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<th>An integral inequality of an intrinsic measure on bounded domains in Cn</th>
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1. Introduction. Let $D$ be a complete hyperbolic bounded domain in $\mathbb{C}^n$ in the sense of [7]. We denote by $M^E_D$ the differential Eisenman-Kobayashi $n$-measure (defined with respect to the unit ball) on $D$. Since we may endow $D$ with a global coordinate system, $M^E_D$ can therefore be viewed as a function. The main goal of this paper is to prove the following theorem.

Theorem. If we assume there exists a neighborhood $N$ of $\partial D$ in $\mathbb{C}^n$ where $M^E_D$ satisfies the growth condition

$$|M^E_D(z)| \geq \frac{k}{(r(z))^{m+s}},$$

where $k$ = positive constant, $r(z) = $ the euclidean distance from $z$ to $\partial D$, $z \in N \cap D$, $m$ and $s$ positive numbers, then we can find a neighborhood $U$ of $\partial D$ in $\mathbb{C}^n$ such that for all $z_0 \in U \cup D$, whenever the closed disk $\{z_0 + \rho z_1 : \rho \in \mathbb{C}, |\rho| \leq 1\}$, $z_1 \in \mathbb{C}^n$, lies in $U \cap D$, the inequality

$$\ln |M^E_D(z_0)| \leq \frac{n}{m\pi} \int_0^{2\pi} \ln |M^E_D(z_0 + e^{i\theta} z_1)| \, d\theta$$

for $|M^E_D|$ holds.

Typical examples satisfying conditions of our theorem include analytic polyhedra, strongly pseudoconvex domains and certain domains of holomorphy with smooth real analytic boundary [1]. For the case of strongly pseudoconvex domains we have the following corollary.

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Corollary. Let $D$ be a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then there exists a neighborhood $U$ of $\partial D$ in $\mathbb{C}^n$ such that for all $z_0 \in U \cap D$, whenever the closed disk

$$\{z_0 + \rho z_1; \rho \in \mathbb{C}, |\rho| \leq 1\}, \quad z_1 \in \mathbb{C}^n,$$

lies in $U \cap D$, the inequality

$$\ln |M_E^D(z_0)| \leq \frac{1}{\pi} \int_0^{2\pi} \ln |M_E^D(z_0 + e^{i\theta} z_1)| \, d\theta$$

holds.

Our proof rests on the boundary assumption of the intrinsic measure and the classical Hartogs’ construction of analytic family of disks [11]. It is quite clear from a minor modification of the proof that our integral inequality also holds on an analytically embedded disk (i.e., the image of a holomorphic embedding $f : B \to D$, here $B = \{z \in \mathbb{C} : |z| < 1\}$, which is homeomorphic up to the boundary $\partial B$ with $f(\partial B) \subset D$ and $f(0) = z_0$). One can see this inequality imposes a restriction on $M_E^D$ when the analytic disk is large and sufficiently close to the boundary. This type of inequality can be generalized to other low dimensional intrinsic measures. A sharpened result can also probably be derived along our line. The arrangement of our paper can be summarized as below.


2. Definition of Eisenman-Kobayashi measure. For the basic definitions and a survey of this subject, one should consult [7, 8]. Since our Eisenman-Kobayashi $n$-measures are defined with respect to the ball in $\mathbb{C}^n$, which is somewhat different from what had been done in [7, 8], we shall include our definition here.

Let $N$ be a complex manifold of dimension $n$. The Eisenman Kobayashi $n$-measure $M_{E}^{N}$ is an $(n, n)$-form $|M_{E}^{N}| \cdot (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge$
\[ dz_n \wedge d\bar{z}_n \] on \( N \), such that \( |M^E_N| \) is defined for all \( x \in N \) as
\[
|M^E_N(x)| = \inf \left\{ \frac{1}{R^{2n}} : \exists f \in \text{Hol} \left( B^n_R, N \right) \text{ such that } f(0) = x, \det (Jf(0)) = 1 \right\},
\]
where \( B^n_R \) is the euclidean ball with center 0 and radius \( R \) in \( C^n \), \( \text{Hol} \left( B^n_R, N \right) \) the set of all holomorphic maps from \( B^n_R \) to \( N \), and \( Jf(0) \) the Jacobian matrix of \( f \) at 0.

