Stabilization of Systems With Probabilistic Interval Input Delays and Its Applications to Networked Control Systems

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***Abstract*—Motivated by the study of a class of networked control systems, this correspondence paper is concerned with the design problem of stabilization controllers for linear systems with stochastic input delays. Different from the common assumptions on time delays, it is assumed here that the probability distribution of the delay taking values in some intervals is known *a priori*. By making full use of the information concerning the probability distribution of the delays, criteria for the stochastic stability and stabilization controller design are derived. Traditionally, in the case that the variation range of the time delay is available, the maximum allowable bound of time delays can be calculated to ensure the stability of the time-delay system. It is shown, via numerical examples, that such a maximum allowable bound could be made larger in the case that the probability distribution of the time delay is known.**

***Index Terms*—Delay, linear matrix inequality (LMI), Lyapunov functional, networked control system (NCS), stabilization.**

I. Introduction

In many practical systems, such as networked and process control systems, the arrival of the measurement and control signals at the receiving point will often take some time, which leads to delayed input. For example, considering the transfer delays of sensor to controller, controller to actuator, and the effect of the data dropout, the linear networked control system (NCS) can be expressed as [31], [32]

$$\dot{x}(t) = Ax(t) + Bu\left(t - \tau(t)\right) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state and control vectors, respectively, A and B are constant matrices with appropriate dimensions, and $\tau(t)$ denotes the time delay which is piecewise continuous and bounded.

In the past several decades, there has been much attention on the study of stability and stabilization of systems with state delay or control input delay [1]–[3], [5], [6], [8], [12], [13], [15]–[17], [20], [21], [26]. Among these references, delay-independent methods [6], [13], [16] or delay-dependent methods [5], [12], [17], [20], [21] were proposed based on the Lyapunov functional approach, Riccati equation method, and linear-matrix-inequality (LMI) technique. For the delay-independent results given in [6], [13], and [16], the input delay term is treated as an uncertainty, which is obviously not suitable for the system (1) when A is not Hurwitz stable. For the delay-dependent results given in [5], [12], [17], [20], and [21], it has been proved that the system (1) is stabilizable provided that the input delay is small enough and, then, the upper bound of the delay guaranteeing the stabilizability of the system has been derived by solving a set of LMIs. It should be noted that, for the analysis and control design of the system, only the information of variation range and/or variation rate of the time delay was employed in [5], [12], [17], [20], and [21]. In some applications, however, such as those systems connected over a wireless network, the observation of communication delay of the data also includes the probability distribution of the delay values. As pointed out in [14] and [23], for a given wireless network, it can be measured that there exists a small number ε such that $\text{Prob}\{\tau(t) > d\} < \varepsilon$, where d is a constant. In this case, the possible value of the delay may be very large, although the probability of the delay taking such a large value is very small. Since the possible value of the delay with a low probability can be very large, it may lie outside the allowable variation range derived by traditional methods [5], [12], [17], [20], [21]. Therefore, the challenging issue is on how to derive some criteria which can exploit the known probability distribution of the delay and obtain a larger allowable variation range of the delay. In [4], when the probability of transfer of the delay from a value to the other is known, the delay-dependent stability analysis and controller design were investigated. However, the delay can only take values in a finite space in [4]. To the best of the authors' knowledge, no result has been reported for the stability analysis and stabilization control of system (1) when both the information of variation range of the time delay and the information of probability of the time-varying delay in an interval are taken into consideration.

In this correspondence paper, we are concerned with the stability analysis and stabilization controller design for the system with a stochastic input delay, which can be viewed as a general model of the system connected over a network medium. In terms of the probability distribution of the delay taking values in an interval, a new model is proposed, which can translate the probabilistic effects of the delay to a parameter matrix of the transformed system. Based on the new model, criteria for the stability and stabilization of the system are derived based on the Lyapunov functional method and LMI techniques.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices, I is the identity matrix of appropriate dimensions, and $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate. Z_{0+} denotes the set of nonnegative integers. The notation $X > 0$ (respectively, $X \geq 0$) for $X \in \mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, positive semidefinite). For a real matrix B and two real symmetric matrices A and C of appropriate dimensions, $\begin{bmatrix} A & * \\ B & C \end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry. $E\{\cdot\}$ denotes the mathematical expectation.

II. System Description and Preliminaries

The following assumption is needed to state our main results.

Assumption 1: There exist constants $\hat{\tau}_1$ and $\hat{\tau}_2$, where $0 \leq \hat{\tau}_1 \leq \hat{\tau}_2$, such that either $\tau(t) \in [0, \hat{\tau}_1]$ or $\tau(t) \in (\hat{\tau}_1, \hat{\tau}_2]$. Furthermore, the probability distribution of $\tau(t)$ taking values in $[0, \hat{\tau}_1]$ and $(\hat{\tau}_1, \hat{\tau}_2]$ is known *a priori*.

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For the stochastic input delay $\tau(t)$, we define the following two random events:

$$\mathcal{F}_1: \quad \tau(t) \in [0, \hat{\tau}_1]$$
$$\mathcal{F}_2: \quad \tau(t) \in (\hat{\tau}_1, \hat{\tau}_2].$$

Then, a random variable can be defined as follows:

$$\delta(t) = \begin{cases} 1, & \mathcal{F}_1 \text{ occurs} \\ 0, & \mathcal{F}_2 \text{ occurs.} \end{cases}$$

Furthermore, define two functions $\tau_1 : \mathbb{R}_+ \to [0, \hat{\tau}_1]$ and $\tau_2 : \mathbb{R}_+ \to [\hat{\tau}_1, \hat{\tau}_2]$ such that

$$\tau_1(t) = \begin{cases} \tau(t), & \text{if } \delta(t) = 1 \\ \hat{\tau}_1, & \text{if } \delta(t) = 0 \end{cases}$$
$$\tau_2(t) = \begin{cases} \hat{\tau}_1, & \text{if } \delta(t) = 1 \\ \tau(t), & \text{if } \delta(t) = 0\,. \end{cases}$$

Using $\delta(t)$ and $\tau_i(t) (i = 1, 2)$, (1) can be equivalently written as

$$\dot{x}(t) = Ax(t) + \delta(t)Bu\left(t - \tau_1(t)\right) + (1 - \delta(t))\, Bu\left(t - \tau_2(t)\right). \quad (2)$$

Remark 1: It can be shown that $\delta(t)$ is a Markovian process. For the following use, it is assumed that $\delta(t)$ follows an unknown but exponential distribution of switchings. Moreover, from the definitions of $\tau_1(t)$, $\tau_2(t)$, and $\delta(t)$, it is shown that the solution of (1) is also that of (2). Therefore, by checking the stability of (2), we can also deduce that of (1). If $\tau(t) \in [0, \hat{\tau}_1]$ for any $t \in \mathbb{R}_+$, (2) becomes $\dot{x}(t) = Ax(t) + Bu(t - \tau_1(t))$, which is a commonly studied system with bounded time-varying input delay. If $\tau(t) \in (\hat{\tau}_1, \hat{\tau}_2]$ for any $t \in \mathbb{R}_+$, (2) becomes $\dot{x}(t) = Ax(t) + Bu(t - \tau_2(t))$, which is the system with interval time-varying input delay.

