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A COMPARISON BETWEEN TWO MODELS FOR PREDICTING ORDERING PROBABILITIES IN MULTI-ENTRY COMPETITIONS

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A COMPARISON BETWEEN TWO MODELS FOR PREDICTING ORDERING PROBABILITIES IN MULTI-ENTRY COMPETITIONS

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Abstract

To predict ordering probabilities of a multi-entry competition (e.g. horse race), two models have been proposed. Harville (1973) proposed a simple and convenient model that people can easily use in practice. Henery (1981) proposed a more sophisticated model but it has no closed form solution. In this paper, we empirically compare the two models using a series of logit models applied to horse-racing data. In horse-racing, many previous studies claimed that the win bet fraction is a reasonable estimate of the winning probability. To consider more complicated bet types (e.g. exacta, place & show), ordering probabilities (e.g. P(horse i wins and horse j finishes second)) are required. The Harville and Henery model assume different running time distribution and produce different sets of ordering probabilities. This paper illustrates that the Harville model is not always as good as the Henery model in predicting ordering probabilities. The theoretical result concludes that if the running time of every horse is normally distributed, the probabilities produced by the Harville model have a systematic bias for the extreme cases (the strongest and weakest horses). We concentrate on horse-racing case but the methodology can be applied to other multi-entry competitions.

Keywords : Ordering probabilities; Running time distributions; Horse races

1. Introduction

In the pari-mutuel betting system of horse-racing, it is useful to predict P(horse i finishes 1st and horse j finishes 2nd) from the simple knowledge of the winning probabilities, i.e. P(horse i wins). For more complicated bets such as the exacta and trifecta, even on-track bettors cannot observe the changes of odds and thus another source of information is required in order to predict the finishing order probabilities. One reasonable estimate of the win probabilities is the win bet fraction. Previous empirical studies showed that the win bet fraction is quite consistent with the true winning probability although a favourite-longshot bias sometimes exists (e.g. Griffith (1949); McGlothlin (1956); Hoerl & Fallin (1974); Ali (1977); Synder (1978); Fabricand (1979); Hausch, Ziemba & Rubinstein (1981); Asch, Malkiel & Quandt (1982); Busche & Hall (1988)).

In this paper, we compare the two models proposed by Harville (1973) and Henery (1981). The former one is simple and easy to use but the latter one is much more complicated. We will consider estimation of ordering probabilities using the two models, including theoretical discussion of the difference between the Harville and Henery models. Section 2 will briefly review the Harville and Henery models. Empirical analysis and theoretical discussion will be given in sections 3 and 4, respectively, with conclusions in section 5.

2. Description of some proposed models

2.1 Harville model

The simplest and most commonly used model to estimate ordering probabilities is the one proposed by Harville (1973). The basic idea is simple. For instance, to predict P(horse i wins and horse j finishes 2nd), we may use :

$$\pi_{ij} = \frac{\pi_i \pi_j}{1 - \pi_i}$$

if π_i and π_j are known (π_i's can be estimated by bet fractions). A similar idea was also mentioned in Plackett (1975). Moreover, it is the ranking model proposed by Luce & Suppes (1965) in the study of choice behaviour. As interpreted in Harville (1973), this model assumes that the event that horse j ranks ahead of all the other horses, save possibly horse i, is independent of the event that horse i wins.

At a first glance, the above formula may seem reasonable and thus, some researchers used this method for estimating probabilities (e.g. Hausch, Ziemba & Rubinstein (1981)). It is also known that some bettors use this method. However, $\pi_{j|i}$ may not be equal to $\pi_j/(1-\pi_i)$ in general. One common argument is mentioned in Hausch, Ziemba and Rubinstein (1981) : "no account is made of the possibility of the Silky Sullivan problem; that is, some horses generally either win or finish out-of-the-money; for these horses the formulas greatly over-estimate the true probability of finishing second or third".

One reasonable way to find these ordering probabilities is to assume an underlying probability distribution for the running times of horses. It can be easily shown that if the running times follow exponential distributions independently with different mean running times, the above formula will be obtained.

McCulloch and Van Zijl (1986) gave a direct test for the Harville model and indicated that the model had a bias. However, their paper depended on the assumption that the show bet fraction for their New Zealand data was the same as the corresponding true ordering probabilities.

