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A general finite strip for the analysis of folded plates

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Abstract

In the conventional semi-analytical finite strip analysis of folded plates, the boundary conditions and the intermediate support conditions must be satisfied *a priori*. The admissible functions used as the longitudinal part of the displacement functions are sometimes difficult to find, and they are valid for specific conditions only. In this paper, a general finite strip is developed for the analysis of folded plate structures. The geometric constraints of the folded plates, such as the conditions at the end and intermediate supports, are modelled by very stiff translational and rotational springs, as appropriate. The complete Fourier series including the constant term are chosen as the longitudinal approximating functions for each of the displacements. As these displacement functions are more general in nature and independent of one another, they are capable of giving more accurate solutions. The potential problem of ill-conditioned matrices is investigated and the appropriate choice of the very stiff springs is also suggested. The formulation is done in such a way to obtain a unified approach, taking full advantage of the power of modern computers. A few numerical examples are presented for comparison with numerical results from published solutions or solutions obtained from the finite element method. The results show that this kind of strips is versatile, efficient and accurate for the analysis of folded plates.

Keywords: folded plates, longitudinal approximating functions, semi-analytical finite strip

1 Introduction

The use of folded plates, as a special kind of shells, is popular in civil engineering and other branches of engineering. Folded plates made of reinforced concrete are extensively used in long-span roofs. The box girder bridges and shear walls can also be considered and analysed as a particular form of folded plates. As many folded plate structures have uniform cross sections, they have been successfully analysed by the conventional semi-analytical finite strip method (Cheung 1969; Cheung and Cheung 1971; Cheung and Delcourt 1997), the spline finite strip method (Fan 1982; Cheung and Fan 1983) as well as the finite strip method using computed shape functions (Cheung and Kong, 1995) and the finite strip method using modified Fourier series (Cheung et al., 1998). A summary of the development of finite strip method was presented by Cheung and Tham (1998). Shahidi et al. (2005) developed a new finite strip method to analyse very large deformations but small strain of thin plates and folded plates by use of the elastic Cosserat theory. Recently, Eccher et al. (2008; 2009) successfully carried out both linear elastic and geometric nonlinear analyses of perforated folded plate structures using the isoparametric spline finite strip method.

In the conventional semi-analytical finite strip method, the boundary conditions and the intermediate support conditions must be satisfied *a priori*. The admissible functions used as the longitudinal part of the displacement functions are sometimes difficult to find, and they are valid for specific conditions only. Moreover, in the in-plane stress analysis of folded plates, the longitudinal displacement functions are very often related to the first derivative of the transverse displacement functions. These place certain restrictions on the general use of the method.

In this paper, a general semi-analytical finite strip is presented for the analysis of folded plate structures using an expansion beyond the span of structure. The folded plate structure is discretized into flat shell finite strips for analysis. The boundary conditions are then introduced by imposing very stiff translational and rotational springs, as appropriate. A few numerical examples are presented for comparison.

2 Longitudinal approximating function

Figure 1 shows an arbitrary function $f(y)$ which is defined over the interval (0, l). To avoid any errors associated with the Gibbs phenomenon, one can construct a function $g(y)$ over the interval ($-l$, 0) to form a new function $\varphi(y)$ so that extension using the period $T=2l$ gives a periodic function which ensures the continuity and smoothness at abscissae of integral multiples of l, as shown in Figure 1.

The function $f(y)$ initially defined over the interval (0, l) can be similarly extended to form $\varphi(y)$ over the interval ($-l$, l) and written in terms of trigonometric functions of $\omega_0 = \pi/l$ as (Liu et al., 1995)

$$f(y) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos(m\omega_0 y) + b_m \sin(m\omega_0 y)\right] \tag{1}$$

where

$$a_m = \frac{1}{l}\int_{-l}^{l}\varphi(y)\cos(m\omega_0 y)dy\,; \quad b_m = \frac{1}{l}\int_{-l}^{l}\varphi(y)\sin(m\omega_0 y)dy \qquad m=0,\ 1,\ 2,\ \ldots \tag{2}$$