Remark 2: The introduction of $\delta(t)$ is motivated by the ideas in [22], [24], [25], and [29], where the signal transmission is subject to random communication delays. However, in this correspondence paper, the random variable $\delta(t)$ is used to account for the probabilistic nature of the time delay taking values in a certain interval. To our knowledge, this correspondence paper represents the first attempt to consider the probabilistic behavior of the time delays falling into a given interval.

Assumption 2: $\delta(t)$ is a Bernoulli distributed sequence with $\mathrm{Prob}\{\delta(t) = 1\} = E\{\delta(t)\} = \delta_0$, and $\mathrm{Prob}\{\delta(t) = 0\} = 1 - E\{\delta(t)\} = 1 - \delta_0$, where $0 \leq \delta_0 \leq 1$ is a constant.

Remark 3: From Assumption 2, it can be shown that $E\{\delta(t) - \delta_0\} = 0$ and $E\{(\delta(t) - \delta_0)^2\} = \delta_0(1 - \delta_0)$.

In this correspondence paper, the designed controller is of a linear type, i.e.,

$$u(t) = Kx(t) \quad (3)$$

where K is a constant matrix to be determined later.

Under control law (3), the closed-loop system of (2) is

$$\dot{x}(t) = Ax(t) + \delta(t)BKx\left(t - \tau_1(t)\right) + \left(1 - \delta(t)\right) BKx\left(t - \tau_2(t)\right). \quad (4)$$

Since (4) is a stochastic system, for the stability analysis of (4), the following definition is needed.

Definition 1: The system (4) is said to be exponentially stable in the mean-square sense if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$E\left\{\|x(t)\|^2\right\} \leq \alpha e^{\beta t} \sup_{-2\hat{\tau}_2 \leq s \leq 0} E\left\{\|\phi(s)\|^2\right\}$$

where $\phi(\cdot)$ is the initial function in (4) defined as $x(t) = \phi(t)$, $t \in [-\hat{\tau}_2, 0]$.

Moreover, we also need the following lemmas. The proof of Lemma 1 can be obtained easily while Lemma 2 can be derived via Lemma 1 and Remark 4.

Lemma 1: Suppose X_1 and X_2 are two constant matrices of same dimensions. Then for a given $h \in [0, 1]$

$$qX_1 + (1 - q)X_2 < 0 \quad (5)$$

holds for any $q \in [h, 1]$ if and only if

$$X_1 < 0 \quad (6)$$
$$hX_1 + (1 - h)X_2 < 0. \quad (7)$$

Remark 4: It can be seen that, when $q \neq h$ (i.e., $q \in (h, 1]$), (6) and (7) are just sufficient (but not necessary) conditions for guaranteeing (5) to hold for any $q \in (h, 1]$.

Lemma 2: Suppose $\tau_i(t)$ $(i = 1, 2)$ are defined as previously and Φ_i, $\Pi_i (i = 1, 2)$, $\Omega = \Omega^{\mathrm{T}}$ are some constant matrices with appropriate dimensions, then

$$\tau_1(t)\Phi_1 + \left(\hat{\tau}_1 - \tau_1(t)\right)\Phi_2 + \left(\tau_2(t) - \hat{\tau}_1\right)\Pi_1 + \left(\hat{\tau}_2 - \tau_2(t)\right)\Pi_2 + \Omega < 0 \quad (8)$$

if the following inequalities hold:

$$\hat{\tau}_1\Phi_1 + (\hat{\tau}_2 - \hat{\tau}_1)\Pi_1 + \Omega < 0 \quad (9)$$
$$\hat{\tau}_1\Phi_1 + (\hat{\tau}_2 - \hat{\tau}_1)\Pi_2 + \Omega < 0 \quad (10)$$
$$\hat{\tau}_1\Phi_2 + (\hat{\tau}_2 - \hat{\tau}_1)\Pi_1 + \Omega < 0 \quad (11)$$
$$\hat{\tau}_1\Phi_2 + (\hat{\tau}_2 - \hat{\tau}_1)\Pi_2 + \Omega < 0. \quad (12)$$

III. Stability Analysis

We rewrite (4) as

$$\begin{aligned}\dot{x}(t) = Ax(t) &+ \delta_0 BKx\left(t - \tau_1(t)\right) + (1 - \delta_0) \\ &\times BKx\left(t - \tau_2(t)\right) + (\delta(t) - \delta_0) \\ &\times \left[BKx\left(t - \tau_1(t)\right) - BKx\left(t - \tau_2(t)\right)\right]. \end{aligned} \quad (13)$$

Next, we will derive some sufficient conditions for guaranteeing the exponential stability in the mean-square sense of (13).

Define

$$y(t) = \mathcal{A}\zeta(t) \quad (14)$$

where $\mathcal{A}$ and $\zeta^{\mathrm{T}}(t)$ are shown at the bottom of the page. Then, (13) can be expressed as

$$\dot{x}(t) = y(t) + \left(\delta(t) - \delta_0\right)\mathcal{B}\zeta(t) \quad (15)$$

where $\mathcal{B} = [0 \quad BK \quad 0 \quad -BK \quad 0 \quad 0]$.

$$\mathcal{A} = [A \quad \delta_0 BK \quad 0 \quad (1 - \delta_0)BK \quad 0 \quad 0]$$
$$\zeta^{\mathrm{T}}(t) = [x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}\left(t - \tau_1(t)\right) \quad x^{\mathrm{T}}(t - \hat{\tau}_1) \quad x^{\mathrm{T}}\left(t - \tau_2(t)\right) \quad x^{\mathrm{T}}(t - \hat{\tau}_2) \quad y^{\mathrm{T}}(t)]$$

Theorem 1: System (4) is exponentially stable in the mean-square sense if, for given constants $\hat{\tau}_1$, $\hat{\tau}_2$, δ_0, and matrix K, there exist matrices $P > 0$, $Q_i > 0$, $R_i > 0$, $Z_i > 0$ $(i = 1, 2)$, N_j, M_j, T_j, W_j $(j = 1, 2, 3, 4, 5, 6)$, and S_l $(l = 1, 2, 3, 4)$ of appropriate dimensions such that the following LMIs hold:

$$\begin{bmatrix} \Xi_{11} & * & * \\ \Xi_{21}^l & \Xi_{22} & * \\ \Xi_{31} & 0 & \Xi_{33} \end{bmatrix} < 0, \qquad l = 1, 2, 3, 4 \tag{16}$$

where

$$\Xi_{21}^1 = \begin{bmatrix} \hat{\tau}_1 N^{\mathrm{T}} \\ \hat{\tau}_1 N^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) T^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) T^{\mathrm{T}} \end{bmatrix} \quad \Xi_{21}^2 = \begin{bmatrix} \hat{\tau}_1 N^{\mathrm{T}} \\ \hat{\tau}_1 N^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) W^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) W^{\mathrm{T}} \end{bmatrix}$$