It is easy to check that for a bounded domain \( D \) in \( C^n \), \( |M^E_D(x)| \neq 0 \) for all \( x \in D \). \( |M^E_D| \) is in general a semicontinuous function \([12]\). When \( D \) is a complete hyperbolic bounded domain, \( |M^E_D| \) can be proved to be continuous. All complete hyperbolic domains in \( C^n \) are pseudoconvex \([7]\).

3. A boundary estimate of \( M^E_D \) on S.P.C. domains and proof of the corollary. Let \( D \) be a strongly pseudoconvex domain in \( C^n \) with smooth boundary. The following will be proved in this section.

**Theorem.** There exist a neighborhood \( U \) of \( \partial D \) in \( C^n \) and a positive constant \( c \) such that for all \( z \in U \cap D \),
\[
|M^E_D(z)| \geq \frac{c}{(r(z))^{n+1}},
\]
where \( r(z) \) is the euclidean distance function from \( z \) to the boundary of \( D \).

Note that since \( \partial D \) is compact and everywhere strongly pseudoconvex, the above statement can be reduced to the following local problem:

For all \( p \in \partial D \), there exists a neighborhood \( V \) of \( p \) in \( C^n \) and a positive constant \( k \) such that for all \( z \in D \cap V \),
\[
|M^E_D(z)| \geq \frac{k}{(r(z))^{n+1}}.
\]

**3.1 Localization lemma.** Let \( D \) be a bounded domain in \( C^n \), \( D_1 \) another domain such that \( D \cap D_1 \) is nonempty.
Definitions. (1) Let \( z, w \) belong to \( D \). Then \( d(z, w) = \inf \{ P(a, b) : \exists f \in \text{Hol}(B_n, D) \text{ s.t. } f(a) = z, f(b) = w, \text{ where } P \text{ is the Kobayashi metric on } B_n \} \); here \( B_n \) is the unit ball in \( C^n \).

(2) Let \( z \) belong to \( D \cap D_1 \), then \( d(z) = \inf \{ d(z, w) : w \in D - D_1 \} \).

Lemma A. Let us denote \( \hat{D} = D \cap D_1 \); then for all \( z \in \hat{D} \), we have

\[
|M_E^{\hat{D}}(z)| \leq (\coth d(z))^2n \cdot |M_E^D(z)|,
\]

where \( M_E^\hat{D} \) and \( M_E^D \) denote the Eisenman-Kobayashi measures on \( \hat{D} \) and \( D \), respectively.

Proof. First of all, let us fix \( z \in \hat{D} \) and let

\[
r = \sup \{ r' : \exists f \in \text{Hol}(B_n^{r'}, \hat{D}) \text{ s.t. } f(0) = z, \det (Jf(0)) = 1 \}.
\]

Then we choose a number \( R \) which is slightly larger than \( r \). From our choice of \( r \) it is obvious that there is an \( f \in \text{Hol}(B_n^R, \hat{D}) \) such that \( f(0) = z, \det (Jf(0)) = 1 \) and it maps a boundary point of \( B_n^R \) to a point belonging to \( D - \hat{D} \). One can see that if \( w \) is such a point belonging to \( D - \hat{D} \), then \( d(z) \leq d(z, w) \). From our definition of \( d \), we observe that

\[
d(z) \leq d(z, w) \leq (1/2) \ln[(1 + r/R)/(1 - r/R)]
\]

(distance-decreasing property under holomorphic mappings; consider \( f \) to be a holomorphic map from \( B_n^R \) to \( D \) \([7]\)). Hence,

\[
1/r \leq \coth d(z) \cdot (1/R),
\]

\[
(1/r)^{2n} \leq (\coth d(z))^{2n} \cdot (1/R)^{2n}.
\]

This inequality is true for all the \( R \)'s satisfying the properties mentioned above. Considering the definition of \( M_E^D(z) \), one can now conclude our desired inequality

\[
|M_E^{\hat{D}}(z)| \leq (\coth d(z))^{2n} \cdot |M_E^D(z)|. \quad \Box
\]

Lemma B. Suppose \( D \) is a strongly pseudoconvex bounded domain in \( C^n \) and \( D_1 \) is a neighborhood of a boundary point of \( D \). Let this
boundary point be $p$, $\hat{D} = D \cap D_1$, and $M^E_D, M^E_{\hat{D}}$ the Eisenman-Kobayashi measures of $\hat{D}$ and $D$, respectively. Then we have

$$\lim_{z \to p} \frac{|M^E_D(z)|}{|M^E_{\hat{D}}(z)|} = 1 \quad \text{for all } z \in \hat{D}.$$ 

