Decentralized H_∞ control for air traffic flow networks modeled by a class of compartmental systems

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***Abstract*— In this paper, we study the decentralized H_∞ control problem for air traffic flow (ATF) networks. By dividing the airspace into a number of interconnected regions, we first characterize the traffic flow dynamics by means of a compartmental system approach, and formulate the ATF control problem in a decentralized way, where the controller gain matrix is required to be diagonal and element-restricted. To solve the problem, a novel characterization is presented to ensure that the closed-loop system is asymptotically stable with a prescribed H_∞ performance. Then, a necessary and sufficient condition for the existence of a required controller is proposed and an iterative linear matrix inequality approach is further developed to solve the design condition. Finally, a simple ATF network is used to show the applicability of the obtained results.**

I. INTRODUCTION

Recently, modeling and control of air traffic flows (ATFs) have drawn considerable research interest in the air traffic management (ATM) community [1] [2] [3] [4] [5]. Basically, there are two types of ATF models, that is, the Lagrangian model [2] and the Eulerian model [3]. The Lagrangian approach models the trajectories of each individual aircraft in the environment, and is useful for certain ATM applications such as conflict detection and resolution. On the other hand, the Eulerian approach models the airspace by dividing it into a number of interconnected regions together with merge and diverge nodes, and describe the air traffic flow dynamics in terms of the flow properties in these regions [1] [3]. Since the traffic flow management problem is more concerned with the aggregating properties of groups of aircraft rather than the dynamic behavior of individual aircraft, the Eulerian approach may be more efficient in real applications. In addition, the computational complexity for Eulerian approach depends on the spatial division of the air traffic environment, not the number of aircraft in it, and thus Eulerian models have become increasingly popular in the control-oriented modeling of ATF networks [1] [3] [4] [5] [6].

Compartmental networks consist of a finite number of homogeneous and well-mixed subsystems (or compartments), which exchange materials with conservation laws between compartments and the environment [7] [8]. In this sense, the Eulerian approach for ATF networks can be exactly characterized in the framework of compartmental networks [3] [4]. In addition to the conservation property, a key physical characterization of such systems is that the transfers between compartments are intrinsically positive. Thus, differential equations for modeling the dynamics of such systems are subject to certain structural constraints, which make their solutions positive orthant invariant. Therefore, compartmental networks possess some unique features that general dynamic network systems do not have, and increasing attention has been paid to the analysis and synthesis of such systems [9].

Moreover, a major feature of the current ATM system is the insufficient sharing of information between decision-makers [10], each having somewhat disparate information to control traffic flows in the corresponding region. Thus, the management policy to be designed should only rely on local information, which can be characterized as a decentralized control problem in AFT networks [4] [5] [11]. In fact, with the advent of complex systems, decentralized control has become increasingly important in real applications, and it has been well known that the decentralized constraint added to the controller generally renders the control design problem to be non-convex, or even intractable [12]. Therefore, considerable efforts have been made to identifying specific classes of tractable problems, which can be easily solvable via convex optimization techniques, see [13] and references therein.

In this paper, we model the air traffic flow networks by means of a compartmental system approach, and view the control strategy as a recirculation rate to adjust the traffic speed in each region. The objective is to design a decentralized controller such that the landing flow of the closed-loop system can track a given flow output in an H_∞ manner. In this sense, the controller gain matrix is required to be diagonal and element-restricted, and conventional approaches for decentralized controller design may not be applicable anymore, mainly because of the additional constraints embedded to the gain matrix. To tackle this, we develop a novel characterization under which the closed-loop system is asymptotically stable with a prescribed H_∞ performance level, and an iterative optimization algorithm is further proposed to solve the design condition. Such a characterization successfully introduces a parameter matrix with fully flexible structure, which can thus be designated to be diagonal and employed to construct a required gain matrix, whereas no conservatism will be introduced accordingly.

The remain of this paper is structured as follows. In Section II, we first formulate the air traffic flow networks in terms of compartmental systems, and propose the corresponding decentralized H_∞ control problem. In Section III, we develop novel theoretical results to design the decentralized H_∞ controller. A 4-compartment network is

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presented to show the efficiency of our results in Section IV. Conclusions are given in Section V.