$$\Xi_{21}^3 = \begin{bmatrix} \hat{\tau}_1 M^{\mathrm{T}} \\ \hat{\tau}_1 M^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) T^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) T^{\mathrm{T}} \end{bmatrix} \quad \Xi_{21}^4 = \begin{bmatrix} \hat{\tau}_1 M^{\mathrm{T}} \\ \hat{\tau}_1 M^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) W^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1) W^{\mathrm{T}} \end{bmatrix}$$

$$\begin{aligned} N^{\mathrm{T}} &= [\, N_1^{\mathrm{T}} \quad N_2^{\mathrm{T}} \quad N_3^{\mathrm{T}} \quad N_4^{\mathrm{T}} \quad N_5^{\mathrm{T}} \quad N_6^{\mathrm{T}} \,] \\ M^{\mathrm{T}} &= [\, M_1^{\mathrm{T}} \quad M_2^{\mathrm{T}} \quad M_3^{\mathrm{T}} \quad M_4^{\mathrm{T}} \quad M_5^{\mathrm{T}} \quad M_6^{\mathrm{T}} \,] \\ T^{\mathrm{T}} &= [\, T_1^{\mathrm{T}} \quad T_2^{\mathrm{T}} \quad T_3^{\mathrm{T}} \quad T_4^{\mathrm{T}} \quad T_5^{\mathrm{T}} \quad T_6^{\mathrm{T}} \,] \\ W^{\mathrm{T}} &= [\, W_1^{\mathrm{T}} \quad W_2^{\mathrm{T}} \quad W_3^{\mathrm{T}} \quad W_4^{\mathrm{T}} \quad W_5^{\mathrm{T}} \quad W_6^{\mathrm{T}} \,] \\ \Xi_{22} &= \mathrm{diag}\,(\, -\hat{\tau}_1 R_1 \quad -\hat{\tau}_1 Z_1 \quad -(\hat{\tau}_2 - \hat{\tau}_1) R_2 \quad -(\hat{\tau}_2 - \hat{\tau}_1) Z_2 \,) \\ \Xi_{33} &= \mathrm{diag}(\, -Z_1 \quad -Z_2 \,) \end{aligned}$$

and Ξ_{11} and Ξ_{31} are shown at the bottom of the page.

Proof: Construct a Lyapunov functional as

$$\begin{aligned} V(x_t) = {} & x^{\mathrm{T}}(t) P x(t) + \int_{t-\hat{\tau}_1}^{t} x^{\mathrm{T}}(s) Q_1 x(s)\, ds \\ & + \int_{t-\hat{\tau}_2}^{t} x^{\mathrm{T}}(s) Q_2 x(s)\, ds \\ & + \int_{t-\hat{\tau}_1}^{t} \int_{s}^{t} y^{\mathrm{T}}(v) R_1 y(v)\, dv ds \\ & + \int_{t-\hat{\tau}_2}^{t-\hat{\tau}_1} \int_{s}^{t} y^{\mathrm{T}}(v) R_2 y(v)\, dv ds \\ & + \delta_0 (1 - \delta_0) \int_{t-\hat{\tau}_1}^{t} \int_{s}^{t} \zeta^{\mathrm{T}}(v) \mathcal{B}^{\mathrm{T}} Z_1 \mathcal{B} \zeta(v)\, dv ds \\ & + \delta_0 (1 - \delta_0) \int_{t-\hat{\tau}_2}^{t-\hat{\tau}_1} \int_{s}^{t} \zeta^{\mathrm{T}}(v) \mathcal{B}^{\mathrm{T}} Z_2 \mathcal{B} \zeta(v)\, dv ds \end{aligned} \tag{17}$$

where $P > 0$, $Q_i > 0$, $R_i > 0$, and $Z_i > 0$ $(i = 1, 2)$. The infinitesimal operator $\mathcal{L}$ of $V(x_t)$ is defined as follows [18]:

$$\mathcal{L} V(x_t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ E\left(V(x_{t+\Delta}) | x_t \right) - V(x_t) \right\}. \tag{18}$$

Then, from (17) and (18) and by using the slack matrix method [9], we can obtain

$$\begin{aligned} & \mathcal{L} V(x_t) \\ & \quad = 2 x^{\mathrm{T}}(t) P y(t) - x^{\mathrm{T}}(t - \hat{\tau}_1) Q_1 x(t - \hat{\tau}_1) \\ & \qquad + x^{\mathrm{T}}(t) (Q_1 + Q_2) x(t) - x^{\mathrm{T}}(t - \hat{\tau}_2) \\ & \qquad \times Q_2 x(t - \hat{\tau}_2) + y^{\mathrm{T}}(t) \left(\hat{\tau}_1 R_1 + (\hat{\tau}_2 - \hat{\tau}_1) R_2 \right) y(t) \\ & \qquad + \delta_0 (1 - \delta_0) \zeta^{\mathrm{T}}(t) \mathcal{B}^{\mathrm{T}} \left(\hat{\tau}_1 Z_1 + (\hat{\tau}_2 - \hat{\tau}_1) Z_2 \right) \mathcal{B} \zeta(t) \\ & \qquad - \int_{t-\hat{\tau}_1}^{t} y^{\mathrm{T}}(s) R_1 y(s) ds - \int_{t-\hat{\tau}_2}^{t-\hat{\tau}_1} y^{\mathrm{T}}(s) R_2 y(s) ds \\ & \qquad - \delta_0 (1 - \delta_0) \int_{t-\hat{\tau}_1}^{t} \zeta^{\mathrm{T}}(s) \mathcal{B}^{\mathrm{T}} Z_1 \mathcal{B} \zeta(s) ds \\ & \qquad - \delta_0 (1 - \delta_0) \int_{t-\hat{\tau}_2}^{t-\hat{\tau}_1} \zeta^{\mathrm{T}}(s) \mathcal{B}^{\mathrm{T}} Z_2 \mathcal{B} \zeta(s) ds \end{aligned}$$

$$\Xi_{11} = \begin{bmatrix} \begin{matrix} N_1 + N_1^T + S_1 A \\ + A^T S_1^T + Q_1 + Q_2 \end{matrix} & * & * & * & * & * \\ \begin{matrix} N_2 - N_1^T + M_1^T \\ + S_2 A + \delta_0 K^T B^T S_1^T \end{matrix} & \begin{matrix} -N_2 - N_2^T + M_2 + M_2^T \\ + \delta_0 S_2 BK + \delta_0 K^T B^T S_2^T \end{matrix} & * & * & * & * \\ N_3 - M_1^T + T_1^T & -N_3 + M_3 - M_2^T + T_2^T & -M_3 - M_3^T - Q_1 + T_3 + T_3^T & * & * & * \\ \begin{matrix} N_4 - T_1^T + W_1^T + S_3 A \\ + (1 - \delta_0) K^T B^T S_1^T \end{matrix} & \begin{matrix} -N_4 + M_4 - T_2^T + W_2^T \\ + \delta_0 S_3 BK + (1 - \delta_0) K^T B^T S_2^T \end{matrix} & -M_4 - T_3^T + W_3^T + T_4 & \begin{matrix} -T_4 - T_4^T + W_4 + W_4^T \\ + (1 - \delta_0) S_3 BK + (1 - \delta_0) K^T B^T S_3^T \end{matrix} & * & * \\ N_5 - W_1^T & -N_5 + M_5 - W_2^T & -M_5 - W_3^T + T_5 & -T_5 + W_5 - W_4^T & -W_5 - W_5^T - Q_2 & * \\ N_6 - S_1^T + S_4 A + P & -N_6 + M_6 - S_2^T + \delta_0 S_4 BK & -M_6 + T_6 & -T_6 + W_6 - S_3^T + (1 - \delta_0) S_4 BK & -W_6 & -S_4 - S_4^T + \hat{\tau}_1 R_1 + (\hat{\tau}_2 - \hat{\tau}_1) R_2 \end{bmatrix}$$