**Proof.** We divide our proof into two steps.

(1) Since the inclusion map $\hat{D} \to D$ is holomorphic, by the volume-decreasing property [7] we have

$$|M^E_D(z)| \geq |M^E_{\hat{D}}(z)| \quad \text{i.e.,} \quad \frac{|M^E_D(z)|}{|M^E_{\hat{D}}(z)|} \geq 1.$$ 

(2) From Lemma (A) we have

$$|M^E_D(z)| \leq (\coth d(z))^{2n} \cdot |M^E_D(z)|.$$ 

Now it is clear from our definitions that

$$d(z, w) \geq d_k(z, w) \quad \forall z, w \in D,$$

where $d_k$ is the Kobayashi metric on $D$ [7]. However, if we set

$$d_k(z, D - \hat{D}) = \inf \{d_k(z, w) : w \in D - \hat{D}\},$$

it is known that

$$\lim_{z \to p} d_k(z, D - \hat{D}) = \infty$$

if $D$ is strongly pseudoconvex (this statement can easily be derived from the result of I. Graham [5]). Thus, $d(z)$ will go to infinity as $z \in \hat{D}$ approaches $p$; consequently, $\coth d(z)$ will tend to 1 as $z \in \hat{D}$ tends to $p$. At this point we obtain another inequality:

$$\lim_{z \to p} \frac{|M^E_D(z)|}{|M^E_{\hat{D}}(z)|} \leq 1.$$ 

Combining these two inequalities, we thereby complete the proof of our localization lemma. \qed
3.2 Proof of our estimate. Before embarking on our proof, we first make two remarks here.

**Remark 1.** An analytic ellipsoid is a strongly pseudoconvex domain $A$ in $\mathbb{C}^n$ which can locally be described as: If $p \in \partial A$ and $W$ is a sufficiently small open neighborhood of $p$ in $\mathbb{C}^n$, then

$$A \cap W = \{ z \in \mathbb{C}^n : g(z) = -z_1 - \bar{z}_1 + \sum_{i,j=1}^{n} b_{ij} z_i \bar{z}_j < 0 \},$$

where $[b_{ij}]_{i,j=1}^{n}$ is a hermitian positive definite matrix.

In our expression $p$ is the origin of the coordinates $\{ z_1, z_2, \ldots, z_n \}$, $z_1$ is the complex normal of $\partial A$ at $p$, and $\{ z_2, \ldots, z_n \}$ is the basis of the maximal complex tangent space $T_p(\partial A)$. It is known that any analytic ellipsoid is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$, and the Eisenman-Kobayashi measure on the unit ball is equal to the volume form of the Bergman metric (with the reservation of the multiple of a constant). The following estimate can thus be obtained from the explicit formula of the Bergman metric on $B_n$ (see [13], for example).

There exists a sufficiently small open neighborhood $W_1$ of $\partial A$ in $\mathbb{C}^n$ and a positive constant $c_1$ such that

$$|M_{E}^A(z)| \approx \frac{c_1}{(r(z))^{n+1}} \quad \forall z \in W_1 \cap A.$$

Furthermore, by the volume-decreasing property again we have the following estimate.

There exists an open neighborhood $W_2$ of $p$ in $\mathbb{C}^n$ and a positive constant $c_1$ such that

$$|M_{E}^A(z)| \geq \frac{c_2}{(r(z))^{n+1}} \quad \forall z \in W_2 = W_2 \cap A.$$

**Remark 2.** Let $D$ be a strongly pseudoconvex boundary domain in $\mathbb{C}^n$ with smooth boundary and $p$ be a given boundary point of $D$ as before. With a similar coordinate system $\{ z_1, z_2, \ldots, z_n \}$ as in Remark 1, in which we can locally characterize $D$ around $p$ as

$$D \cap U_1 = \{ z \in D; G(z) < 0 \},$$
where $U_1$ is a neighborhood of $p$ in $\mathbb{C}^n$, and

$$G(z) = -z_1 - \bar{z}_1 + \sum a_{ij} z_i \bar{z}_j + 2 \text{Re} \left\{ \sum_{i \geq 2} \frac{\partial^2 G}{\partial z_i \partial \bar{z}_j}(p) z_i \bar{z}_j \right\} + O(|z|^3)$$

with respect to the above coordinate system. Since $D$ is strongly pseudoconvex, $(a_{ij})$ is a hermitian positive definite matrix. We can apply our localization in Lemma B to make the assertion: For all $\varepsilon > 0$, there exists a neighborhood $U_2$ of $p$ in $\mathbb{C}^n$ such that

$$\left| \frac{|M^E_{U_2}(z)|}{|M^B_{U_2}(z)|} - 1 \right| < \varepsilon$$

for all $z \in \hat{U}_2 = U_2 \cap D$.