For convenience, the terms in Eq. (1) are renumbered as a unified set of longitudinal approximating functions $Y_m(y)$ with another index variable m running from 1 to $R = 2r+1$ as

$$Y_m(y) = \begin{cases} 1 & m = 1 \\ \cos(k\omega_0 y) & m = 2k;\ k = 1, 2, \ldots, r \\ \sin(k\omega_0 y) & m = 2k+1;\ k = 1, 2, \ldots, r \end{cases} \cdot \tag{3}$$

Theoretically this set of approximating functions $Y_m(y)$ can approach any well behaved function defined over the interval (0, l) effectively. The extended period of $2l$ is an optimum value, as further increase in period will not improve the accuracy any more

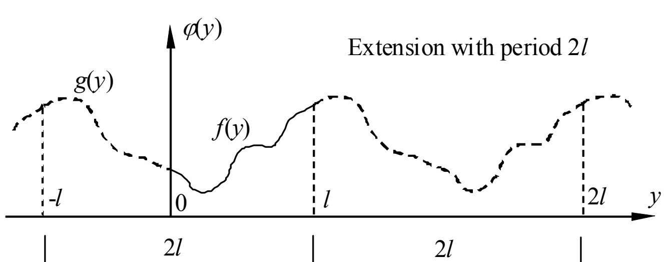

Figure 1. An arbitrary function f(y) defined over the interval (0, l) and its periodic extension.

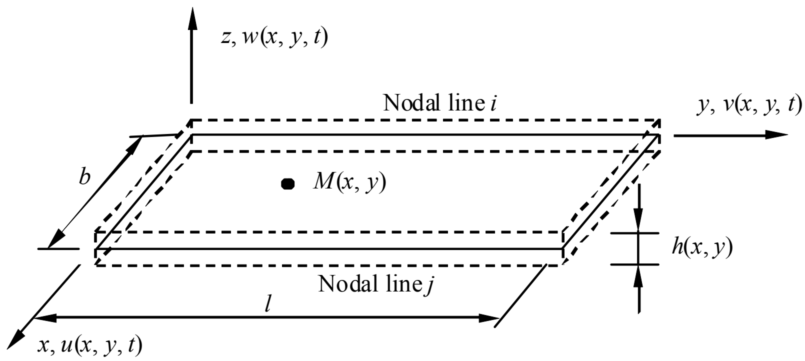

Figure 2. A typical flat shell strip in local coordinate system.

3 Finite strip formulation

3.1 Finite strip discretisation

Figure 2 shows a typical flat shell finite strip of length l, breadth b and thickness h. The displacement vector $\mathbf{u}(x,y,t)$ at an arbitrary point $M(x,y)$ located on the mid-surface of the strip comprises the components $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ in the x, y and z directions respectively, where t stands for time. In line with the normal finite strip approach, the displacement vector $\mathbf{u}(x,y,t)$ can be written in terms of the shape function matrices $\mathbf{c}_u(x)$, $\mathbf{c}_v(x)$ and $\mathbf{c}_w(x)$, the displacement functions $Y_m(y)$ and the nodal displacement vector $\delta_m(t)$ as (Cheung and Tham 1998)

$$\mathbf{u}(x,y,t) = \begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t \end{Bmatrix} = \sum_{m=1}^{R} \begin{bmatrix} \mathbf{c}_u(x) \\ \mathbf{c}_v(x) \\ \mathbf{c}_w(x) \end{bmatrix} Y_m(y)\, \boldsymbol{\delta}_m(t) = \sum_{m=1}^{R} \mathbf{H}_m(x,y)\, \boldsymbol{\delta}_m(t) \tag{4}$$

where $\mathbf{H}_m(x, y)$ is the interpolation function matrix for the m-th mode, and

$$\boldsymbol{\delta}_m(t) = \begin{bmatrix} u_i(t) & v_i(t) & w_i(t) & \theta_i(t) & u_j(t) & v_j(t) & w_j(t) & \theta_j(t) \end{bmatrix}^{\mathrm{T}}. \tag{5}$$

Once the displacement functions are chosen, the stiffness matrices, mass matrices, load vectors, etc. can be established and the problem solved in the usual finite strip method (Cheung and Tham, 1998).