$$\Xi_{31} = \begin{bmatrix} 0 & \sqrt{\hat{\tau}_1 \delta_0 (1 - \delta_0)} Z_1 BK & 0 & -\sqrt{\hat{\tau}_1 \delta_0 (1 - \delta_0)} Z_1 BK & 0 & 0 \\ 0 & \sqrt{(\hat{\tau}_2 - \hat{\tau}_1) \delta_0 (1 - \delta_0)} Z_2 BK & 0 & -\sqrt{(\hat{\tau}_2 - \hat{\tau}_1) \delta_0 (1 - \delta_0)} Z_2 BK & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
&+2\zeta^{\mathrm{T}}(t)N\left[x(t)-x\left(t-\tau_1(t)\right)-\int_{t-\tau_1(t)}^{t}\dot{x}(s)ds\right]\\
&+2\zeta^{\mathrm{T}}(t)M\left[x\left(t-\tau_1(t)\right)-x(t-\hat{\tau}_1)-\int_{t-\hat{\tau}_1}^{t-\tau_1(t)}\dot{x}(s)ds\right]\\
&+2\zeta^{\mathrm{T}}(t)T\left[x(t-\hat{\tau}_1)-x\left(t-\tau_2(t)\right)-\int_{t-\tau_2(t)}^{t-\hat{\tau}_1}\dot{x}(s)ds\right]\\
&+2\zeta^{\mathrm{T}}(t)W\left[x\left(t-\tau_2(t)\right)-x(t-\hat{\tau}_2)-\int_{t-\hat{\tau}_2}^{t-\tau_2(t)}\dot{x}(s)ds\right]\\
&+2\gamma^{\mathrm{T}}(t)S\left[Ax(t)+\delta_0BKx\left(t-\tau_1(t)\right)\right.\\
&\qquad\left.+(1-\delta_0)BKx\left(t-\tau_2(t)\right)-y(t)\right] \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
\gamma^{\mathrm{T}}(t)&=\left[\,x^{\mathrm{T}}(t)\quad x^{\mathrm{T}}\left(t-\tau_1(t)\right)\quad x^{\mathrm{T}}\left(t-\tau_2(t)\right)\quad y^{\mathrm{T}}(t)\,\right]\\
S^{\mathrm{T}}&=\left[\,S_1^{\mathrm{T}}\quad S_2^{\mathrm{T}}\quad S_3^{\mathrm{T}}\quad S_4^{\mathrm{T}}\,\right].
\end{aligned}$$

Note that, from (15)

$$\begin{aligned}
&-2\zeta^{\mathrm{T}}(t)N\int_{t-\tau_1(t)}^{t}\dot{x}(s)ds\\
&\quad\le \tau_1(t)\zeta^{\mathrm{T}}(t)N\left(R_1^{-1}+Z_1^{-1}\right)N^{\mathrm{T}}\zeta(t)\\
&\quad+\int_{t-\tau_1(t)}^{t}y^{\mathrm{T}}(s)R_1y(s)ds\\
&\quad+\int_{t-\tau_1(t)}^{t}\left(\delta(s)-\delta_0\right)^2\zeta^{\mathrm{T}}(s)\mathcal{B}^{\mathrm{T}}Z_1\mathcal{B}\zeta(s)ds. \qquad (20)
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
&-2\zeta^{\mathrm{T}}(t)M\int_{t-\hat{\tau}_1}^{t-\tau_1(t)}\dot{x}(s)\,ds\\
&\quad\le \left(\hat{\tau}_1-\tau_1(t)\right)\zeta^{\mathrm{T}}(t)M\left(R_1^{-1}+Z_1^{-1}\right)M^{\mathrm{T}}\zeta(t)\\
&\quad+\int_{t-\hat{\tau}_1}^{t-\tau_1(t)}y^{\mathrm{T}}(s)R_1y(s)\,ds\\
&\quad+\int_{t-\hat{\tau}_1}^{t-\tau_1(t)}\left(\delta(s)-\delta_0\right)^2\zeta^{\mathrm{T}}(s)\mathcal{B}^{\mathrm{T}}Z_1\mathcal{B}\zeta(s)\,ds \qquad (21)\\
&-2\zeta^{\mathrm{T}}(t)T\int_{t-\tau_2(t)}^{t-\hat{\tau}_1}\dot{x}(s)\,ds\\
&\quad\le \left(\tau_2(t)-\hat{\tau}_1\right)\zeta^{\mathrm{T}}(t)T\left(R_2^{-1}+Z_2^{-1}\right)T^{\mathrm{T}}\zeta(t)\\
&\quad+\int_{t-\tau_2(t)}^{t-\hat{\tau}_1}y^{\mathrm{T}}(s)R_2y(s)\,ds\\
&\quad+\int_{t-\tau_2(t)}^{t-\hat{\tau}_1}\left(\delta(s)-\delta_0\right)^2\zeta^{\mathrm{T}}(s)\mathcal{B}^{\mathrm{T}}Z_2\mathcal{B}\zeta(s)\,ds \qquad (22)\\
&-2\zeta^{\mathrm{T}}(t)W\int_{t-\hat{\tau}_2}^{t-\tau_2(t)}\dot{x}(s)\,ds\\
&\quad\le \left(\hat{\tau}_2-\tau_2(t)\right)\zeta^{\mathrm{T}}(t)W\left(R_2^{-1}+Z_2^{-1}\right)W^{\mathrm{T}}\zeta(t)\\
&\quad+\int_{t-\hat{\tau}_2}^{t-\tau_2(t)}y^{\mathrm{T}}(s)R_2y(s)\,ds\\
&\quad+\int_{t-\hat{\tau}_2}^{t-\tau_2(t)}\left(\delta(s)-\delta_0\right)^2\zeta^{\mathrm{T}}(s)\mathcal{B}^{\mathrm{T}}Z_2\mathcal{B}\zeta(s)\,ds. \qquad (23)
\end{aligned}$$

