, e.g. [17, pages 133-134].

Since it was introduced by [3], [4], and [5], the class $\mathcal{S}(\gamma)$ has been extensively investigated by many researchers and applied to various fields. This class is often used to model claim-size distributions; see, e.g. [11], [18], and [28].

Closely related is the class $\mathcal{R}_{-\infty}$ of distributions with rapidly-varying tails, characterized by the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 0, \qquad y > 1.$$

Clearly, if $F \in \mathcal{L}(\gamma)$ for some $\gamma > 0$ then $F \in \mathcal{R}_{-\infty}$. A lot of distributions in the class $\mathcal{S}(0)$ such as lognormal and Weibull distributions still belong to the class $\mathcal{R}_{-\infty}$.

For the compound Poisson model, the conditional joint distribution of the n occurrence times given a fixed time of observation, $N_t = n$, lends easier evaluation of the weighted sum in (2.3). The inter-play of this conditional distribution and the asymptotic approximation of convolution-equivalent tails entails the first main result in this paper:

Theorem 3.1. *In the compound Poisson model with constant force of interest $\delta > 0$, if $F \in \mathcal{S}(\gamma)$ for some $\gamma \geq 0$ then it holds for every $0 < t < \infty$ that*

$$\psi(x,t) \sim \lambda \exp\left\{ \frac{\lambda}{\delta} \int_{\gamma \mathrm{e}^{-\delta t}}^{\gamma} \frac{\mathrm{E}\mathrm{e}^{sX} - 1}{s} \mathrm{d}s - \frac{\gamma c}{\delta} \right\} \int_0^t \overline{F}\left(x \mathrm{e}^{\delta s}\right) \mathrm{d}s. \tag{3.3}$$

It is tempting to plug in $t = \infty$ on both sides of (3.3) to get an asymptotic expression for the infinite-time absolute ruin probability. But in general, the repeated limits with respect to $x \to \infty$ and $t \to \infty$ of the ratio of both sides of (3.3) may depend on the order of limits, yielding different results. It turns out, however, that this intuitive plug-in result in the compound Poisson model is valid as a consequence of our next main result for the general renewal risk model.

Theorem 3.2. *In the renewal risk model with constant force of interest $\delta > 0$, if $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$ then*

$$\mathrm{E}\mathrm{e}^{\gamma S_\infty} < \infty, \quad \textit{where} \ \ S_\infty = \sum_{k=1}^{\infty} X_k \prod_{i=1}^{k} Y_i$$

and

$$\psi(x,\infty) \sim \mathrm{E}\mathrm{e}^{\gamma S_\infty} \Pr\left(XY > x + c/\delta\right), \tag{3.4}$$

where $Y = \mathrm{e}^{-\delta\theta}$ is the generic random variable of the sequence $\{Y_k = \mathrm{e}^{-\delta\theta_k}, k = 1, 2, \ldots\}$.

The expectation $\mathrm{E}\mathrm{e}^{\gamma S_\infty}$ appearing in relation (3.4) is generally unknown for $\gamma > 0$. However, if we go back to the compound Poisson model then this quantity is explicitly available, as shown in the following last main result of the paper:

Theorem 3.3. *In the compound Poisson model with constant force of interest $\delta > 0$, if $F \in \mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$ for some $\gamma \geq 0$ then it holds that*

$$\psi(x,\infty) \sim \lambda \exp\left\{\frac{\lambda}{\delta}\int_0^\gamma \frac{\mathrm{E}\mathrm{e}^{sX} - 1}{s}\mathrm{d}s - \frac{\gamma c}{\delta}\right\} \int_0^\infty \overline{F}\left(x\mathrm{e}^{\delta s}\right)\mathrm{d}s. \tag{3.5}$$

As remarked above, relation (3.5) corresponds to relation (3.3) with $t = \infty$.