Notation: Let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the n-column real vectors; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n \times m$. For any real symmetric matrices P and Q, the notation $P \geq Q$ (respectively, $P > Q$) means that the matrix $P - Q$ is positive semi-definite (respectively, positive definite). For a matrix $A \in \mathbb{R}^{m\times n}$, a_{ij} denotes the element located at the ith row and the jth column. I represents the identity matrix with an appropriate dimension. The superscript "T" denotes matrix transpose. For a given matrix $B \in \mathbb{R}^{n\times m}$ with $\text{rank}(B) = r$, $B^{\perp} \in \mathbb{R}^{m\times(m-r)}$ denotes the right orthonormal complement of B by $BB^{\perp} = 0$ and $B^{\perp}\left(B^{\perp}\right)^T = I$. In addition, $\text{Her}\left(M\right) \triangleq M^T + M$ is defined for any matrix $M \in \mathbb{R}^{n\times n}$. $|\cdot|$ represents the Euclidean norm for vectors and $\|\cdot\|$ represents the spectral norm for matrices. For a transfer function matrix $G(s)$, $\|G\|_\infty$ represents the H_∞ norm of $G(s)$. The symbol $\#$ is used to represent a matrix which can be inferred by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

II. Model Description and Problem Formulation

Modeling of air traffic flow networks has been an important issue in the air traffic management community, here we present a Eulerian framework to describe unidirectional air traffic flows by means of compartmental networks. Basically, the Eulerian model can be achieved by division of the airspace into a number of interconnected regions, and characterize flow dynamics in terms of flow properties in these regions. In this paper, we view each region as a compartment, and use them interchangeably hereafter. We assume that the flow out of a region (outflow) can diverge and enter other regions, and similarly, flows from multiple regions (inflow) can converge and enter one subsequent region. We view each region as a compartment, and use them subsequently without any distinction hereafter. We describe the topology of the network by directed graph $G = (V, E)$, where V is the set of compartments (nodes), and E the set of links (edges). We index the compartments by $i \in \mathcal{N} \triangleq \{1, \ldots, n\}$. For any $i \in \mathcal{N}$, we call $\mathcal{M}(i)$ the set of nodes which enter into compartment i, and $\mathcal{L}(i)$ the set of nodes into which compartment i enters. We call $\mathcal{I}$ the set of sources (inlet ports) in the network and $\mathcal{O}$ the set of sinks (outlet ports), at which we might want to perform optimization. An example of this type of ATF networks is given in Fig. 1. In this example, $\mathcal{N} = \{A_1, A_2, A_3, A_4\}$, $\mathcal{I} = \{I_1, I_2\}$ and $\mathcal{O} = \{O_1\}$. There is only one landing airport here, but the formulation allows multiple landing airports in this paper.

For compartment i, its state $x_i(t)$ represents the number of aircraft at time t, thus must be positive all the time. The traversal time τ_i is computed from the average aircraft speed v_i and the region dimension Ω_i as $\tau_i = \Omega_i / v_i$, therefore, the natural flow rate for compartment i can be defined as $x_i\left(t\right)/\tau_i$. Following the principal of flow conservation, we denote α_{ji}, $0 \leq \alpha_{ji} \leq 1$, as the fraction of the inflow rate from region j that flows into region i. Thus, we have

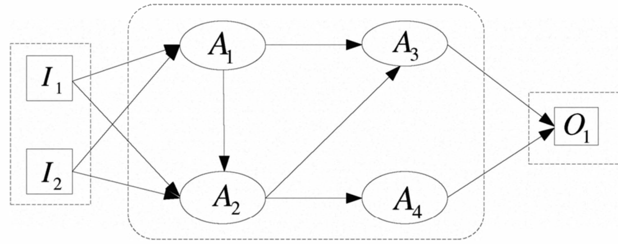

Fig. 1: Example of a 4-compartment air traffic network.