Combining (19)–(23) and taking the expectation on both sides of (19), we obtain

$$\begin{aligned}
&E\left\{\mathcal{L}V(x_t)\right\}\\
&\le E\left\{\zeta^{\mathrm{T}}(t)\left[\Xi_{11}+\delta_0(1-\delta_0)\mathcal{B}^{\mathrm{T}}\left(\hat{\tau}_1Z_1+(\hat{\tau}_2-\hat{\tau}_1)Z_2\right)\mathcal{B}\right.\right.\\
&\quad+\tau_1(t)N\left(R_1^{-1}+Z_1^{-1}\right)N^{\mathrm{T}}\\
&\quad+\left(\hat{\tau}_1-\tau_1(t)\right)M\left(R_1^{-1}+Z_1^{-1}\right)M^{\mathrm{T}}\\
&\quad+\left(\tau_2(t)-\hat{\tau}_1\right)T\left(R_2^{-1}+Z_2^{-1}\right)T^{\mathrm{T}}\\
&\quad\left.\left.+\left(\hat{\tau}_2-\tau_2(t)\right)W\left(R_2^{-1}+Z_2^{-1}\right)W^{\mathrm{T}}\right]\zeta(t)\right\}. \qquad (24)
\end{aligned}$$

By Lemma 2, it can be seen that (16) with $l=1,2,3,4$ are sufficient conditions for guaranteeing that the right-hand side of (24) is less than zero. Moreover, it is easy to show that, under the conditions in (16), there exists a small enough constant $\lambda>0$ such that the right-hand side of (24) is less than $-\lambda I$. Therefore, from (24), we have

$$E\left\{\mathcal{L}V(x_t)\right\}\le -\lambda E\left\{\zeta^{\mathrm{T}}(s)\zeta(s)\right\}.$$

Then, using a method similar to that of [19], we can show that

$$E\left\{x^{\mathrm{T}}(t)x(t)\right\}\le \bar{\alpha}e^{-\varepsilon t}\sup_{-2\hat{\tau}_2\le s\le 0}E\left\{\|\phi(s)\|^2\right\} \qquad (25)$$

where $\bar{\alpha}=(\alpha/\lambda_{\min}(P))$. ■

Remark 5: In the special case, when $\tau(t)\in[\hat{\tau}_1,\hat{\tau}_2]$ for any $t\in\mathbb{R}_+$, (4) can be expressed as

$$\dot{x}(t)=Ax(t)+BKx(t-\tau_2(t)). \qquad (26)$$

For (16) with $l=1,2$, letting $\delta_0=0$, deleting the rows and columns corresponding to Ξ_{31} and Ξ_{33} and to Z_j $(j=1,2)$, and deleting the second row and second column and replacing M_i by N_i $(i=1,3,4,5,6)$, we can obtain a result, denoted as Theorem $1'$, to check the exponential stability of (26). Due to this correspondence paper's space limitation, the detailed statement of Theorem $1'$ is omitted here.

IV. Stabilization Controller Design

In this section, the design criteria for the feedback control gain K will be derived based on Theorem 1 and Remark 5.

Theorem 2: System (4) with $K=YX^{-\mathrm{T}}$ is exponentially stable in the mean-square sense, if, for given constants $\hat{\tau}_1$, $\hat{\tau}_2$, δ_0, and ρ_m $(m=2,3,4)$, there exist matrices $\hat{P}>0$, $\hat{Q}_i>0$, $\hat{R}_i>0$, $\hat{Z}_i>0$ $(i=1,2)$, X, Y, $\hat{N}_j$, $\hat{M}_j$, $\hat{\mathrm{T}}_j$, and $\hat{W}_j$ $(j=1,2,3,4,5,6)$ of appropriate dimensions such that the following matrix equalities hold:

$$\begin{bmatrix} \Gamma_{11} & * & * \\ \Gamma_{21}^{l} & \Gamma_{22} & * \\ \Gamma_{31} & 0 & \Gamma_{33} \end{bmatrix} < 0, \qquad l = 1,2,3,4 \tag{27}$$

where

$$\Gamma_{21}^{1} = \begin{bmatrix} \hat{\tau}_1 \hat{N}^{\mathrm{T}} \\ \hat{\tau}_1 \hat{N}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{\mathrm{T}}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{\mathrm{T}}^{\mathrm{T}} \end{bmatrix} \quad \Gamma_{21}^{2} = \begin{bmatrix} \hat{\tau}_1 \hat{N}^{\mathrm{T}} \\ \hat{\tau}_1 \hat{N}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{W}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{W}^{\mathrm{T}} \end{bmatrix}$$

$$\Gamma_{21}^{3} = \begin{bmatrix} \hat{\tau}_1 \hat{M}^{\mathrm{T}} \\ \hat{\tau}_1 \hat{M}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{\mathrm{T}}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{\mathrm{T}}^{\mathrm{T}} \end{bmatrix} \quad \Gamma_{21}^{4} = \begin{bmatrix} \hat{\tau}_1 \hat{M}^{\mathrm{T}} \\ \hat{\tau}_1 \hat{M}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{W}^{\mathrm{T}} \\ (\hat{\tau}_2 - \hat{\tau}_1)\hat{W}^{\mathrm{T}} \end{bmatrix}$$

$$\hat{N}^{\mathrm{T}} = [\, \hat{N}_1^{\mathrm{T}} \quad \hat{N}_2^{\mathrm{T}} \quad \hat{N}_3^{\mathrm{T}} \quad \hat{N}_4^{\mathrm{T}} \quad \hat{N}_5^{\mathrm{T}} \quad \hat{N}_6^{\mathrm{T}} \,]$$

$$\hat{M}^{\mathrm{T}} = [\, \hat{M}_1^{\mathrm{T}} \quad \hat{M}_2^{\mathrm{T}} \quad \hat{M}_3^{\mathrm{T}} \quad \hat{M}_4^{\mathrm{T}} \quad \hat{M}_5^{\mathrm{T}} \quad \hat{M}_6^{\mathrm{T}} \,]$$

$$\hat{\mathrm{T}}^{\mathrm{T}} = [\, \hat{\mathrm{T}}_1^{\mathrm{T}} \quad \hat{\mathrm{T}}_2^{\mathrm{T}} \quad \hat{\mathrm{T}}_3^{\mathrm{T}} \quad \hat{\mathrm{T}}_4^{\mathrm{T}} \quad \hat{\mathrm{T}}_5^{\mathrm{T}} \quad \hat{\mathrm{T}}_6^{\mathrm{T}} \,]$$

$$\hat{W}^{\mathrm{T}} = [\, \hat{W}_1^{\mathrm{T}} \quad \hat{W}_2^{\mathrm{T}} \quad \hat{W}_3^{\mathrm{T}} \quad \hat{W}_4^{\mathrm{T}} \quad \hat{W}_5^{\mathrm{T}} \quad \hat{W}_6^{\mathrm{T}} \,]$$

$$\Gamma_{22} = \mathrm{diag}(-\hat{\tau}_1 \hat{R}_1 \quad -\hat{\tau}_1 \hat{Z}_1 \quad -(\hat{\tau}_2 - \hat{\tau}_1)\hat{R}_2 \quad -(\hat{\tau}_2 - \hat{\tau}_1)\hat{Z}_2)$$

$$\Gamma_{33} = \mathrm{diag}\,(\, -X^{\mathrm{T}} \hat{Z}_1^{-1} X \quad -X^{\mathrm{T}} \hat{Z}_2^{-1} X \,)$$

and Γ_{11} and Γ_{31} are shown at the bottom of the page.