4. Proofs

4.1. **Lemmas.**

Lemma 4.1. *Let F, F_1, and F_2 be three distributions on $[0,\infty)$ such that $F \in \mathcal{S}(\gamma)$ and that the limit $l_i = \lim_{x\to\infty} \overline{F_i}(x)/\overline{F}(x)$ exists and is finite for $i = 1, 2$. Then*

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}(x)} = l_1 \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_2(\mathrm{d}y) + l_2 \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_1(\mathrm{d}y).$$

Proof. See [21, Proposition 2]. □

Lemma 4.2. *Let F_1 and F_2 be two distributions on $[0,\infty)$. If $F_1 \in \mathcal{S}(\gamma)$, $F_2 \in \mathcal{L}(\gamma)$, and $\overline{F_2}(x) = O\left(\overline{F_1}(x)\right)$, then $F_1 * F_2 \in \mathcal{S}(\gamma)$ and*

$$\overline{F_1 * F_2}(x) \sim \overline{F_1}(x) \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_2(\mathrm{d}y) + \overline{F_2}(x) \int_{0-}^{\infty} \mathrm{e}^{\gamma y} F_1(\mathrm{d}y).$$

Proof. See [6, Corollary 1]. □

Lemma 4.3. *Let $\{N_t, t \geq 0\}$ be a Poisson process with occurrence times T_k, $k = 1, 2, \ldots$ and let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables independent of $\{N_t, t \geq 0\}$. Given $N_t = n$ for arbitrarily fixed $t > 0$ and $n = 1, 2, \ldots$, the sum $\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta T_k}$ is equal in distribution to the sum $\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k}$, where the random vector $(U_1, \ldots, U_n)$ consists of i.i.d. random variables uniformly distributed on $(0, 1)$ and is independent of the vector $(X_1, \ldots, X_n)$.*

Proof. According to [22, Theorem 2.3.1], the conditional distribution of $(T_1, \ldots, T_n)$ given $N_t = n$ is the same as the distribution of the random vector $(tU_{(1,n)}, \ldots, tU_{(n,n)})$, where $U_{(1,n)}$, …, $U_{(n,n)}$ denote the order statistics of the n random variables U_1, …, U_n. Furthermore, since in the sum $\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_{(k,n)}}$ the vector $(X_1, \ldots, X_n)$ consists of i.i.d. random variables and is independent of $(U_{(1,n)}, \ldots, U_{(n,n)})$, by rearrangement this sum is equal in distribution to the sum $\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k}$. □

Lemma 4.4. *For two independent nonnegative random variables X and Y, if X follows a distribution $F \in \mathcal{S}(\gamma)$ and Y follows a distribution with an upper endpoint*

$$1 = \sup \{y : \Pr(Y \leq y) < 1\},$$

then the product XY still follows a distribution in the class $\mathcal{S}(\gamma)$.

Proof. See [26, Theorem 1.1]. □

Lemma 4.5. *Let F be a distribution on $[0, \infty)$. If $F \in \mathcal{S}(\gamma)$, then*

(i) it holds for each fixed $n = 1, 2, \ldots$ that

$$\overline{F^{n*}}(x) \sim n \left(\int_{0-}^{\infty} \mathrm{e}^{\gamma y} F(\mathrm{d}y) \right)^{n-1} \overline{F}(x); \tag{4.1}$$

(ii) for every $\varepsilon > 0$ there exists some constant $C_\varepsilon > 0$ such that the inequality

$$\frac{\overline{F^{n*}}(x)}{\overline{F}(x)} \leq C_\varepsilon \left(\int_{0-}^{\infty} \mathrm{e}^{\gamma y} F(\mathrm{d}y) + \varepsilon \right)^{n} \tag{4.2}$$

holds for all $n = 1, 2, \ldots$ and all x.

Proof. See [5, page 665]. □

4.2. **Proof of Theorem 3.1.** Starting with (2.3) and conditioning on N_t, we have

$$\psi(x, t) = \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta T_k} > x + \frac{c}{\delta} \,\middle|\, N_t = n \right) \Pr(N_t = n).$$

By means of Lemma 4.3, we can have a sequence of i.i.d. random variables, $\{U, U_k, k = 1, 2, \ldots\}$, uniformly distributed on the interval $(0, 1)$ and independent of $\{X, X_k, k = 1, 2, \ldots\}$, such that

$$\psi(x, t) = \sum_{n=1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k} > x + \frac{c}{\delta} \right) \Pr(N_t = n).$$

By Lemma 4.4, the products $X_k \mathrm{e}^{-\delta t U_k}$, $k = 1, 2, \ldots$, are i.i.d. with common distribution belonging to the class $\mathcal{S}(\gamma)$. Therefore by Lemma 4.5(i), it holds for each $n = 1, 2, \ldots$ that

$$\Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k} > x + \frac{c}{\delta}\right) \sim n \mathrm{e}^{-\gamma c/\delta} \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}}\right)^{n-1} \Pr\left(X \mathrm{e}^{-\delta t U} > x\right).$$