2.2 Henery model

Henery (1981) suggested to assume that the running times are independent normal with unit variance, i.e. $T_i \sim N(\theta_i,1)$ independently. The resulting probabilities are obviously the same as that of a general constant variance model. Under the Henery model,

$$P\,[T_1 < T_2 < \ldots < T_n] = \int_{-\infty}^{\infty}\phi(t_1-\theta_1)\ \ldots \int_{t_{n-1}}^{\infty}\phi(t_n-\theta_n)\ dt_n \ldots dt_1$$

where $\phi(.)$ is the density function of standard normal distribution,

However, computing the above probability is difficult and even computing π_{ij} is not easy because, unlike the Harville model, no closed form solution exists. Henery suggested to use the following approximation :

$$P\,[T_1 < T_2 < \ldots < T_n] = \Phi\,\{\,\Phi^{-1}\,P\,[T_1 < T_2 < \ldots < T_n]\,\}$$

$$\simeq \Phi\,[\,\xi + \frac{1}{n!\phi(\xi)} \sum_i \theta_i\, \mu_{i:n}] \qquad (1)$$

where $\xi = \Phi^{-1}(1/n!)$, $\mu_{i;n}$ is the expected value of the ith standard normal order statistic in a sample of size n.

(1) is obtained by using Taylor's expansion about $\underset{\sim}{\theta}=\underset{\sim}{0}$ for the term inside the large bracket.

Using similar methods,

$$P\,[T_i \text{ is smallest}] \simeq \Phi\,[\,z_0 + \frac{\theta_i\, \mu_{1:n}}{(n-1)\phi(z_0)}\,] \qquad (2)$$

where $z_0 = \Phi^{-1}(1/n)$.

Henery (1981) also suggested another approximation method but that produces many negative probabilities in our experience, thus we only consider the kind of approximations mentioned above in this paper.

Hence, by using (2), we can have estimates of $\underset{\sim}{\theta}$ if $\underset{\sim}{\pi}$ is known or the win bet fractions are good estimates of $\underset{\sim}{\pi}$. Then, we may substitute the estimated values of $\underset{\sim}{\theta}$ in appropriate equations to obtain estimates of ordering probabilities.

For example,

$$\pi_{ij} = P\,(T_i < T_j < \text{others})$$

$$\simeq \Phi\,\{\,a + \gamma\,[\,\theta_i\mu_{1;n} + \theta_j\mu_{2;n} + \frac{(\theta_i + \theta_j)(\mu_{1;n} + \mu_{2;n})}{n-2}\,]\}$$

where $a = \Phi^{-1}[\frac{1}{n(n-1)}]$ and $\gamma = \frac{1}{n(n-1)\phi(a)}$ here.

In practice, to satisfy the unit-sum constraint, simple scaling is usually necessary.

As exponential and normal distributions may be considered as special cases of the gamma distribution (with 2 extreme values of shape parameters), the Harville and Henery models can be considered as special cases of the model proposed by Stern (1990) who suggests gamma running times with a fixed shape parameter.

3. Conditional logistic analysis for Harville & Henery models

Bacon-Shone, Lo & Busche (1992,a) based on a complicated model-fitting process, suggested using the simple constant-β model in order to analyse win bet data, i.e.

$$\pi_i = \frac{P_i^{\beta}}{\sum_r P_r^{\beta}}$$

where π_i = P(horse i wins),

P_i = Win bet fraction of horse i,

i.e. the proportion of win bet on horse i,

β is a parameter to be estimated by maximum likelihood assuming the win event follows a multinomial distribution.

The above model can be rewritten as follows :

$$\ln(\pi_i / \pi_k) = \beta \ln(P_i / P_k) \qquad \text{for any } i,k \ (i \neq k)$$

which means the multivariate logit of the win probability depends on the logit of the bet fractions in a very simple way. Using a similar structure for conditional probabilities, we have :

$$\ln(\pi_{j|i} / \pi_{k|i}) = \mu \ln(P_{j|i} / P_{k|i}) \quad \text{for any } i,j,k \ (i \neq j \neq k)$$

where $\pi_{j|i}$ = P (horse j finishes second | horse i wins)

$$= \frac{\text{P (horses i \& j finish first \& second resp.)}}{\text{P (horse i wins)}}$$

$$= \pi_{ij} / \pi_i$$

Similarly, we define :

π_{ijk} = P(horse i finishes 1st, j finishes 2nd and k finishes 3rd)

and other notations such as $\pi_{k|ij}$, $\pi_{m|ijk}$, etc. are self-explanatory.

To study how good the Henery model is when compared to Harville model, we can fit the following series of models for conditional probabilities :

$$\begin{aligned} \text{logit } \pi_{j|i} &= \mu \text{ logit } P_{j|i} \\ \text{logit } \pi_{k|ij} &= \omega \text{ logit } P_{k|ij} \\ \text{logit } \pi_{l|ijk} &= \zeta \text{ logit } P_{l|ijk} \\ &\vdots \qquad \vdots \end{aligned} \tag{3}$$

where all P's on the right hand side of (3) are the conditional probabilities estimated by the Harville model and all the logits are multivariate. Replacing all these P's by the conditional probabilities estimated by the Henery model, we are able to observe the more general patterns of bias for the two models.