Proof: First, note that (16) equals to

$$\begin{bmatrix} \Xi_{11} & * & * \\ \Xi_{21}^{l} & \Xi_{22} & * \\ \hat{\Xi}_{31} & 0 & \hat{\Xi}_{33} \end{bmatrix} < 0, \qquad l = 1,2,3,4 \tag{28}$$

where

$$\hat{\Xi}_{33} = \mathrm{diag}\,(\, -Z_1^{-1} \quad -Z_2^{-1} \,)$$

and $\hat{\Xi}_{31}$ is shown at the bottom of the page.

Define $S_1 = X^{-1}$, $S_2 = \rho_2 X^{-1}$, $S_3 = \rho_3 X^{-1}$, and $S_4 = \rho_4 X^{-1}$, where $\rho_4 \neq 0$ and X is a nonsingular matrix. Pre and postmultiplying (28) with $(\underbrace{X\ X\ \ldots\ X}_{10}\ I\ I\,)$ and its transpose, respectively, and defining $\hat{P} = XPX^{\mathrm{T}}$, $\hat{N}_i = XN_iX^{\mathrm{T}}$, $\hat{\mathrm{T}}_i = XT_iX^{\mathrm{T}}$, $\hat{W}_i = XW_iX^{\mathrm{T}}$, $\hat{M}_i = XM_iX^{\mathrm{T}}$, $\hat{Q}_i = XQ_iX^{\mathrm{T}}$, $\hat{R}_i = XR_iX^{\mathrm{T}}$, $\hat{Z}_i = XZ_iX^{\mathrm{T}}$, and $Y = KX^{\mathrm{T}}$, by using Theorem 1, the result follows. ■

Remark 6: Along a similar line of Remark 5, we can conclude a result based on Theorem 2, which can be used to solve the stabilization feedback gain K for (26). We denote the result as Theorem 2′.

Remark 7: It should be noted that (27) is a nonconvex feasibility problem, since $X^{\mathrm{T}} \hat{Z}_i^{-1} X$ $(i = 1,2)$ appear in (27). Define G_i $(i = 1,2)$ such that

$$X^{\mathrm{T}} \hat{Z}_i^{-1} X > G_i, \qquad i = 1,2. \tag{29}$$

Then, the solvability of (27) can be replaced by (27)′ and (29), where (27)′ is derived from (27) by changing Γ_{33} as $\Gamma_{33}' = (\, -G_1 \quad -G_2 \,)$. To efficiently solve it, we may employ the cone-complementarity method [7].

By Schur complements, (29) is equivalent to

$$\begin{bmatrix} \hat{Z}_i^{-1} & X^{-\mathrm{T}} \\ X^{-1} & G_i^{-1} \end{bmatrix} > 0.$$

Define $\bar{Z}_i = \hat{Z}_i^{-1}$, $\bar{G}_i = G_i^{-1}$, and $\bar{X} = X^{-1}$. Similar to [7], we provide the following linearization algorithm for the solvability of Theorem 3.

$$\Gamma_{11} = \begin{bmatrix}
\begin{matrix}\hat{N}_1 + \hat{N}_1^T + AX^T \\ +XA^T + \hat{Q}_1 + \hat{Q}_2\end{matrix} & * & * & * & * & * \\
\begin{matrix}\hat{N}_2 - \hat{N}_1^T + \hat{M}_1^T \\ +\rho_2 AX^T + \delta_0 Y^T B^T\end{matrix} & \begin{matrix}-\hat{N}_2 - \hat{N}_2^T + \hat{M}_2 + \hat{M}_2^T \\ +\delta_0\rho_2 BY + \delta_0\rho_2 Y^T B^T\end{matrix} & * & * & * & * \\
\hat{N}_3 - \hat{M}_1^T + \hat{T}_1^T & -\hat{N}_3 + \hat{M}_3 - \hat{M}_2^T + \hat{T}_2^T & -\hat{M}_3 - \hat{M}_3^T - \hat{Q}_1 + \hat{T}_3 + \hat{T}_3^T & * & * & * \\
\begin{matrix}\hat{N}_4 - \hat{T}_1^T + \hat{W}_1^T \\ +\rho_3 AX^T + (1-\delta_0)Y^T B^T\end{matrix} & \begin{matrix}-\hat{N}_4 + \hat{M}_4 - \hat{T}_2^T + \hat{W}_2^T \\ +\delta_0\rho_3 BY + (1-\delta_0)\rho_2 Y^T B^T\end{matrix} & -\hat{M}_4 - \hat{T}_3^T + \hat{W}_3^T + \hat{T}_4 & \begin{matrix}-\hat{T}_4 - \hat{T}_4^T + \hat{W}_4 + \hat{W}_4^T \\ +(1-\delta_0)\rho_3 BY + (1-\delta_0)\rho_3 Y^T B^T\end{matrix} & * & * \\
\hat{N}_5 - \hat{W}_1^T & -\hat{N}_5 + \hat{M}_5 - \hat{W}_2^T & -\hat{M}_5 - \hat{W}_3^T + \hat{T}5 & -\hat{T}_5 + \hat{W}_5 - \hat{W}_4^T & -\hat{W}_5 - \hat{W}_5^T - \hat{Q}_2 & * \\
\hat{N}_6 - X + \rho_4 AX^T + \hat{P} & -\hat{N}_6 + \hat{M}_6 - \rho_2 X + \delta_0\rho_4 BY & -\hat{M}_6 + \hat{T}_6 & -\hat{T}_6 + \hat{W}_6 - \rho_3 X + (1-\delta_0)\rho_4 BY & -\hat{W}_6 & -\rho_4 X^T - \rho_4 X + \hat{\tau}_1 \hat{R}_1 + (\hat{\tau}_2 - \hat{\tau}_1)\hat{R}_2
\end{bmatrix}$$

$$\Gamma_{31} = \begin{bmatrix} 0 & \sqrt{\hat{\tau}_1 \delta_0 (1-\delta_0)}BY & 0 & -\sqrt{(\hat{\tau}_2 - \hat{\tau}_1)\delta_0(1-\delta_0)}BY & 0 & 0 \\ 0 & \sqrt{(\hat{\tau}_2 - \hat{\tau}_1)\delta_0(1-\delta_0)}BY & 0 & -\sqrt{\hat{\tau}_1 \delta_0 (1-\delta_0)}BY & 0 & 0 \end{bmatrix}$$

$$\hat{\Xi}_{31} = \begin{bmatrix} 0 & \sqrt{\hat{\tau}_1 \delta_0 (1-\delta_0)}BK & 0 & -\sqrt{\hat{\tau}_1 \delta_0 (1-\delta_0)}BK & 0 & 0 \\ 0 & \sqrt{(\hat{\tau}_2 - \hat{\tau}_1)\delta_0(1-\delta_0)}BK & 0 & -\sqrt{(\hat{\tau}_2 - \hat{\tau}_1)\delta_0(1-\delta_0)}BK & 0 & 0 \end{bmatrix}$$

TABLE I
MAXIMUM ALLOWABLE $\hat{\tau}_2$

Methods / $\hat{\tau}_1$	0	0.2	0.4	0.6	0.8	1
Wu *et al.* [27] and Xu *et al.* [28]	0.6614	NA	NA	NA	NA	NA
Jiang *et al.* [10]	0.6916	0.7869	0.8839	0.9847	1.0909	1.2038
Yue *et al.* [30]	0.7761	0.8582	0.9413	1.0281	1.1214	1.2233
Theorem 2	0.9028	0.9711	1.0414	1.1150	1.1938	1.2809

Algorithm 1 (For Theorem 2)

Given constants ρ_m $(m = 2, 3, 4)$ and let N denote the maximum number of iterations.