$\sum_{k\in\mathcal{L}(i)} \alpha_{ik} = 1$ for any $i \in \mathcal{N}$, since all aircraft leaving region i must enter the subsequent regions. Similarly, we let $0 \leq \beta_{ji} \leq 1$ the fraction of the inflow rate from the inlet airport I_j that flows into region i, and $0 \leq \gamma_{ji} \leq 1$ the fraction of the outflow rate from region j that enters into airport O_i. See [1] [5] for more details about Eulerian modeling.

The control input for each compartment i, denoted as $u_i(t)$, is viewed as a recirculation rate and can be implemented by adjusting the traffic speed inside region i. That is, the control actions are modeled as taking part of the natural outflow of the region and recirculating it back into the same region. Therefore, the total outflow rate of compartment i can be obtained as

$$f_i(t) = \frac{x_i\left(t\right)}{\tau_i} - u_i(t). \tag{1}$$

Thus, the dynamics of compartment i can be described by

$$\begin{aligned} \dot{x}_i &= -\left[\frac{x_i}{\tau_i} - u_i\right] + \sum_{j\in\mathcal{M}(i)} \alpha_{ji}\left[\frac{x_j}{\tau_j} - u_j\right] \\ &\quad + \sum_{j\in\mathcal{I}} \beta_{ji} w_j,\ i = 1, \ldots, n. \end{aligned} \tag{2}$$

The inflow rate entering into outlet ports is defined as $z = \left[z_1, \ldots, z_q\right]^T$ with

$$z_i = \sum_{\mathcal{L}(j)=\mathcal{O}} \gamma_{ji} f_j,$$

which can be employed as a performance output.

Traffic flow management usually relies on a distributed set of decision-makers, which further implies that the control strategy should be designed in a decentralized manner. With this in mind, we shall look for a general state-feedback controller for (2), which obeys the decentralized information structure constraint requiring that each compartment i is controlled using only its own state information. That is, the controller should be designed as

$$u_i = k_i x_i \text{ for } i = 1, \ldots, n. \tag{3}$$

For convenience, denote $K = \text{diag}\left(k_1, \ldots, k_n\right)$. In what follows, we shall formulate the constraint added to k_is. On one hand, the controlled outflow rate $f_i(t)$ in (1) should be positive according to its physical meaning. Because $x_i \geq 0$ represents the number of aircraft in section i, we have

$$k_i \leq \frac{1}{\tau_i} \text{ for } i = 1, \ldots, n. \tag{4}$$

In fact, (4) will preserve the positivity of system (2), see [14] for more details about positive systems. On the other hand, there should be a maximal average speed $\bar{v}_i$ for each region i, which corresponds to a minimal traversal time $\underline{\tau}_i = \Omega_i/\bar{v}_i$. Thus, $f_i(t)$ should be upper bounded by $x_i(t)/\underline{\tau}_i$, which further yields

$$\frac{1}{\tau_i} - \frac{1}{\underline{\tau}_i} \le k_i \text{ for } i = 1, \ldots, n. \tag{5}$$

Conditions (4) and (5) can be referred to as the positivity condition and capacity condition, respectively. Therefore, we have $\underline{K} \le K \le \bar{K}$ where

$$\begin{aligned} \underline{K} &= \operatorname{diag}\left(\frac{1}{\tau_1} - \frac{1}{\underline{\tau}_1}, \ldots, \frac{1}{\tau_n} - \frac{1}{\underline{\tau}_n}\right), & (6) \\ \bar{K} &= \operatorname{diag}\left(\frac{1}{\tau_1}, \ldots, \frac{1}{\tau_n}\right). & (7) \end{aligned}$$

To facilitate later developments, we combine the individual dynamics for each compartment in (2), and introduce the following general linear time-invariant system:

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bu(t) + B_w w(t) \\ z(t) &=& Cx(t) + Du(t), \end{array} \right. \tag{8}$$

where the system matrices A, B, B_w, C, D can be obtained directly from the interconnection of subsystems (2).

The control objective is to find a structural controller $u(t) = Kx(t)$ with $\underline{K} \le K \le \bar{K}$ such that system (8) is asymptotically stable in closed-loop and the output z behaves satisfactorily. More specifically, we assume that the reference output is generated by

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) &=& F\xi(t) + Gw(t), \\ z_{ref}(t) &=& H\xi(t), \end{array} \right. \tag{9}$$

where $\xi(t) \in \mathbb{R}^r$ for any $t > 0$. Of course, the system parameters in (9) must be selected such that the reference output $z_{ref}(t)$ is meaningful with respect to the physical system to be controlled.