To simplify our analysis, the approximation method proposed by Henery (1981) is employed in this section since exact computations involve lots of higher dimensional integrations. For each race,

$$P(T_{11}<\ldots<T_{1q} \mid T_{11}<\ldots<T_{1,q-1})$$

$$\simeq \frac{\Phi\{ C_q + \upsilon_q [\sum_{r=1}^{q} \theta_{1r} \mu_{r;n} + \sum_{r=1}^{q} \theta_{1r} \sum_{s=1} \mu_{s;n} / (n-q)] \}}{\Phi\{ C_{q-1} + \upsilon_{q-1} [\sum_{r=1}^{q-1} \theta_{1r} \mu_{r;n} + \sum_{r=1}^{q-1} \theta_{1} \sum_{s=1}^{q-1} \mu_{s;n} / (n-q+1)] \}} \tag{4}$$

for q = 2,3,...,n-1

where $C_q = \Phi^{-1}(1/{}_nP_q)$, $\upsilon_q = \dfrac{1}{\phi(C_q)\,{}_nP_q}$,

${}_nP_q = n(n-1)\ldots(n-q+1)$,

$\mu_{i;n}$ = ith expected standard normal order statistic, and

n = total number of horses in the race.

Scaling is required to adjust the formula in (4) so that all conditional probabilities sum to one.

We have chosen 600 8-horse-races from a Hong Kong (1981-89) data set for this analysis. The data were collected from The Royal Hong Kong Jockey Club (1981-90). In our case, n=8 and q=2,3,...,7. The results are shown in Table 1 and Table 2.

Table 1

Conditional analysis for Harville model

q	2	3	4	5	6	7
parameter estimates	0.8551	0.6675	0.5271	0.4480	0.3616	0.2369
$l(1)$	-1055.22	-1015.53	-949.34	-827.77	-689.49	-463.94
l(para. est.)	-1052.60	-1000.77	-917.85	-791.94	-639.83	-409.33