3.2 Stiffness matrix $\mathbf{K}_{mn}^{fss}$ of the flat shell strip

The stiffness matrix $\mathbf{K}_{mn}^{fss}$ of the flat shell strip can be worked out by integrating over the area A_e as

$$\mathbf{K}_{mn}^{fss} = \iint_{A_e} \mathbf{B}_m^{\mathrm{T}} \mathbf{D} \mathbf{B}_n \, dxdy \tag{6}$$

in terms of the elasticity matrix $\mathbf{D}$ and the strain matrices $\mathbf{B}_m$ and $\mathbf{B}_n$. The stiffness matrix $\mathbf{K}_{mn}^{fss}$ of the flat shell strip can also be assembled using the elements k_{ij}^{p} of the stiffness matrix $\mathbf{K}_{mn}^{p}$ of the plane stress strip, and the elements k_{ij}^{b} of the stiffness matrix $\mathbf{K}_{mn}^{b}$ of the plate bending strip as

$$
\mathbf{K}_{mn}^{fss} = \begin{bmatrix}
k_{11}^p & k_{12}^p & 0 & 0 & k_{13}^p & k_{14}^p & 0 & 0 \\
k_{21}^p & k_{22}^p & 0 & 0 & k_{23}^p & k_{24}^p & 0 & 0 \\
0 & 0 & k_{11}^b & k_{12}^b & 0 & 0 & k_{13}^b & k_{14}^b \\
0 & 0 & k_{21}^b & k_{22}^b & 0 & 0 & k_{23}^b & k_{24}^b \\
k_{31}^p & k_{32}^p & 0 & 0 & k_{33}^p & k_{34}^p & 0 & 0 \\
k_{41}^p & k_{42}^p & 0 & 0 & k_{43}^p & k_{44}^p & 0 & 0 \\
0 & 0 & k_{31}^b & k_{32}^b & 0 & 0 & k_{33}^b & k_{34}^b \\
0 & 0 & k_{41}^b & k_{42}^b & 0 & 0 & k_{43}^b & k_{44}^b
\end{bmatrix}. \tag{7}
$$

3.3 Mass matrix $\mathbf{M}_{mn}^{fss}$ of the flat shell strip

The mass matrix $\mathbf{M}_{mn}^{fss}$ of the flat shell strip can be worked out as

$$
\mathbf{M}_{mn}^{fss} = \iint_{A_e} \rho h(x,y)\mathbf{H}_m^{\mathrm{T}}(x,y)\mathbf{H}_n(x,y)dxdy \tag{8}
$$

where ρ is the mass density of the strip. Likewise the mass matrix $\mathbf{M}_{mn}^{fss}$ of the flat shell strip can be assembled using the elements m_{ij}^p of the mass matrix $\mathbf{M}_{mn}^p$ of the plane stress strip, and the elements m_{ij}^b of the mass matrix $\mathbf{M}_{mn}^b$ of the plate bending strip in a manner similar to stiffness matrices.

3.4 Support conditions

Unlike the conventional semi-analytical finite strip method in which the support conditions have been satisfied *a priori* by the choice of displacement functions, the support conditions have to be introduced afterwards in the present method. The rigid supports can be considered as very stiff elastic supports. Figure 3 shows a flat shell strip with a point spring support located at the point (x_{pi}, y_{pi}), with spring stiffness of k_{pxi}, k_{pyi} and k_{pzi} along the directions of coordinate axes. The potential energy U^{ps} stored in the elastic point supports can be expressed as

$$
U^{ps} = \frac{1}{2}\sum_i \left[k_{pxi}u^2(x_{pi},y_{pi},t) + k_{pyi}v^2(x_{pi},y_{pi},t) + k_{pzi}w^2(x_{pi},y_{pi},t)\right]. \tag{9}
$$

Substituting Eq. (6) into Eq. (9) gives

$$
U^{ps} = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\boldsymbol{\delta}_m^{\mathrm{T}}\mathbf{K}_{mn}^{ps}\boldsymbol{\delta}_n \tag{10}
$$