Step 1: Find a feasible solution $\{X, \bar{X}, \bar{Z}_1, \hat{Z}_1, \bar{Z}_2, \hat{Z}_2, \bar{G}_1, G_1, \bar{G}_2, G_2\}$ to LMIs (27)$'$ and

$$\begin{bmatrix} \bar{Z}_i & \bar{X}^{\mathrm{T}} \\ \bar{X} & \bar{G}_i \end{bmatrix} > 0 \quad \begin{bmatrix} \hat{Z}_i & I \\ I & \bar{Z}_i \end{bmatrix} \geq 0$$
$$\begin{bmatrix} G_i & I \\ I & \bar{G}_i \end{bmatrix} \geq 0, \qquad (i = 1, 2)$$
$$\begin{bmatrix} X & I \\ I & \bar{X} \end{bmatrix} \geq 0. \tag{30}$$

If no feasible solution, EXIT and try other constants ρ_m $(m = 2, 3, 4)$. Set $k = 0$.

Step 2: Solve the following minimization problem:

$$\min \quad \mathrm{tr}\Bigg(X_k\bar{X} + \bar{X}_k X + \sum_{i=1}^{2} \left(\hat{Z}_{ik}\bar{Z}_i + \bar{Z}_{ik}\hat{Z}_i + G_{ik}\bar{G}_i + \bar{G}_{ik}G_i \right) \Bigg) \tag{31}$$

subject to LMIs (27)$'$ and (30).

Step 3: If $X^{\mathrm{T}}\hat{Z}_i^{-1}X > G_i$ $(i = 1, 2)$ are satisfied, a controller is given by $K = YX^{-\mathrm{T}}$. Otherwise, set $k = k + 1$. If $k < N$, go to Step 2; otherwise, EXIT (no feasible solution is found).

V. NUMERICAL EXAMPLES

In this section, two numerical examples will be employed to show the applications of our method. In Example 1, stability criteria in Section III will be used to solve the maximum allowable value of the time delay under different cases. The applications of criteria in this correspondence paper to NCSs will be shown in Example 2.

Example 1: Consider the system

$$\dot{x}(t) = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix} x(t - \tau(t)) \tag{32}$$

we are now concerned with the following two cases of the time delay $\tau(t)$:

Case 1) $0 \leq \hat{\tau}_1 \leq \tau(t) \leq \hat{\tau}_2$.
Table I compares our results with existing ones.

Case 2) Probability distribution of $\tau(t)$ is known.

In this case, Table II shows the upper bounds of $\tau(t)$ for different values of δ_0 and $\hat{\tau}_1$.

TABLE II
MAXIMUM ALLOWABLE $\hat{\tau}_2$

δ_0	0.7	0.8	0.9	0.99
$\hat{\tau}_1 = 0.1$	2.4149	3.3373	5.6150	25.4429
$\hat{\tau}_1 = 0.4$	1.9634	2.5469	3.9919	16.7526
$\hat{\tau}_1 = 0.7$	1.4875	1.7586	2.3213	7.4844

TABLE III
COMPUTATION RESULTS FOR DIFFERENT VALUES OF $\hat{\tau}_1$ AND $\delta_0 = 0.8$

$\hat{\tau}_1$	$\hat{\tau}_2$	K	$\hat{\tau}_2$(by using Theorem 1)
20	237	$[-0.0013 \quad -0.0108]$	2791
50	209	$[-0.0023 \quad -0.0215]$	1439
80	165	$[-0.0024 \quad -0.0235]$	1321

Example 2: Consider a feedback control system [33]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \tag{33}$$

which is assumed that the sensor, controller, and actuator are connected by a common network medium. For a given linear controller $u(t) = [-3.75 \quad -11.5]x(t)$, the asymptotic stability of closed-loop system was investigated [11], [12], [21], [31], [33] under the consideration of the effect of the network conditions, such as the transmission delay of the data packet. The upper bounds of the transmission delay for guaranteeing the asymptotic stability of the system are, respectively, 4.5×10^{-4} [33], 0.0538 [21], 0.7805 [12], 0.8695 [31], and 0.9412 [11]. By using Theorem 1$'$, it can be computed that the allowable upper bound of the transmission delay is 1.0432.

As pointed out in [31], an NCS can also be transformed into a system with interval time-varying delay. When assuming that the lower bound of the delay is not zero, the allowable upper bounds of the delay are given in [11] for the system. It was computed that the upper bound of the delay is 0.9635 for the case of $\hat{\tau}_1 = 0.2$ [11] or 0.9916 for the case of $\hat{\tau}_1 = 0.3$ [32]. By using Theorem 1$'$, the obtained upper bounds of the delay are 1.0467 for the case of $\hat{\tau}_1 = 0.2$ and 1.0487 for the case of $\hat{\tau}_1 = 0.3$.

By using Theorem 2$'$ and considering $\rho_2 = 0.2$, $\rho_3 = 10$, we obtain that the upper bound of the time delay is 67 for the case of $\hat{\tau}_1 = 0$ and the corresponding feedback gain $K = [-0.0022 \quad -0.0218]$. Similarly, by using the method developed in [31] and employing Theorem 1$'$ with $K = [-0.0022 \quad -0.0218]$, it can be obtained that the allowable upper bound of the time delay is 608. Choosing $\rho_2 = 0.2$, $\rho_3 = 0.12$, and $\rho_3 = 20$ and using Algorithm 1, we can obtain the computation results for different values of $\hat{\tau}_1$ and $\delta_0 = 0.8$. Then, by using Theorem 1, we can finally compute the allowable upper bound of the time delay for the corresponding feedback gains (see Table III).

It can be found by Table III that, when the probability distribution of the time delay can be observed, using Theorem 2 can lead to a larger allowable upper bound of the delay than that using Theorem 2$'$ under the similar requirements for the system performance.

VI. Conclusion

In this correspondence paper, the stabilization problem has been investigated for a class of linear systems with stochastic input delay. Based on the information of probability distribution of the time delay, a new model of the system, which has probability-distribution-dependent parameter matrices, has been proposed. In terms of the new model, some stability and stabilization criteria for the system have been derived by solving a set of LMIs. To derive the LMI conditions for determining the stability and stabilizability of the system, the convexity of a matrix equation has been employed to give less conservative results. Our main results have been applied to a commonly employed NCS. Examples have shown that the proposed method can lead to less conservative results than those obtained by existing ones. Furthermore, if the probability distribution of the delay is known *a priori*, the allowable upper bound of the delay may be larger than those derived for the case when only the variation range of the delay is used.