Let $\hat{x}(t) = \left[x^T(t), \xi^T(t)\right]^T$ and $e(t) = z(t) - z_{ref}(t)$, then, from (8) and (9), we obtain the following augmented system:

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}(t) &=& \hat{A}\hat{x}(t) + \hat{B}w(t), \\ e(t) &=& \hat{C}\hat{x}(t), \end{array} \right. \tag{10}$$

where

$$\hat{A} = \bar{A} + \bar{M}K\bar{J}, \ \hat{B} = \bar{B}, \ \hat{C} = \bar{C} + \bar{N}K\bar{J}$$

with

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B_w \\ G \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & -H \end{bmatrix}, \\ \bar{M} &= \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \bar{J} = \begin{bmatrix} I & 0 \end{bmatrix}, \ \bar{N} = D. \end{aligned}$$

In this paper, we characterize the optimality criterion for controller satisfactory in terms of the H_∞ norm of system (10). More specifically, we want to find structural controllers such that $\|\mathcal{T}_{ew}\|_\infty < \delta$, where $\mathcal{T}_{ew}$ is the transfer function from w to e, and $\delta > 0$ is a given H_∞ performance level.

Based on the foregoing discussions, we now formulate the decentralized H_∞ control problem for system (2) below.

Problem ATFC (Air Traffic Flow Control): Given a reference output in (9), design a decentralized controller $u(t) = Kx(t)$ for system (8) such that the following requirements are fulfilled simultaneously.

(1) K is diagonal with $\underline{K} \le K \le \bar{K}$, where $\underline{K}$ and $\bar{K}$ are defined in (6) and (7).
(2) The system in (10) is asymptotically stable and satisfies $\|\mathcal{T}_{ew}\|_\infty < \delta$, where δ is a given H_∞ performance level.

The following result gives a fundamental characterization on the stability of (10) with H_∞ performance, which can be viewed as a special case of the well-known bound real lemma in [15].

Lemma 1: The system in (10) is asymptotically stable with $\|\mathcal{T}_{ew}\|_\infty < \delta$, if and only if there exists a matrix $P > 0$ such that

$$\Pi = \begin{bmatrix} P\hat{A} + \hat{A}^T P & P\hat{B} & \hat{C}^T \\ \# & -\delta I & 0 \\ \# & \# & -\delta I \end{bmatrix} < 0. \tag{11}$$

III. Main Results

A. Novel Analysis Characterization and Design Condition

In this section, we first develop a novel characterization in Theorem 1 under which system (10) will be asymptotically stable and satisfies the H_∞ performance $\|\mathcal{T}_{ew}\|_\infty < \delta$. Then, we further establish a necessary and sufficient condition for the existence of the desired structural controllers in Theorem 2.

Theorem 1: Given the decentralized control gain K. The system in (10) is asymptotically stable with H_∞ performance $\|\mathcal{T}_{ew}\|_\infty < \delta$, if and only if there exists a matrix $P > 0$ and *any matrix* $X > 0$ such that

$$\Sigma \triangleq \begin{bmatrix} \mathcal{P}^T\mathcal{A} + \mathcal{A}^T\mathcal{P} & \mathcal{P}^T\mathcal{B} & \mathcal{C}^T \\ \# & -\delta I & 0 \\ \# & \# & -\delta I \end{bmatrix} < 0, \tag{12}$$

where

$$\begin{aligned} \mathcal{P} &= \begin{bmatrix} P & 0 \\ -\frac{1}{2}XK\bar{J} & \frac{1}{2}X \end{bmatrix}, \\ \mathcal{A} &= \begin{bmatrix} \bar{A} & \bar{M} \\ K\bar{J} & -I \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}, \\ \mathcal{C} &= \begin{bmatrix} \bar{C} & \bar{N} \end{bmatrix}. \end{aligned}$$

Proof: (Sufficiency) Define a nonsingular transformation matrix as follows:

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ K\bar{J} & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}. \tag{13}$$

Pre- and post-multiplying (12) by T^T and T, we obtain

$$\bar{\Sigma} \triangleq T^T\Sigma T = \begin{bmatrix} P\hat{A} + \hat{A}^T P & P\hat{B} & \hat{C}^T & P\bar{M} \\ \# & -\delta I & 0 & 0 \\ \# & \# & -\delta I & \bar{N} \\ \# & \# & \# & -X \end{bmatrix} < 0. \tag{14}$$

It follows from Lemma 1 and the third leading principal matrix of $\bar{\Sigma}$ that the system in (10) is asymptotically stable with H_∞ performance guaranteed.

(Necessity) If the system in (10) is asymptotically stable and satisfies the H_∞ performance, then it can be seen from Lemma 1 that there exist a matrix $P > 0$ such that

$$\Pi = \begin{bmatrix} P\hat{A} + \hat{A}^T P & P\hat{B} & \hat{C}^T \\ \# & -\delta I & 0 \\ \# & \# & -\delta I \end{bmatrix} < 0.$$

Then, for *any* $S > 0$, there exists a sufficiently large scalar ϱ such that

$$-\varrho S - \begin{bmatrix} P\bar{M} \\ 0 \\ \bar{N} \end{bmatrix}^T \Pi \begin{bmatrix} P\bar{M} \\ 0 \\ \bar{N} \end{bmatrix} < 0. \tag{15}$$

By choosing $X = \varrho S$ and applying Schur complement [16] to (15), we have

$$\Sigma = T^{-T} \bar{\Sigma} T^{-1} < 0,$$

which completes the whole proof. □

Remark 1: From (11) in Lemma 1, one can see that the controller gain matrix K is coupled with the Lyapunov matrix P, and thus additional constraints on P may be induced when K is parametrized, which will inevitably make the corresponding result conservative [4]. The advantage of condition (12) is twofold: First, we separate P from K, which can be further parametrized by any $X > 0$; Second, the existence of X is naturally satisfied, and it can be designated to be with any structure, which will greatly facilitate the design of K subsequently.

We are now in a position to establish a necessary and sufficient condition for the existence of the desired structural controllers, which is stated in the following theorem.

Theorem 2: Problem ATFC is solvable *if and only if* there exist matrices $P > 0$, U, and diagonal matrices $X > 0$, L such that

$$\begin{aligned} \Sigma(U) &\triangleq \begin{bmatrix} \Pi_1 & P\bar{M} + \bar{J}^T L^T & P\bar{B} & \bar{C}^T \\ \# & -X & 0 & \bar{N}^T \\ \# & \# & -\delta I & 0 \\ \# & \# & \# & -\delta I \end{bmatrix} \\ &< 0, \end{aligned} \tag{16}$$

$$X\underline{K} \le L \le X\bar{K}, \tag{17}$$

where

$$\Pi_1 = \mathrm{Her}\left(P\bar{A} - U^T L \bar{J}\right) + U^T X U.$$

Under this condition, a desired decentralized controller can be chosen as

$$K = X^{-1} L. \tag{18}$$

Proof: Expanding (12), we have

$$\begin{bmatrix} \Pi_2 & P\bar{M} + \bar{J}^T K^T X & P\bar{B} & \bar{C}^T \\ \# & -X & 0 & \bar{N}^T \\ \# & \# & -\delta I & 0 \\ \# & \# & \# & -\delta I \end{bmatrix} < 0, \tag{19}$$

where

$$\Pi_2 = P\bar{A} + \bar{A}^T P - \bar{J}^T K^T X K \bar{J}.$$

In what follows, we shall prove the sufficiency part and the necessity part, respectively.

(Sufficiency) First, since X diagonal positive definite and L diagonal, it follows from (18) that K is diagonal. Second, (17) and (18) indicate that $\underline{K} \le K \le \bar{K}$ directly. Moreover, substitute $L = XK$ into (16) and observe that, for any U,

$$-\bar{J}^T K^T X K \bar{J} \le U^T X U - \mathrm{Her}\left(U^T L \bar{J}\right), \tag{20}$$

we have that (19) holds, which further indicates that (12). Then, based on Theorem 1, we conclude that the system in (10) is asymptotically stable and satisfies the H_∞ performance. This completes the sufficiency proof.