Table 2

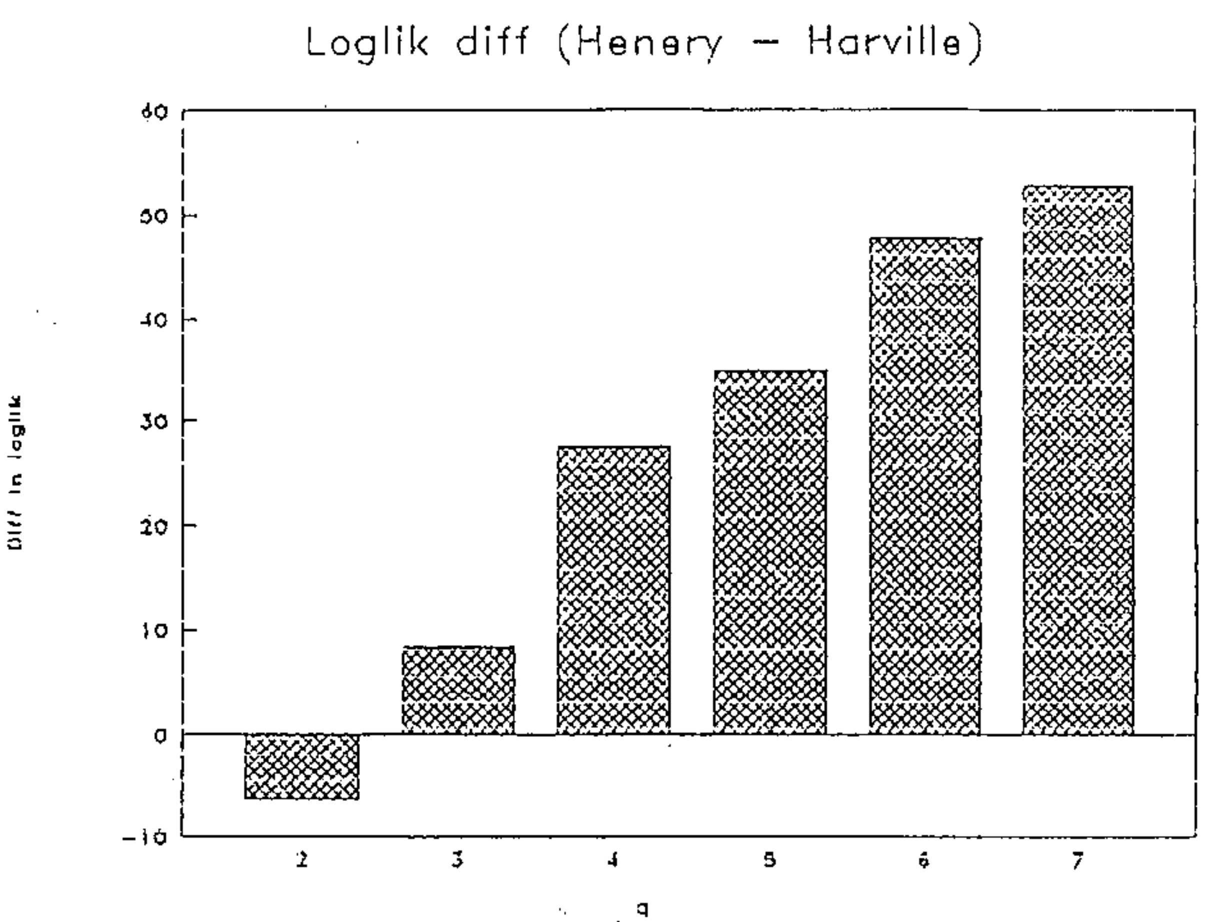
Conditional analysis for Henery model

q	2	3	4	5	6	7
parameter estimates	1.1358	1.1002	1.0372	1.1747	0.9681	0.7134
$l(1)$	-1061.47	-1007.18	-921.83	-793.04	-641.77	-411.29
l(para. est.)	-1060.11	-1006.65	-921.78	-792.21	-641.75	-410.45

(N.B. : $l(1)$ means log likelihood when the appropriate parameter equals one.)

It is easier to observe the pattern of systematic differences between the two models in fig. 1-3. From these tables and figures, it is clear that the Harville model shows a systematic bias since the estimated parameters are decreasing smoothly when q increases. That means that the bias produced by the Harville model is more serious if the conditional probability of a horse finishing in lower order is to be estimated.

Figure 1



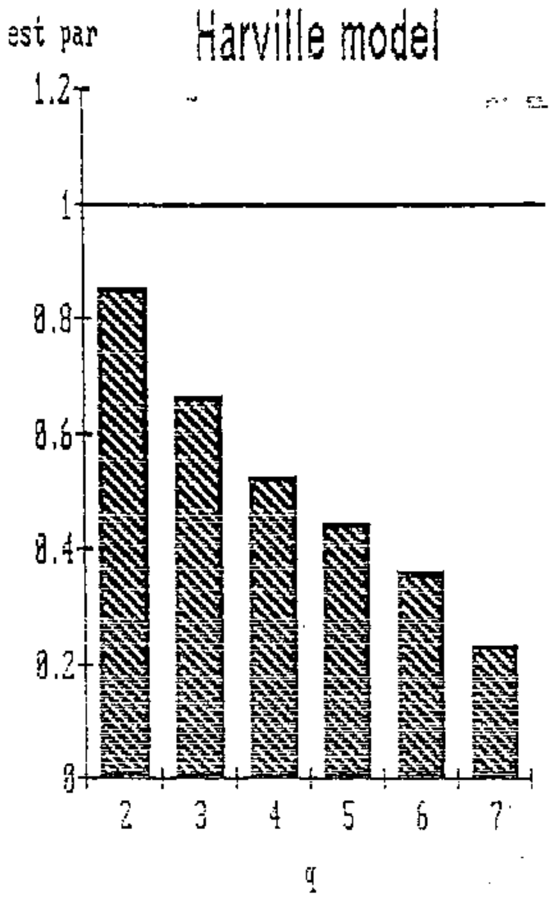


Figure 2

Estimated parameters under the Harville model

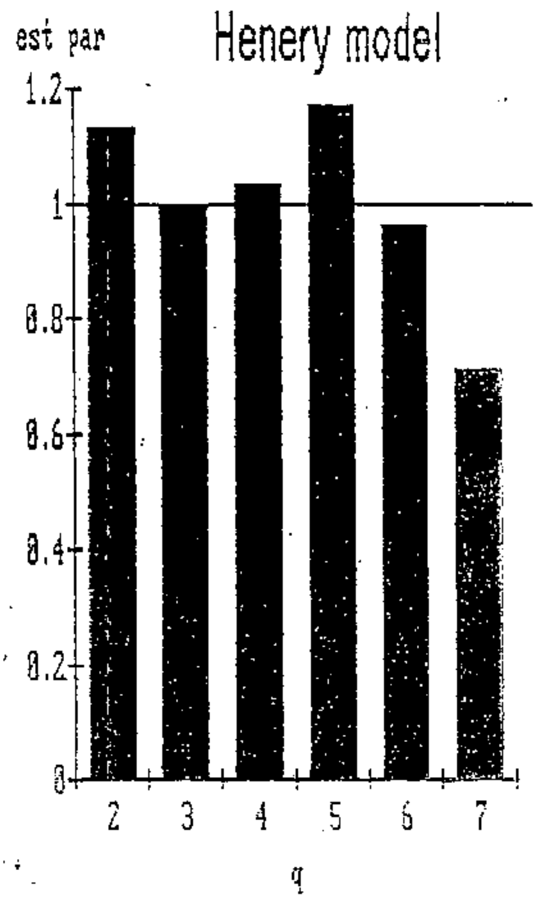


Figure 3

Estimated parameters under the Henery model

On the other hand, the estimated parameters for the Henery model are quite close to one for different q. Moreover, the log likelihoods of the two models also show that the Henery model is generally much better especially when q is large, but not better than the Harville with estimated parameters ! (Although the method of comparing log likelihood values directly is naive given the non-nested models, detailed Cox tests (Cox(1962)) reported in Bacon-Shone, Lo & Busche (1992,b) for simpler comparisons support these conclusions.)