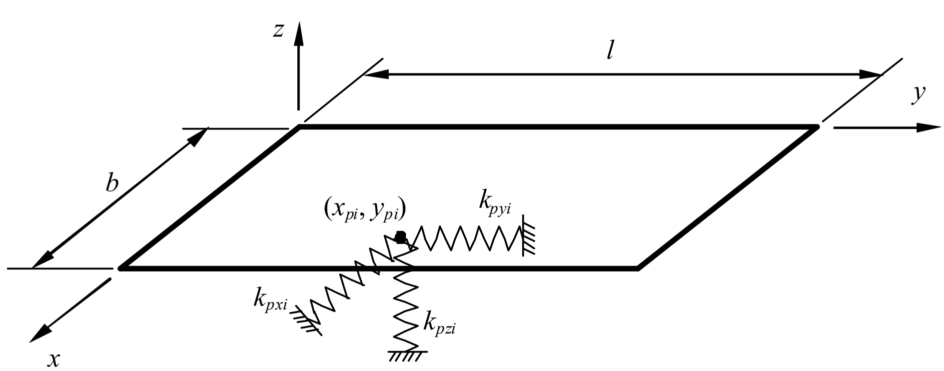

Figure 3. A flat shell strip with an elastic point support at (x_{pi}, y_{pi}).

where $\mathbf{K}_{mn}^{ps}$ is the contribution of the elastic point support to the stiffness matrix of the flat shell strip given by

$$\mathbf{K}_{mn}^{ps} = \sum_i \left[k_{pxi} \mathbf{c}_u^{\mathrm{T}}(x_{pi}) \mathbf{c}_u(x_{pi}) + k_{pyi} \mathbf{c}_v^{\mathrm{T}}(x_{pi}) \mathbf{c}_v(x_{pi}) + k_{pzi} \mathbf{c}_w^{\mathrm{T}}(x_{pi}) \mathbf{c}_w(x_{pi}) \right] Y_m(y_{pi}) Y_n(y_{pi}) . \qquad (11)$$

The contribution of other types of support to the stiffness matrix can be similarly worked out.

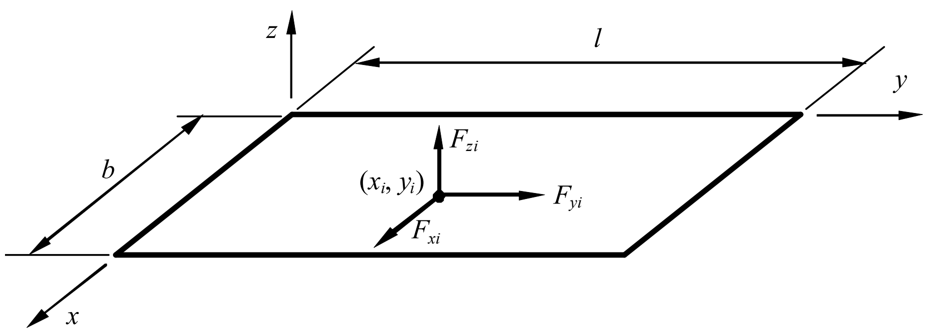

Figure 4. A flat shell strip under a point load at (x_i, y_i).

3.5 Load vectors

Figure 4 shows a flat shell strip subjected to a point load $\mathbf{F}_i = \{F_{xi}, F_{yi}, F_{zi}\}$ at the point (x_i, y_i). The potential energy U^{pl} associated with the point load can be expressed as

$$U^{pl} = -\sum_i \mathbf{F}_i^{\mathrm{T}} \cdot \mathbf{u}(x_i, y_i, t) . \qquad (12)$$

Substituting Eq. (6) into Eq. (12), one has

$$U^{pl} = -\sum_{m=1}^{R} \boldsymbol{\delta}_m^{\mathrm{T}} \cdot \mathbf{f}_m^{pl} \qquad (13)$$

where $\mathbf{f}_{mn}^{pl}$ is the load vector corresponding to the point load $\mathbf{F}_i$ and it appears as

$$\mathbf{f}_m^{pl} = \sum_i \mathbf{H}_m^{\mathrm{T}}(x_i, y_i) \cdot \mathbf{F}_i . \qquad (14)$$

The load vectors for other cases can be worked out in a similar manner.

3.6 Formulation and solution of the governing equation

When a plate strip is of constant thickness, the elements of the strip stiffness and mass matrices can be expressed in terms of the following basic integral

$$I_{ij} = \int_0^l Y_i(y) Y_j(y) dy \qquad (i = 1, 2, \ldots, R; j = 1, 2, \ldots, R). \qquad (15)$$

Note that the integral I_{ij} is symmetrical and hence $I_{ij} = I_{ji}$. Because of the orthogonality of the harmonic functions, most of the off-diagonal values are zero. As the above stiffness matrices, mass matrices and load vectors have been derived based on the local coordinate system. They should be transformed to the global coordinate system before being assembled together to form the corresponding global matrices and vectors for solution of the static and dynamic problems (Cheung and Tham, 1998).