(Necessity) If *Problem ATFC* is solvable, then for the given system in (10), it follows from Theorem 1 that there exists $P > 0$, and *diagonal* $X > 0$ such that (12), or equivalently, (19) holds. By choosing $U = K\bar{J}$ and letting $L = XK$, one has that (16) holds. In addition, condition (17) can be obtained directly because $\underline{K} \le K \le \bar{K}$ and $L = XK$ with X positive diagonal. This completes the whole proof. □

Remark 2: It should be noted that although X is specified to be positive diagonal in Theorem 2, the design condition for the existence of a required controller is still necessary and sufficient. Moreover, due to the fact that the structure of X is rather flexible, it can be designated to be a positive scalar matrix, or even an identity matrix, whereas no conservatism will be introduced consequently. Therefore, if the controller gain matrix K is required to satisfy some constraints, such as block diagonal, triangular, positive or symmetric, they can be readily dealt with in the same framework proposed in this paper.

B. Computational Approaches

Our aim in this section is to derive numerically tractable algorithms to synthesize the required controllers with the help of convex optimization techniques. Indeed, although the controller can be designed according to Theorem 2, it is still difficult to verify (16) due to the existence of cubic terms in Π_1. However, it can be noted that when matrix U is fixed, (16) turns out to be linear with respect to the other variables. Therefore, a natural way is to fix U, and solve (16)–(17) by existing LMI techniques [16]. For $\Sigma(U)$ defined in (16), it follows from the proof in Theorem 2 that

$$\Sigma(X^{-1} L \bar{J}) \le \Sigma(U)$$

holds for any fixed P, X, and L. Therefore, it can be concluded that the scalar μ satisfying

$$\Sigma(U) < \mu I$$

achieves its minimum when $U = X^{-1} L \bar{J}$. Thus, the following iterative algorithm can be proposed to solve the decentralized H_∞ control problem.

Algorithm ATFC:

1) (*Initialization*) Set $j = 1$. For a given H_∞ performance level $\delta > 0$, select the initial matrix U_1 such that the following auxiliary system

$$\begin{cases} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{M}u(t) + \bar{B}w(t), \\ e(t) &= \bar{C}\bar{x}(t) + \bar{N}u(t), \end{cases} \tag{21}$$

with

$$u(t) = U_1 \bar{x}(t) \tag{22}$$

is asymptotically stable with $\|\bar{\mathcal{T}}_{ew}\|_\infty < \delta$, where $\bar{\mathcal{T}}_{ew}$ is the transfer function from w to e in (21).

2) (*Iteration*) For fixed U_j, solve the following convex optimization problem for the parameters in $\Lambda \triangleq \{P > 0, X > 0 \text{ diagonal}, L \text{ diagonal}\}$:

$$\mu_j^* = \min_{\lambda} \mu_j$$
$$\text{s.t. (17) and } \Xi(U_j) < \mu_j I.$$

Denote the corresponding value of X and L as X_j and L_j, respectively.

3) (*Criterion*) If

$$\mu_j^* \le 0,$$

then a desired controller matrix K is obtained as $K = X_j^{-1} L_j$ and STOP,
else if $|(\mu_j^* - \mu_{j-1}^*)/\mu_j^*| < \varepsilon$, where ε is a prescribed tolerance, then go to Step 4,
else update U_{j+1} as

$$U_{j+1} = X_j^{-1} L_j \bar{J},$$

set $j = j + 1$, and go to Step 2.

4) (*Termination*) A solution to *Problem ATFC* may not exist. STOP.

It should be noted that the sequence μ_j^* is monotonically decreasing with respect to j, that is, $\mu_{j+1}^* \le \mu_j^*$, and thus if μ_j^* does not converge to a positive scalar, then it will finally be nonpositive after a sufficient number of iterations, which corresponds to the stopping criterion in Step 3. Also, the initial matrix U_1 can be viewed as a state-feedback H_∞ controller matrix, and be constructed by existing convex optimization approaches. We present the following result with proof omitted.