4. Theoretical investigation of the Harville and Henery models

In this section, the difference in estimating the conditional probability of horse j finishing second given that horse i finishes first (denoted by $\pi_{j|i}$) by Harville and Henery models will be theoretically investigated under the assumption that the Henery model is correct. That is, we will study the following difference :

$$\pi_{j|i} - \frac{\pi_j}{1 - \pi_i} \qquad (5)$$

Let θ_i = Expected running times of horse i = $E(T_i)$.
Without loss of generality, we may assume $\theta_1 < \theta_2 < \dots < \theta_n$.
(Note that $n \geq 3$. Otherwise, there is no need to discuss $\pi_{j|i}$.)
To study (5), we need the following lemmas.

Lemma 1

Let u,v and w are non-negative functions. Moreover, u is non-decreasing and v/w is non-increasing, then,

$$\frac{\int uv}{\int uw} \leq \frac{\int v}{\int w}$$

Or, if u is non-decreasing and v/w is non-decreasing, then

$$\frac{\int uv}{\int uw} \geq \frac{\int v}{\int w} \qquad (6)$$

For the proof, see Gutmann and Maymin (1987).

Lemma 2

Define the following function :

$$J(v;\ \theta_j) = \frac{\phi(v+\theta_j-\theta_i)\prod_{r\neq ij}\Phi(v-\theta_i+\theta_r)}{\sum_{s\neq ij}\phi(v+\theta_s-\theta_i)\prod_{t\neq is}\Phi(v-\theta_i+\theta_t)} \tag{7}$$

then, $J(v;\ \theta_a)$ is non-decreasing in v, and

$J(v;\ \theta_b)$ is non-increasing in v,

where $a = \underset{r\neq i}{\mathrm{Min}}\ r$, i.e. $\theta_a = \underset{r\neq i}{\mathrm{Min}}\ \theta_r$ and

$b = \underset{r\neq i}{\mathrm{Max}}\ r$, i.e. $\theta_b = \underset{r\neq i}{\mathrm{Max}}\ \theta_r$

The proof of Lemma 2 is given in Appendix A.

Theorem 1

$$\pi_{a|i} - \frac{\pi_a}{1-\pi_i} \le 0 \quad \text{and} \quad \pi_{b|i} - \frac{\pi_b}{1-\pi_i} \ge 0$$

where

$a = \underset{r\neq i}{\mathrm{Min}}\ r$, i.e. $\theta_a = \underset{r\neq i}{\mathrm{Min}}\ \theta_r$ and

$b = \underset{r\neq i}{\mathrm{Max}}\ r$, i.e. $\theta_b = \underset{r\neq i}{\mathrm{Max}}\ \theta_r$

Proof:

Consider the difference (7),

$$\pi_{j|i} - \frac{\pi_j}{1-\pi_i} = \frac{1}{\pi_i}\left[\ \pi_{ij} - \frac{\pi_i\ \pi_j}{1-\pi_i}\ \right]$$

$$\pi_{ij} = \int_{-\infty}^{\infty}[1 - \Phi(u-\theta_j+\theta_i)]\prod_{s\neq ij}\Phi(u-\theta_j+\theta_s)\ \phi(u)\ du$$

$$= \int_{-\infty}^{\infty}\prod_{s\neq ij}\Phi(u-\theta_j+\theta_s)\ \phi(u)\ du - \int_{-\infty}^{\infty}\prod_{r\neq j}\Phi(u-\theta_j+\theta_r)\ \phi(u)\ du$$

$$= \pi_{j(i)} - \pi_j$$

where $\pi_{j(i)} = P(T_j < \underset{s\neq ij}{\mathrm{Min}}\ \{T_s\})$

i.e. the probability of horse j wins if horse i is removed from the race.

Therefore,

$$\pi_{j|i} - \frac{\pi_j}{1-\pi_i} = \frac{1}{\pi_i}\left(\pi_{j(i)} - \frac{\pi_j}{1-\pi_i}\right)$$

Define $$g_{j|i} = \pi_{j(i)} - \frac{\pi_j}{1-\pi_i} \tag{8}$$

Thus, it suffices to show that $g_{a|i} \le 0$ and $g_{b|i} \ge 0$.