References

[1] Q. Han, X. Yu, and K. Gu, "On computing the maximum time-delay bound for stability of linear neutral systems," *IEEE Trans. Autom. Control*, vol. 49, no. 12, pp. 2281–2285, Dec. 2004.

[2] Q. Han, "On stability of linear neutral systems with mixed time delays: A discretized Lyapunov functional approach," *Automatica*, vol. 41, no. 7, pp. 1209–1218, Jul. 2005.

[3] Q. Han, "On designing time-varying delay feedback controllers for master–slave synchronization of Lur'e systems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 54, no. 7, pp. 1573–1583, Jul. 2007.

[4] W.-H. Chen, J.-X. Xu, and Z. H. Guan, "Guaranteed cost control for uncertain Markovian jump systems with mode-dependent time-delays," *IEEE Trans. Autom. Control*, vol. 48, no. 12, pp. 2270–2277, Dec. 2003.

[5] E. Cheres, Z. J. Palmor, and S. Gutman, "Min–max predictor control for uncertain systems with input delays," *IEEE Trans. Autom. Control*, vol. 35, no. 2, pp. 210–214, Feb. 1990.

[6] H. H. Choi and M. J. Chung, "Observer-based H_∞ controller design for state delayed linear systems," *Automatica*, vol. 32, no. 2, pp. 1073–1075, Jul. 1996.

[7] L. El Ghaoui, F. Oustry, and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems," *IEEE Trans. Autom. Control*, vol. 42, no. 8, pp. 1171–1176, Aug. 1997.

[8] H. Gao, J. Lam, C. Wang, and S. Xu, "H_∞ model reduction for discrete time-delay systems: Delay-independent and dependent approaches," *Int. J. Control*, vol. 77, no. 4, pp. 321–335, 2004.

[9] Y. He, M. Wu, J. H. She, and G. P. Liu, "Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic-type uncertainties," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 828–832, May 2004.

[10] X. F. Jiang and Q. L. Han, "On H_∞ control for linear systems with interval time-varying delay," *Automatica*, vol. 41, no. 12, pp. 2099–2106, Dec. 2005.

[11] X. F. Jiang and Q. L. Han, "Delay-dependent robust stability for uncertain linear systems with interval time-varying delay," *Automatica*, vol. 42, no. 6, pp. 1059–1065, Jun. 2006.

[12] D. S. Kim, Y. S. Lee, W. H. Kwon, and H. S. Park, "Maximum allowable delay bounds of networked control systems," *Control Eng. Pract.*, vol. 11, no. 11, pp. 1301–1313, Nov. 2003.

[13] J. H. Kim, E. T. Jeung, and H. B. Park, "Robust control for parameter uncertain delay systems in state and control input," *Automatica*, vol. 32, no. 9, pp. 1337–1339, Sep. 1996.

[14] E. Knightly, "H-bind: A new approach to providing statistical performance guarantees to VBR traffic," in *Proc. IEEE INFOCOM*, San Francisco, CA, Mar. 1996, pp. 1091–1099.

[15] O. I. Kosmidou and Y. S. Boutalis, "A linear matrix inequality approach for guaranteed cost control of systems with state and input delays," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 36, no. 5, pp. 936–942, Sep. 2006.

[16] X. Li, M. Huay, B. Huang, and D. G. Fisher, "Delay-dependent robust H_∞ control of uncertain linear systems with input delay," in *Proc. Amer. Control Conf.*, San Diego, CA, 1999, pp. 1861–1865.

[17] R. C. Luo and L. Y. Chung, "Stabilization for linear uncertain system with time latency," *IEEE Trans. Ind. Electron.*, vol. 49, no. 4, pp. 905–910, Aug. 2002.

[18] X. Mao, "Exponential stability of stochastic delay interval systems with Markovian switching," *IEEE Trans. Autom. Control*, vol. 47, no. 10, pp. 1604–1612, Oct. 2002.

[19] X. Mao, N. Koroleva, and A. Rodkina, "Robust stability of uncertain stochastic differential delay equations," *Syst. Control Lett.*, vol. 35, no. 5, pp. 325–336, 1998.

[20] Y. S. Moon, P. Park, and W. H. Kwon, "Robust stabilization of uncertain input-delayed systems using reduction method," *Automatica*, vol. 37, no. 2, pp. 307–312, Feb. 2001.

[21] H. S. Park, Y. H. Kim, D. S. Kim, and W. H. Kwon, "A scheduling method for network-based control systems," *IEEE Trans. Control Syst. Technol.*, vol. 10, no. 3, pp. 318–330, May 2002.

[22] A. Ray, "Output feedback control under randomly varying distributed delays," *J. Guid. Control Dyn.*, vol. 17, no. 4, pp. 701–711, 1994.

[23] S. Y. Wang, R. Nathuji, R. Bettati, and W. Zhao, "Providing statistical delay guarantees in wireless networks," in *Proc. 24th ICDCS*, 2004, pp. 48–55.

[24] Z. Wang, D. W. C. Ho, and X. Liu, "Robust filtering under randomly varying sensor delay with variance constraints," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 51, no. 6, pp. 320–326, Jun. 2004.

[25] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H_∞ filtering for stochastic time-delay systems with missing measurements," *IEEE Trans. Signal Process.*, vol. 54, no. 7, pp. 2579–2587, Jul. 2006.

[26] Z. Wang, D. W. C. Ho, and X. Liu, "A note on the robust stability of uncertain stochastic fuzzy systems with time-delays," *IEEE Trans. Syst., Man Cybern. A, Syst., Humans*, vol. 34, no. 4, pp. 570–576, Jul. 2004.

[27] M. Wu, Y. He, J. H. She, and G. P. Liu, "Delay-dependent criteria for robust stability of time-varying delay systems," *Automatica*, vol. 40, no. 8, pp. 1435–1439, Aug. 2004.

[28] S. Xu, J. Lam, and Y. Zou, "New results on delay-dependent robust H_∞ control for systems with time-varying delay," *Automatica*, vol. 42, no. 2, pp. 343–348, Feb. 2006.

[29] F. Yang, Z. Wang, Y. S. Hung, and M. Gani, "H_∞ control for networked systems with random communication delays," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 511–518, Mar. 2006.

[30] D. Yue and Q. Han, "Network-based robust H_∞ filtering for uncertain linear systems," *IEEE Trans. Signal Process.*, vol. 54, no. 11, pp. 4293–4301, Nov. 2006.

[31] D. Yue, Q. Han, and P. Chen, "State feedback controller design of networked control systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 51, no. 11, pp. 640–644, Nov. 2004.

[32] D. Yue, Q. Han, and J. Lam, "Network-based robust H_∞ control of systems with uncertainty," *Automatica*, vol. 41, no. 6, pp. 999–1007, 2005.

[33] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Syst. Mag.*, vol. 21, no. 1, pp. 84–99, Feb. 2001.