Theorem 3: System (21) with (22) is asymptotically stable with $\|\bar{\mathcal{T}}_{ew}\|_\infty < \delta$, if and only if there exist matrices $Q > 0$ and V such that

$$\begin{bmatrix} \mathrm{Her}(\bar{A}Q + \bar{M}V) & \bar{B} & V^T \bar{N}^T + Q\bar{C}^T \\ \# & -\delta I & 0 \\ \# & \# & -\delta I \end{bmatrix} < 0. \tag{23}$$

Under this condition, an initial choice of U can be given by $U_1 = VQ^{-1}$.

Theorem 3 presents a necessary condition to check the feasibility of *Problem ATFC*, and shows how the initial matrix U_1 can be constructed. Therefore, if one cannot find such a matrix U_1, then one can conclude immediately that there does not exist a solution to *Problem ATFC*.

IV. Application Example

In this section, we provide an example to illustrate the effectiveness of the proposed decentralized H_∞ controller design method. We consider the simple ATF network shown in Fig. 1, and want to regulate the arrivals at airport O_1 which constitute the system outflows. The network dynamics can be represented by the following state-space model:

$$\left\{ \begin{aligned} \dot{x}(t) = {} & \begin{bmatrix} -\frac{1}{\tau_1} & 0 & 0 & 0 \\ \frac{\beta_{12}}{\tau_1} & -\frac{1}{\tau_2} & 0 & 0 \\ \frac{1-\beta_{12}}{\tau_1} & \frac{\beta_{23}}{\tau_2} & -\frac{1}{\tau_3} & 0 \\ 0 & \frac{1-\beta_{23}}{\tau_2} & 0 & -\frac{1}{\tau_4} \end{bmatrix} x(t) \\ & + \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta_{12} & 1 & 0 & 0 \\ -(1-\beta_{12}) & -\beta_{23} & 1 & 0 \\ 0 & -(1-\beta_{23}) & 0 & 1 \end{bmatrix} u(t) \\ & + \begin{bmatrix} \gamma_1 & \gamma_2 \\ 1-\gamma_1 & 1-\gamma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} w(t) \\ z(t) = {} & \begin{bmatrix} 0 & 0 & \frac{1}{\tau_3} & \frac{1}{\tau_4} \end{bmatrix} x(t) \\ & + \begin{bmatrix} 0 & 0 & -1 & -1 \end{bmatrix} u(t), \end{aligned} \right. \tag{24}$$

where $x(t) = [x_1(t), \ldots, x_4(t)]^T$ denotes the number of aircraft in regions A_1 to A_4, and $u(t) = [u_1(t), \ldots, u_4(t)]^T$ is the control input added to regulate the traffic output $z(t)$. Assume that the estimated parameters for (24) are

$$\tau_1 = \tau_2 = 0.2, \ \tau_3 = \tau_4 = 0.4,$$
$$\beta_{12} = 0.25, \ \beta_{23} = 0.5, \ \gamma_1 = 0.2, \ \gamma_2 = 0.5,$$

and the reference output is generated by

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & -0.5\xi(t) + [1, 1]\, w(t) \\ z_{ref}(t) & = & 0.5\xi(t) \end{array} \right. . \tag{25}$$

In this example, we specify the H_∞ performance level as $\delta = 0.6$, and the aim is to match the required arrival rate $z_{ref}(t)$ by designing a decentralized controller in (24). For simplicity, we assume that the aircraft in each region travels at the maximal speed, that is, $\underline{K} = 0$, and the problem reduces to find a diagonal K with $0 \le K \le \mathrm{diag}(5, 5, 2.5, 2.5)$ such that system (24) is asymptotically stable and satisfies $\|\mathcal{T}_{ew}\|_\infty < 0.6$ with $e(t) = z(t) - z_{ref}(t)$.

It can be easily verified that the H_∞ performance level is 0.8 when there is no controller added to system (24). Also, it is worth pointing out that traditional H_∞ controller design methods cannot guarantee us to obtain a desired controller, since they may violate the positivity constraint or capacity constraint in general. In what follows, we shall use the method proposed in this paper to design a required controller.