**Proof of our main estimate.** To start, we fix a boundary point $p \in \partial D$ and choose the coordinate system $\{z_1, \ldots, z_n\}$ as in Remark 2. Now we construct an analytic ellipsoid $A_s$ whose defining equation around the point $p$ is given by

$$g_s = -z_1 - \bar{z}_1 + \sum (a_{ij} - s \cdot \delta_{ij}) z_i \bar{z}_j,$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases},$$

and $s$ is a well-chosen positive constant such that $A_s$ satisfies the following properties:

(i) there exists a neighborhood $V$ of $p$ in $\mathbb{C}^n$ such that $V \cap A_s = \{z : g_s(z) < 0\}$;

(ii) with the same $V$ in (i), we have $V \cap A_s \supset V \cap D$.

Graphically, our situation can be illustrated by the picture on the top of the next page.

If we denote by $V_1 = A_s \cap V$ (white area) and $V_2 = D \cap V$ (shaded area), we have by volume decreasing property

$$|M^E_{V_2}(z)| \geq |M^E_{V_1}(z)| \quad \forall z \in V_2.$$
Moreover, if $V$ is sufficiently small we can use Remarks 1 and 2 above to conclude that

$$|M_E^D(z)| \geq \frac{k}{(r(z))^{n+1}} \quad \forall z \in \mathcal{N}_p$$

where $\mathcal{N}_p = \{\text{the axis Re}(z_1)\} \cap V$, $k$ is a suitable constant.

Finally we have to observe that all of our processes described in this section are \textit{uniform} in the following sense. We can choose a sufficiently small neighborhood $T \subset \partial D$ of $p$ in such a way that all the \textit{sizes} of domains of comparison and constants can be unchanged so that all the above arguments will remain valid for all $q \in T$. Then we further refine our $V = \cup_{q \in T}\{\mathcal{N}_q\}$. This completes the whole proof. \qed

\textit{Remark} . More precise boundary estimates for both Eisenman-Kobayashi and Carathéodory measures were carried out in [16] following the original work of Graham [4,5] in the case of metrics. The localization lemma in the case of the Kobayashi metric was first used by Royden [12] and Graham [4,5]. Some other related results can be found in [13, 2, 6, 3].
Proof of the Corollary. Applying our theorem, we let $m = n$ and $s = 1$.

4. Proof of our main statement. Take the neighborhood $N$ as in the assumption of our theorem. Since $D$ is complete hyperbolic in the sense of Kobayashi, consequently $\ln |M^E_D|$ is a continuous function on

$$\{z_0 + \rho z_1 : \rho \in \mathbb{C}, |\rho| \leq 1\} \subset N \cap D.$$  

We can always find a real valued function $h$, defined and continuous on $|\rho| \leq 1$, harmonic in $|\rho| < 1$, and equal to $(1/m) \ln |M^E_D|$ on $|\rho| = 1$, that is,

$$h(\rho) = (1/m) \ln |M^E_D(z_0 + \rho z_1)| \quad \forall |\rho| = 1.$$

Let $h^*$ be a harmonic conjugate of $h$, set $g = h + ih^*$. Then $g$ is continuous on $|\rho| \leq 1$ and holomorphic in $|\rho| < 1$. Next, let $b$ be any vector in $\mathbb{C}^n$ with $||b|| = 1$ and $\lambda_0$ any real number satisfying $0 < \lambda_0 < 1$. Consider the analytic disk

$$\Sigma_\lambda : \rho \rightarrow z_0 + \rho z_1 + \lambda e^{-g(\rho)} b,$$

in $\mathbb{C}^n$ where $|\rho| \leq 1$ and $\lambda$ fixed, $0 \leq \lambda \leq \lambda_0$.