Rewrite :

$$g_{j|i} = \frac{\pi_{j(i)}\left(\sum_{s\neq ij}\pi_s + \pi_j\right) - \pi_j}{1-\pi_i}$$

$$= \frac{\sum_{s\neq ij}\pi_s \sum_{s\neq ij}\pi_{s(i)}}{1-\pi_i}\left[\frac{\pi_{j(i)}}{\sum_{s\neq ij}\pi_{s(i)}} - \frac{\pi_j}{\sum_{s\neq ij}\pi_s}\right]$$

Now, we consider :

$$\frac{\pi_{j(i)}}{\sum_{s\neq ij}\pi_{s(i)}} - \frac{\pi_j}{\sum_{s\neq ij}\pi_s}$$

$$= \frac{\int_{-\infty}^{\infty}\prod_{r\neq ij}\Phi(u-\theta_j+\theta_r)\,\phi(u)\,du}{\sum_{s\neq ij}\int_{-\infty}^{\infty}\prod_{t\neq sj}\Phi(u-\theta_s+\theta_t)\,\phi(u)\,du}$$

$$- \frac{\int_{-\infty}^{\infty}\prod_{r\neq j}\Phi(u-\theta_j+\theta_r)\,\phi(u)\,du}{\sum_{\neq ij}\int_{-\infty}^{\infty}\prod_{t\neq s}\Phi(u-\theta_s+\theta_t)\,\phi(u)\,du}$$

$$= \frac{\int_{-\infty}^{\infty} \prod_{r\neq i\,j} \Phi(v-\theta_i+\theta_r)\ \phi(v-\theta_i+\theta_j)\ dv}{\int_{-\infty}^{\infty} \sum_{s\neq i\,j} \prod_{t\neq s\,i} \Phi(v-\theta_i+\theta_t)\ \phi(v-\theta_i+\theta_s)\ dv}$$

$$- \frac{\int_{-\infty}^{\infty} \Phi(v) \prod_{r\neq i\,j} \Phi(v-\theta_i+\theta_r)\ \phi(v-\theta_i+\theta_j)\ dv}{\int_{-\infty}^{\infty} \Phi(v) \sum_{s\neq i\,j} \prod_{t\neq s\,i} \Phi(v-\theta_i+\theta_t)\ \phi(v-\theta_i+\theta_s)\ dv}$$

by change of variables using :

$v = u-\theta_j+\theta_i$ in the numerator, and

$v = u-\theta_s+\theta_i$ in the denominator.

$$\left.\begin{array}{l} \le 0 \quad \text{when } j=a \\ \\ \ge 0 \quad \text{when } j=b \end{array}\right\} \text{ by using Lemma 2 together with Lemma 1}$$

Hence, $g_{a|i} \le 0$ and $g_{b|i} \ge 0$ and the required result follows.

When n>3, we have only shown that the above result is valid for extreme values of j. But for a<j<b, the difference may be greater than or smaller than zero depending on the particular set of $(\theta_1,\theta_2,\ldots,\theta_n)$.

The above theorem means that if the running times satisfy the assumption of the Henery model, the Harville model will overestimate the conditional probability of the most favourite horse finishing second and underestimate the conditional probability of the longshot finishing second, providing a possible explanation of the "Silky Sullivan" effect.

5. Conclusion

The results obtained in this paper support the conclusion that the Harville model has a systematic bias in estimating ordering probabilities based on our data set. On the other hand, according to our data analysis, the Henery model does not cause any systematic bias and thus it should be more reliable. Our theoretical result in section 4 analytically supports the systematic bias caused by the Harville model when the Henery model holds.

APPENDIX A : Proof of Lemma 2

To prove lemma 2, we have to prove the following statement first.

Define :

$h(x) = \frac{\phi(x)}{\Phi(x)}$, where $\phi(.)$ and $\Phi(.)$ are standard normal pdf and cdf, respectively.

Then, it can be shown that $h'(x) > -1$ for $x \in \mathbb{R}$.

Proof :

$$h'(x) = \frac{-x\Phi(x)\phi(x)-\phi(x)^2}{\Phi(x)^2} = -x\frac{\phi(x)}{\Phi(x)} - [\frac{\phi(x)}{\Phi(x)}]^2$$

When $x \geq 0$,

consider the function $\frac{\Phi(x)}{\phi(x)}[\frac{\Phi(x)}{\phi(x)} - x]$,

the minimum of this function occurs when $x=0$ since it is easy to show that both $\Phi(x)/\phi(x)$ and $\Phi(x)/\phi(x)-x$ are increasing functions for $x \geq 0$. Thus,

$$\frac{\Phi(x)}{\phi(x)}[\frac{\Phi(x)}{\phi(x)} - x] \geq [\frac{\Phi(0)}{\phi(0)}]^2 = 1.571 > 1$$

=>

$$[\frac{\Phi(x)}{\phi(x)}]^2 - x\frac{\Phi(x)}{\phi(x)} - 1 > 0 \qquad \text{(A.1)}$$

Multiply (A.1) by $[\phi(x)/\Phi(x)]^2$, we have :

$h'(x) > -1$ for $x \geq 0$.