4 Numerical examples

In the examples of folded plate structures investigated, it is assumed that a diaphragm is provided at each support to prevent the distortion of the cross section.

4.1 *Effect of support stiffness on the solution accuracy*

To investigate the potential problem of ill-conditioned matrices, a clamped beam is modelled as a strip with translational and rotational springs having the same numerical stiffness at the two ends. Free vibration analysis is carried out using 21 terms and the first 5 frequencies are compared with the exact solutions. Figure 5 shows the variation of percentage errors with the ratio of the added stiffness to the maximum element of the stiffness matrix. It shows that when the ratio is in the range of 10^3 to 10^8, accurate solutions can be obtained using double precision computation of a personal computer.

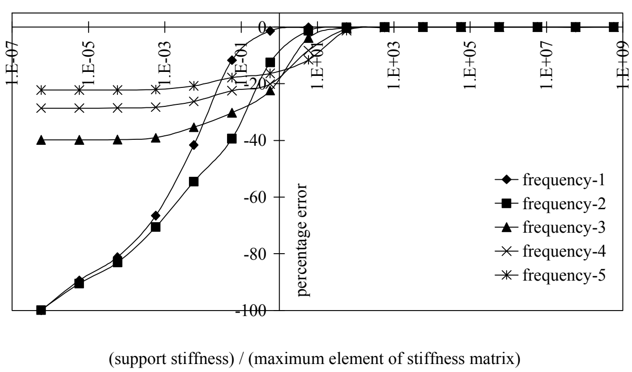

Figure 5. Effect of support stiffness on solution accuracy.

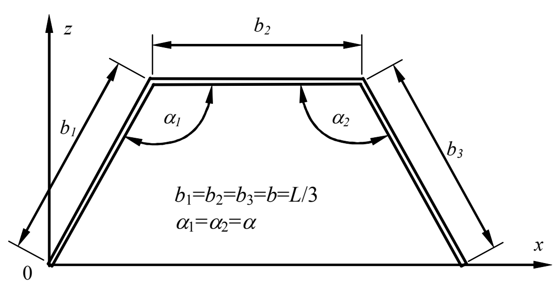

Figure 6. Cross section of a folded plate with two folds.

4.2 *Free vibration of a cantilever folded plate with two folds*

Figure 6 shows the cross section of a cantilever folded plate of length L with two folds. The crank angle α may take values of 90°, 120° and 150°. A total of 9 strips of equal width and 13 terms have been used. The Poisson's ratio μ is 0.3. The dimensionless frequency parameter λ_i is defined as $\lambda_i = \omega_i L\sqrt{\rho(1-\mu^2)/E}$ where ω_i is the i^{th} natural circular frequency, ρ is the mass density and E is Young's modulus. The lowest three frequencies obtained using the present method are compared with the finite element-transfer matrix solutions of Liu and Huang (1992) and the finite strip solutions of Kong (1994) in Table 1. Good agreement is observed among various solutions. The maximum difference among the three sets of results is 3%, and in most cases the difference is within 0.5%.

Table 1. Natural frequencies of a cantilever folded plate with two folds.

Crank angle α (degree)	Mode No.	Frequencies λ_i		
		Present	Liu & Huang (1992)	Kong (1994)
90	1	0.1237	0.1249	0.1245
	2	0.1254	0.1260	0.1258
	3	0.2593	0.2579	0.2598
120	1	0.0973	0.1000	0.0980
	2	0.1245	0.1241	0.1249
	3	0.2585	0.2571	0.2589
150	1	0.0672	0.0687	0.0675
	2	0.1140	0.1145	0.1145
	3	0.2063	0.2109	0.2078

5 Conclusions

A general finite strip is developed for the analysis of folded plate structures. The geometric constraints of the folded plates, such as the conditions at the end and intermediate supports, are modelled by very stiff springs. The complete Fourier series including the constant term are chosen as the longitudinal approximating functions for each of the displacements. Numerical examples are presented for comparison with available numerical results. The results show that this kind of strips is versatile, efficient and accurate for the analysis of folded plates. The method is especially useful in the analysis of folded plates with complicated support conditions, such as box girder bridges.

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