Applying the dominated convergence theorem justified by Lemma 4.5(ii), we obtain

$$\begin{aligned}
\psi(x,t) &\sim \mathrm{e}^{-\gamma c/\delta} \sum_{n=1}^{\infty} n \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}}\right)^{n-1} \Pr\left(X \mathrm{e}^{-\delta t U} > x\right) \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} \\
&= \lambda t \mathrm{e}^{-\gamma c/\delta} \exp\left\{\lambda t \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}} - 1\right)\right\} \Pr\left(X \mathrm{e}^{-\delta t U} > x\right) \\
&= \lambda \exp\left\{\frac{\lambda}{\delta} \int_{\gamma \mathrm{e}^{-\delta t}}^{\gamma} \frac{\mathrm{E}\mathrm{e}^{sX} - 1}{s} \mathrm{d}s - \frac{\gamma c}{\delta}\right\} \int_0^t \overline{F}\left(x \mathrm{e}^{\delta s}\right) \mathrm{d}s.
\end{aligned}$$

This leads to (3.3).

4.3. **Proof of Theorem 3.2.** Our proof below is motivated by an idea of Grey [15] in his proof of Theorem 1. Let Z be a random variable with distribution F and independent of $\{(X,Y), (X_k, Y_k), k = 1, 2, \ldots\}$, and denote the distribution of $Y = \mathrm{e}^{-\delta\theta}$ by G, which is supported on $(0,1)$. Then,

$$\begin{aligned}
\Pr\left((Z+X)Y > x\right) &= \int_0^1 \Pr\left(Z + X > x/y\right) G(\mathrm{d}y) \\
&\sim 2\mathrm{E}\mathrm{e}^{\gamma X} \int_0^1 \overline{F}\left(x/y\right) G(\mathrm{d}y) \\
&= o(1)\overline{F}(x),
\end{aligned}$$

where the second step is due to $F \in \mathcal{S}(\gamma)$ and the last step due to $F \in \mathcal{R}_{-\infty}$. Therefore, there is some $x_0 > 0$ large enough such that, for all $x > x_0$,

$$\Pr\left((Z+X)Y > x\right) \leq \overline{F}(x). \tag{4.3}$$

Construct a new conditional random variable $X^* = (Z | Z > x_0)$, whose distribution still belongs to the intersection $\mathcal{S}(\gamma) \cap \mathcal{R}_{-\infty}$. Then, it is easy to see that

$$(X^* + X)Y \overset{d}{\leq} X^*, \tag{4.4}$$

or, equivalently, for all x,

$$\Pr\left((X^* + X)Y > x\right) \leq \Pr\left(Z > x | Z > x_0\right). \tag{4.5}$$

Actually, when $x \le x_0$ relation (4.5) is trivial as the right-hand side becomes equal to 1, while when $x > x_0$, by (4.3),

$$\begin{aligned}\Pr\left((X^* + X)Y > x\right) &= \frac{\Pr\left((Z+X)Y > x, Z > x_0\right)}{\Pr\left(Z > x_0\right)} \\ &\le \frac{\Pr\left((Z+X)Y > x\right)}{\Pr\left(Z > x_0\right)} \\ &\le \frac{\Pr\left(Z > x\right)}{\Pr\left(Z > x_0\right)} \\ &= \Pr\left(Z > x | Z > x_0\right).\end{aligned}$$

Thus, relation (4.5) always holds. Relation (4.4) leads to

$$(X^* + X_1)Y_1 \overset{d}{\le} X^*, \qquad (X^* + X_2)Y_2 \overset{d}{\le} X^*.$$