Now, for $x<0$,

let $x = -y$ and thus $y>0$.

Then,

$$[\frac{\Phi(x)}{\phi(x)}]^2 - x\frac{\Phi(x)}{\phi(x)} - 1$$

$$= [\frac{1-\Phi(y)}{\phi(y)}]^2 + y\frac{1-\Phi(y)}{\phi(y)} - 1$$

$$= [\frac{\Phi(y)}{\phi(y)}]^2 - y\frac{\Phi(y)}{\phi(y)} - 1 + \frac{1-2\Phi(y)+y\phi(y)}{\phi(y)^2}$$

It is easy to show that :

$a(y) = 1 - 2\Phi(y) + y\phi(y) > 0$

since $a'(y) = -(1+y^2)\phi(y) < 0 \Rightarrow a(y)$ is decreasing in y $(y>0)$

$\Rightarrow a(y) > a(0) = 1/2 > 0$

Hence, by (A.1), it is also true that :

$$[\frac{\Phi(x)}{\phi(x)}]^2 - x\frac{\Phi(x)}{\phi(x)} - 1 > 0 \quad \text{for } x<0 \tag{A.2}$$

Finally, from (A.1) and (A.2), we have :

$$h'(x) > -1 \quad \text{for } x \in \mathbb{R}.$$

Now, we are going to prove lemma 2.

Consider the derivative of $J(v;\theta_j)$ in (7) with respect to v :

$$\frac{d\, J(v;\,\theta_j)}{d\, v} \{ \sum_{s\neq lj}\phi(v+\theta_s-\theta_l)\prod_{t\neq ls}\Phi(v-\theta_l+\theta_t) \}^2$$

$$= \phi(v+\theta_j-\theta_l)\prod_{r\neq lj}\Phi(v-\theta_l+\theta_r)$$

$$\left\{ \left[\sum_{s\neq lj}\phi(v+\theta_s-\theta_l)\prod_{t\neq ls}\Phi(v-\theta_l+\theta_t) \right] \left[\sum_{r\neq lj}\frac{\phi(v-\theta_l+\theta_r)}{\Phi(v-\theta_l+\theta_r)} - (v+\theta_j-\theta_l) \right] \right.$$

$$\left. - \sum_{s\neq lj}\left[\phi(v+\theta_s-\theta_l)\prod_{t\neq ls}\Phi(v-\theta_l+\theta_t) \left[\sum_{t\neq sl}\frac{\phi(v-\theta_l+\theta_t)}{\Phi(v-\theta_l+\theta_t)} - (v+\theta_s-\theta_l) \right]\right] \right\}$$

$$= \phi(v+\theta_j-\theta_l)\prod_{r\neq lj}\Phi(v-\theta_l+\theta_r)$$

$$\left\{ \sum_{s\neq lj}\phi(v+\theta_s-\theta_l)\prod_{t\neq ls}\Phi(v-\theta_l+\theta_t) \left[\sum_{r\neq lj}\frac{\phi(v-\theta_l+\theta_r)}{\Phi(v-\theta_l+\theta_r)} - \sum_{t\neq sl}\frac{\phi(v-\theta_l+\theta_t)}{\Phi(v-\theta_l+\theta_t)} \right.\right.$$

$$\left.\left. - (\theta_j - \theta_s) \right] \right\}$$

$$= \phi(v+\theta_j-\theta_i)\prod_{r\neq ij}\Phi(v-\theta_i+\theta_r)$$

$$\left\{ \sum_{s\neq ij} C_s \left[\frac{\phi(v-\theta_i+\theta_s)}{\Phi(v-\theta_i+\theta_s)} - \frac{\phi(v-\theta_i+\theta_j)}{\Phi(v-\theta_i+\theta_j)} - (\theta_j - \theta_s) \right] \right\}$$

where

$$C_s = \phi(v+\theta_s-\theta_i)\prod_{t\neq is}\Phi(v-\theta_i+\theta_t)$$

Recall that $h(x) = \phi(x)/\Phi(x)$

then,

if $j=a$,

$$\frac{d\ J(v;\ \theta_a)}{d\ v} \left\{ \sum_{s\neq ia}\phi(v+\theta_s-\theta_i)\prod_{t\neq is}\Phi(v-\theta_i+\theta_t) \right\}^2$$

$$= \phi(v+\theta_a-\theta_i)\prod_{r\neq ia}\Phi(v-\theta_i+\theta_r)$$

$$\left\{ \sum_{s\neq ia} C_s \left[[h'(v_0) + 1]\ (\theta_s - \theta_a) \right] \right\}$$

by the mean value theorem,

where $v_0 \in (v-\theta_i+\theta_a,\ v-\theta_i+\theta_s)$

≥ 0 by the above result.