By resorting to Theorem 3, an initial matrix U_1 can be first obtained as

$$U_1 = \begin{bmatrix} 4.3921 & 0 & 0 & 0 & -0.0478 \\ 0 & 4.2332 & 0 & 0 & -0.0422 \\ 0 & 0 & 2.2246 & 0 & -0.2943 \\ 0 & 0 & 0 & 2.3739 & -0.1918 \end{bmatrix}.$$

Then, by running *Algorithm ATFC*, a feasible solution for X and L can be achieved with

$$X = 10^3 \times \mathrm{diag}(0.9742, 0.9146, 2.8303, 2.8678),$$
$$L = 10^3 \times \mathrm{diag}(4.1187, 3.9110, 5.5094, 5.5736),$$

which further yields the decentralized controller gain matrix as

$$K = X^{-1}L = \mathrm{diag}\,(4.2279, 4.1537, 1.9466, 1.9435)\,.$$

We assume that the inflow rates from airports I_1 and I_2 are sinusoidal, and have a mean of 10 aircraft/time-unit and an amplitude of 3 within the 20 time units as follows:

$$w(t) = \begin{cases} \begin{bmatrix} 10 + 3\sin t \\ 10 + 3\cos t \end{bmatrix}, & 0 \le t \le 20 \\ 0, & \text{otherwise} \end{cases},$$

which are plotted in Fig. 2.

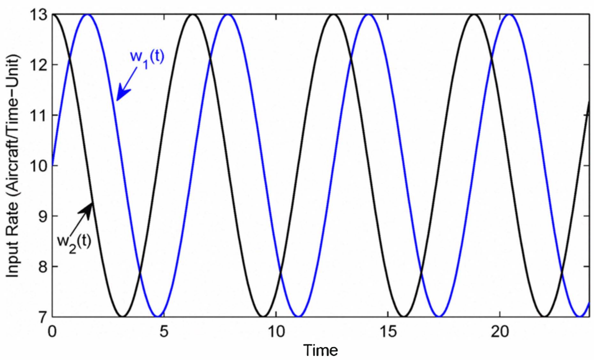

Fig. 2: Flow rates from airports I_1 and I_2.

The performance of the closed-loop flow is evaluated in a simulation, where the initial conditions used in the simulation are

$$x(0) = [2, 6, 1, 5]^T, \; \xi(0) = 2.$$

Fig. 3, which shows the landing rates of the closed-loop system for the desired, uncontrolled and controlled cases, indicates that the controlled outflow rate are well regulated to match the desired one.

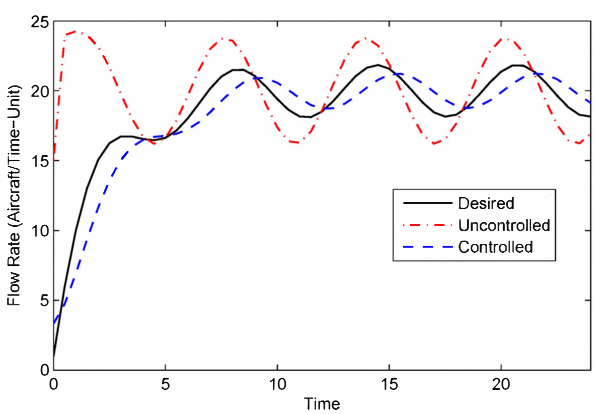

Fig. 3: Landing rate of the closed-loop system.

V. Conclusion

The decentralized H_∞ control problem for air traffic flow networks, which can be characterized in virtue of a class of compartmental systems, has been studied in this paper. In the formulation, the controller gain matrix is required to be diagonal and element-restricted. A matrix-inequality-based characterization has been developed to ensure the asymptotic stability of the controlled system with a prescribed H_∞ performance level, and a necessary and sufficient condition for the existence of a desired controller has been established accordingly. Then, an iterative LMI algorithm, which can be easily solved by existing convex optimization techniques, has been further proposed to solve the design condition. Finally, a 4-compartment air traffic network has been presented to illustrate the effectiveness of the theoretical results.

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