It follows that

$$\left(\left(X^* + X_2\right)Y_2 + X_1\right)Y_1 \overset{d}{\le} X^*.$$

Hence, $S_{T_1} = X_1Y_1 \overset{d}{\le} X^*$ and $S_{T_2} = X_1Y_1 + X_2Y_2Y_1 \overset{d}{\le} X^*$. Repeating these iterations we obtain $S_{T_n} \overset{d}{\le} X^*$ for every $n = 1, 2, \ldots$. Letting $n \to \infty$ yields

$$S_\infty \overset{d}{\le} X^*,$$

which implies, as a by-product, that $\mathrm{E}e^{\gamma S_\infty} < \infty$. Let $\tilde{S}_\infty$ be a copy of S_∞ independent of $\{(X_k, Y_k), k = 1, 2, \ldots\}$. Then, for every $n = 1, 2, \ldots$,

$$S_\infty \overset{d}{=} S_{T_n} + \tilde{S}_\infty \prod_{i=1}^{n} Y_i.$$

Therefore,

$$S_\infty \overset{d}{\le} S_{T_n} + X^* \prod_{i=1}^{n} Y_i.$$

From this we obtain

$$\begin{aligned}\Pr\left(S_\infty > x\right) &\le \Pr\left(S_{T_n} + X^* \prod_{i=1}^{n} Y_i > x\right) \\ &= \int_0^1 \Pr\left(X_1 + \sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i > \frac{x}{y}\right) G(\mathrm{d}y). \qquad (4.6)\end{aligned}$$

Let $n \geq 2$ in (4.6). By Lemma 4.2,

$$\begin{aligned}
&\Pr\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i > x\right) \\
&\leq \Pr\left(\left(\sum_{k=2}^{n} X_k + X^*\right) Y_2 > x\right) \\
&= \int_0^1 \Pr\left(\sum_{k=2}^{n} X_k + X^* > \frac{x}{y}\right) G(\mathrm{d}y) \\
&\sim \left(\mathrm{E}e^{\gamma X^*}(n-1)\left(\mathrm{E}e^{\gamma X}\right)^{n-2} + \frac{\left(\mathrm{E}e^{\gamma X}\right)^{n-1}}{\overline{F}(x_0)}\right) \int_0^1 \Pr\left(X > x/y\right) G(\mathrm{d}y) \\
&= o(1)\overline{F}(x),
\end{aligned}$$

where the last step is due to $F \in \mathcal{R}_{-\infty}$. Now we apply Lemma 4.1 to continue the derivation of (4.6) to find that

$$\begin{aligned}
\Pr\left(S_\infty > x\right) &\lesssim \int_0^1 \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i\right)} \Pr\left(X_1 > x/y\right) G(\mathrm{d}y) \\
&= \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i\right)} \Pr\left(XY > x\right),
\end{aligned}$$

or, equivalently,

$$\limsup_{x\to\infty} \frac{\Pr\left(S_\infty > x\right)}{\Pr\left(XY > x\right)} \leq \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i\right)}.$$

Clearly, $\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i + X^* \prod_{i=2}^{n} Y_i$ converges to S_∞ in distribution as $n \to \infty$. Therefore, by the dominated convergence theorem, the expectation on the right-hand side above converges to $\mathrm{E}e^{\gamma S_\infty}$ as $n \to \infty$. This establishes the asymptotic upper bound as

$$\limsup_{x\to\infty} \frac{\Pr\left(S_\infty > x\right)}{\Pr\left(XY > x\right)} \leq \mathrm{E}e^{\gamma S_\infty}.$$

It is easier to construct the corresponding asymptotic lower bound. Similarly as above,

$$\begin{aligned}
\Pr\left(S_\infty > x\right) &\geq \Pr\left(S_{T_n} > x\right) \\
&= \int_0^1 \Pr\left(X_1 + \sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i > \frac{x}{y}\right) G(\mathrm{d}y) \\
&\sim \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i\right)} \Pr\left(XY > x\right),
\end{aligned}$$

or, equivalently,

$$\liminf_{x\to\infty} \frac{\Pr\left(S_\infty > x\right)}{\Pr\left(XY > x\right)} \geq \mathrm{E}e^{\gamma\left(\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i\right)}.$$

Clearly, $\sum_{k=2}^{n} X_k \prod_{i=2}^{k} Y_i$ converges to S_∞ in distribution as $n \to \infty$. Therefore, the expectation on the right-hand side above converges to $\mathrm{E}\mathrm{e}^{\gamma S_\infty}$ as $n \to \infty$ too. We have

$$\liminf_{x\to\infty} \frac{\Pr(S_\infty > x)}{\Pr(XY > x)} \geq \mathrm{E}\mathrm{e}^{\gamma S_\infty}.$$

This ends the proof of Theorem 3.2.