On the other hand, if $j=b$,

$$\frac{d\ J(v;\ \theta_b)}{d\ v} \left\{ \sum_{s\neq ib}\phi(v+\theta_s-\theta_i)\prod_{t\neq is}\Phi(v-\theta_i+\theta_t) \right\}^2$$

$$= \phi(v+\theta_b-\theta_i)\prod_{r\neq ib}\Phi(v-\theta_i+\theta_r)$$

$$\left\{ \sum_{s\neq ib} C_s \left[[-h'(v_0) - 1]\ (\theta_b - \theta_s) \right] \right\}$$

by the mean value theorem,

where $v_0 \in (v-\theta_i+\theta_s,\ v-\theta_i+\theta_b)$

≤ 0

We have shown that

$$\frac{d\ J(v;\ \theta_a)}{d\ v} \geq 0 \quad \text{and} \quad \frac{d\ J(v;\ \theta_b)}{d\ v} \leq 0$$

hence the result follows.

REFERENCES

Ali, M.M. (1977) Probability and utility estimates for racetrack bettors, *Journal of Political Economy*, 84 (August) , pp. 803-15.

Asch,Peter;Malkiel,B. and Quandt,R. (1982) Racetrack betting and informed behavior, *Journal of Financial Economics*, 10 (June), pp. 187-94.

Bacon-Shone,J.; Lo,V.S.Y. and Busche,K. (1992,a) "Modelling the winning probability." Research report 10, Department of Statistics, University of Hong Kong.

Bacon-Shone,J.; Lo,V.S.Y. and Busche,K. (1992,b) Logistic analyses for complicated bets. Research report 11, Department of Statistics, University of Hong Kong.

Busche,K. and Hall,C.D. (1988) "An exception to the risk preference anomaly." *Journal of Business*, 61 ,337-46.

Cox. D.R. (1962) Further results on tests of separate families of hypotheses. *Journal of Royal Statistical Society B*, 24, No.2, pp. 406-424.

Fabricand,B.P. (1979) *The science of winning, A random walk on the road to riches* (Van Nostrand Reinhold).

Figlewski,S. (1979) Subjective information and market efficiency in a betting model, *Journal of Political Economy*, 87 (February) , pp. 75-88.

Griffith,R.M. (1949) Odds adjustments by American horse-racing bettors, *American Journal of Psychology*, 62 , pp. 290-294.

Gutmann,S. & Maymin,Z. (1987) Is the selected population the best ?, *The Annals of Statistics*, 15, No.1 , pp. 456-461.

Harville,D.A. (1973) Assigning probabilities to the outcomes of multi-entry competitions, *Journal of the American Statistical Association*, 68 (June) , pp. 312-16.

Hausch,D.B.; Ziemba,W.T. and Rubinstein,M. (1981) Efficiency of the market for racetrack betting, *Management Science*, 27 (December) , pp. 1435-52.

Henery,R.J. (1981) Permutation probabilities as models for horse races, *Journal of Royal Statistical Society B*, 43, No.1 , pp. 86-91.

Hoerl,A.E. and Fallin,H.K. (1974) Reliability of subjective evaluations in a high incentive situation, *Journal of Royal Statistical Society, Series A*, 137 , pp. 227-230.

Luce, R. Duncan and Suppes, P. (1965) Preference, utility and subjective probability in Luce,R.D., Bush,R.R. and Galanter,E., ed., *Handbook of Mathematical Psychology* Vol. III, Chapter 19 , pp. 249-410.

McCulloch, B. and Van Zijl, T. (1986) Direct test of Harville's multi-entry competitions model on race-track betting data, *Journal of Applied Statistics*, Vol. 13, No.2, pp. 213-220.

Metzger, Mary Ann (1985) Biases in betting,An application of laboratory findings, *Psychological Reports*, 56 , pp. 883-888.

Plackett,R.L. (1975) The analysis of permutations. *Applied Statistics*, 24 , pp. 193-202.

Snyder, Wayne W. (1978) Horse racing ,Testing the efficient markets model, *Journal of Finance*, 33 (September) , pp. 1109-1118.

Stern, Hal (1990) Models for distributions on permutations, *Journal of the American Statistical Association*, 85, No. 410 (June), pp. 558-564.

The Royal Hong Kong Jockey Club (1981-90) *Hong Kong racing record for the racing season 1980/81*; ...... ; *Hong Kong racing record for the racing season 1989/90.*