Claim. $\cup_{\lambda_0 \leq \lambda \leq \lambda_0} \partial \Sigma_\lambda \subset D$.

Since for all $z \in \partial \Sigma_\lambda$, $z$ is the image of some $\rho$ with $|\rho| = 1$,

$$||z - (z_0 + \rho z_1)|| = ||\lambda e^{-g(\rho)} b|| = \lambda e^{-h(\rho)} \leq \lambda_0 e^{-(1/m) \ln |M^E_D(z_0 + \rho z_1)|} = \lambda_0 \cdot |M^E_D(z_0 + \rho z_1)|^{-1/m}.$$

By our assumption, since $z_0 + \rho z_1 \in N \cap D$, we have

$$|M^E_D(z_0 + \rho z_1)| \geq \frac{k}{(r(z_0 + \rho z_1))^{m+s}}.$$  

Hence,

$$\frac{1}{|M^E_D(z_0 + \rho z_1)|^{1/m}} \leq r(z_0 + \rho z_1) \cdot \left[\frac{(r(z_0 + \rho z_1))^{s}}{k}\right]^{1/m}.$$
Moreover, we can choose \( U \subset N \) to be sufficiently small so that the inequality
\[
\left[ \frac{(r(z_0 + \rho z_1))^s}{k} \right]^{1/m} < 1
\]
also holds. Therefore, we obtain
\[
||z - (z_0 + \rho z_1)|| < \lambda_0 \cdot r(z_0 + \rho z_1).
\]
However, \( \lambda_0 \) lies between zero and one; this yields
\[
||z - (z_0 + \rho z_1)|| < r(z_0 + \rho z_1),
\]
which is independent of \( z \) and \( \lambda \). Hence, the inequality holds for all \( z \in \cup \partial \Sigma_{\lambda} \), that is, \( \cup \partial \Sigma_{\lambda} \) is bounded. Furthermore, for all points \( z \in \cup \partial \Sigma_{\lambda} \) and \( w \in \partial D \),
\[
||z - w|| \geq ||(z_0 + \rho z_1) - w|| - ||z - (z_0 + \rho z_1)||,
\]
thus
\[
r(z) \geq r(z_0 + \rho z_1) - ||z - (z_0 + \rho z_1)|| > 0,
\]
hence
\[
\bigcup_{0 \leq \lambda \leq \lambda_0} \partial \Sigma_{\lambda} \subset \subset D.
\]
This verifies our claim. Applying Kontinuitätssatz, we therefore obtain
\[
z_0 + \rho z_1 + \lambda e^{-g(\rho)} b \in D \quad \forall 0 \leq \lambda < 1, \quad |\rho| \leq 1.
\]
Hence,
\[
z_0 + \lambda e^{-g(\rho)} b e^{i\theta} \in D \quad \forall 0 \leq \lambda < 1, \quad 0 \leq \theta \leq 2\pi,
\]
where \( b \) is in arbitrary direction. That is, the ball \( B(z_0, ||\lambda e^{-g(\rho)} b||) \subset D \), for all \( 0 \leq \lambda < 1 \). It implies that
\[
B(z_0, ||e^{-g(\rho)} b||) \subset D;
\]
consequently,
\[
B(z_0, e^{-h(0)}) \subset D.
\]
Hence, the function \( f : B_n^{e^{-h(0)}} \to D \) (recall that \( b \) is in arbitrary direction) such that \( f(z) = z_0 + z \) is well defined and holomorphic. Note that \( f(0) = z_0 \) and \( \det(Jf(0)) = 1 \), hence

\[
\left| M_{E}^{D}(z_0) \right| = \inf \left\{ \frac{1}{R^{2n}} : \exists f \in \text{Hol}(B_n^{R}, D), \text{such that } f(0) = z_0, \det(Jf(0)) = 1 \right\}
\]

\[
\leq \frac{1}{e^{-2n \cdot h(0)}} = e^{2n \cdot h(0)}.
\]

Therefore,

\[
\ln \left| M_{E}^{D}(z_0) \right| \leq 2n \cdot h(0) = 2n \cdot \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \, d\theta
\]

\[
= \frac{n}{m\pi} \int_0^{2\pi} \ln \left| M_{E}^{D}(z_0 + e^{i\theta}z_1) \right| \, d\theta,
\]

which is exactly what we want to show. \( \square \)

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