4.4. **Proof of Theorem 3.3.** We first calculate the factor $\mathrm{E}\mathrm{e}^{\gamma S_\infty}$ of (3.4) in the framework of the compound Poisson model. As in the proof of Theorem 3.1, applying Lemma 4.3 to

$$S_t = \sum_{k=1}^{N_t} X_k \prod_{i=1}^{k} Y_i = \sum_{k=1}^{N_t} X_k \mathrm{e}^{-\delta T_k}$$

we have a sequence of i.i.d. random variables, $\{U, U_k, k = 1, 2, \ldots\}$, uniformly distributed on the interval $(0, 1)$ and independent of $\{X, X_k, k = 1, 2, \ldots\}$, such that

$$\begin{aligned}
\mathrm{E}\mathrm{e}^{\gamma S_t} &= \sum_{n=0}^{\infty} \mathrm{E}\left(\mathrm{e}^{\gamma \sum_{k=1}^{N_t} X_k \mathrm{e}^{-\delta T_k}} \;\middle|\; N_t = n\right) \Pr(N_t = n)\\
&= \sum_{n=0}^{\infty} \mathrm{E}\left(\mathrm{e}^{\gamma \sum_{k=1}^{n} X_k \mathrm{e}^{-\delta t U_k}}\right) \Pr(N_t = n)\\
&= \sum_{n=0}^{\infty} \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}}\right)^n \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}\\
&= \exp\left\{\lambda t \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}} - 1\right)\right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{E}\mathrm{e}^{\gamma S_\infty} &= \lim_{t\to\infty} \mathrm{E}\mathrm{e}^{\gamma S_t}\\
&= \lim_{t\to\infty} \exp\left\{\lambda t \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta t U}} - 1\right)\right\}\\
&= \exp\left\{\lim_{t\to\infty} \lambda \int_0^t \left(\mathrm{E}\mathrm{e}^{\gamma X \mathrm{e}^{-\delta s}} - 1\right) \mathrm{d}s\right\}\\
&= \exp\left\{\frac{\lambda}{\delta} \int_0^\gamma \frac{\mathrm{E}\mathrm{e}^{sX} - 1}{s} \mathrm{d}s\right\}. \qquad (4.7)
\end{aligned}$$

We then calculate the probability $\Pr(XY > x + c/\delta)$ in (3.4). Since $F \in \mathcal{R}_{-\infty}$, by [26, Lemma 3.1(i)] it holds for every $\varepsilon > 0$ that

$$\Pr(XY > x) \sim \int_0^\varepsilon \overline{F}\left(x\mathrm{e}^{\delta s}\right) \lambda \mathrm{e}^{-\lambda s} \mathrm{d}s$$

and that

$$\int_0^\varepsilon \overline{F}\left(x\mathrm{e}^{\delta s}\right) \mathrm{d}s \sim \int_0^\infty \overline{F}\left(x\mathrm{e}^{\delta s}\right) \mathrm{d}s$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\Pr(XY > x) \sim \lambda \int_0^\infty \overline{F}\left(x\mathrm{e}^{\delta s}\right) \mathrm{d}s.$$

Note that the distribution of XY still belongs to the class $\mathcal{S}(\gamma)$ according to Lemma 4.4. Hence,

$$\Pr\left(XY > x + c/\delta\right) \sim \mathrm{e}^{-\gamma c/\delta} \Pr\left(XY > x\right) \sim \lambda \mathrm{e}^{-\gamma c/\delta} \int_0^\infty \overline{F}\left(x\mathrm{e}^{\delta s}\right) \mathrm{d}s. \tag{4.8}$$

Plugging (4.7) and (4.8) into (3.4) yields relation (3.5).

Acknowledgments. The authors would like to thank the anonymous referee for his/her careful reading of an earlier version of this paper.

References

[1] Asmussen, S. Subexponential asymptotics for stochastic processes: extremal behavior, stationary distributions and first passage probabilities. Ann. Appl. Probab. 8 (1998), no. 2, 354–374.
[2] Cai, J. On the time value of absolute ruin with debit interest. Adv. in Appl. Probab. 39 (2007), no. 2, 343–359.
[3] Chistyakov, V. P. A theorem on sums of independent positive random variables and its applications to branching random processes. Theory Probab. Appl. 9 (1964), 640–648.
[4] Chover, J.; Ney, P.; Wainger, S. Functions of probability measures. J. Analyse Math. 26 (1973a), 255–302.
[5] Chover, J.; Ney, P.; Wainger, S. Degeneracy properties of subcritical branching processes. Ann. Probability 1 (1973b), 663–673.
[6] Cline, D. B. H. Convolution tails, product tails and domains of attraction. Probab. Theory Relat. Fields 72 (1986), no. 4, 529–557.
[7] Cline, D. B. H.; Samorodnitsky, G. Subexponentiality of the product of independent random variables. Stochastic Process. Appl. 49 (1994), no. 1, 75–98.
[8] Dassios, A.; Embrechts, P. Martingales and insurance risk. Comm. Statist. Stochastic Models 5 (1989), no. 2, 181–217.
[9] Dickson, D. C. M.; Egídio dos Reis, A. D. The effect of interest on negative surplus. Insurance Math. Econom. 21 (1997), no. 1, 1–16.
[10] Embrechts, P.; Schmidli, H. Ruin estimation for a general insurance risk model. Adv. in Appl. Probab. 26 (1994), no. 2, 404–422.
[11] Embrechts, P.; Veraverbeke, N. Estimates for the probability of ruin with special emphasis on the possibility of large claims. Insurance Math. Econom. 1 (1982), no. 1, 55–72.
[12] Gerber, H. U. Der Einfluss von Zins auf die Ruinwahrscheinlichkeit. Bull. Swiss Assoc. Actuaries 71 (1971), 63–70.
[13] Gerber, H. U. An Introduction to Mathematical Risk Theory. University of Pennsylvania, Philadelphia, 1979.
[14] Gerber, H. U.; Yang, H. Absolute ruin probabilities in a jump diffusion risk model with investment. N. Am. Actuar. J. 11 (2007), no. 3, 159–169.
[15] Grey, D. R. Regular variation in the tail behaviour of solutions of random difference equations. Ann. Appl. Probab. 4 (1994), no. 1, 169–183.
[16] Kalashnikov, V.; Konstantinides, D. Ruin under interest force and subexponential claims: a simple treatment. Insurance Math. Econom. 27 (2000), no. 1, 145–149.
[17] Klüppelberg, C. Subexponential distributions and integrated tails. J. Appl. Probab. 25 (1988), no. 1, 132–141.
[18] Klüppelberg, C. Estimation of ruin probabilities by means of hazard rates. Insurance Math. Econom. 8 (1989), no. 4, 279–285.
[19] Klüppelberg, C.; Stadtmüller, U. Ruin probabilities in the presence of heavy-tails and interest rates. Scand. Actuar. J. 1998, no. 1, 49–58.

[20] Konstantinides, D.; Tang, Q.; Tsitsiashvili, G. Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails. Insurance Math. Econom. 31 (2002), no. 3, 447–460.
[21] Rogozin, B.A.: Sgibnev M.S. Banach algebras of measures on the real axis with the given asymptotics of distributions at infinity. Siberian Math. J. 40 (1999), no 3, 565–576.
[22] Ross, S. M. Stochastic Processes. John Wiley & Sons, Inc., New York, 1983.
[23] Sundt, B.; Teugels, J. L. Ruin estimates under interest force. Insurance Math. Econom. 16 (1995), no. 1, 7–22.
[24] Sundt, B.; Teugels, J. L. The adjustment function in ruin estimates under interest force. Insurance Math. Econom. 19 (1997), no. 2, 85–94.
[25] Tang, Q. The finite-time ruin probability of the compound Poisson model with constant interest force. J. Appl. Probab. 42 (2005), no. 3, 608–619.
[26] Tang, Q. On convolution equivalence with applications. Bernoulli 12 (2006), no. 3, 535–549.
[27] Tang, Q.; Tsitsiashvili, G. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. Stochastic Process. Appl. 108 (2003), no. 2, 299–325.
[28] Tang, Q.; Tsitsiashvili, G. Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. Adv. in Appl. Probab. 36 (2004), no. 4, 1278–1299.
[29] Yang, H.; Zhang, Z.; Lan, C. On the time value of absolute ruin for a multi-layer compound Poisson model under interest force. Statist. Probab. Lett. 78 (2008), no. 13, 1835–1845.
[30] Zhu, J.; Yang, H. Estimates for the absolute ruin probability in the compound Poisson risk model with credit and debit interest. J. Appl. Probab. 45 (2008), no. 3, 818–830.