Delocalization of Relativistic Dirac Particles in Disordered One-Dimensional Systems and Its Implementation with Cold Atoms

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We study theoretically the localization of relativistic particles in disordered one-dimensional chains. It is found that the relativistic particles tend to delocalization in comparison with the nonrelativistic particles with the same disorder strength. More intriguingly, we reveal that the massless Dirac particles are entirely delocalized for any energy due to the inherent chiral symmetry, leading to a well-known result that particles are always localized in one-dimensional systems for arbitrary weak disorders to break down. Furthermore, we propose a feasible scheme to detect the delocalization feature of the Dirac particles with cold atoms in a light-induced gauge field.

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Since the pioneering work of Anderson [1], a substantial amount of effort has been devoted to the understanding of transport properties of electrons in disordered systems [2]. A very significant advance along this direction is the scaling theory proposed by Thouless *et al.* [3] and Abrahams *et al.* [4]. In the scaling theory, it is argued that the quantity $\beta \equiv d\ln g/d\ln L$ is a monotonic and nonsingular function of g only, with g a dimensionless conductance and L the sample size. Currently there exists a well-known result that the conductance g approaches zero as the sample size L goes to infinity for any disordered one-dimensional system. In particular, an arbitrary weak disorder strength leads to the localization of all states of electrons in one-dimensional chains [4–6].

Actually, an implicit precondition for the above results is that the particles are governed by the Schrödinger equation since the (quasi)particles addressed in condensed matter systems are in general nonrelativistic. Notably, relativistic Dirac particles have recently attracted a significant amount of attention because the quasiparticles in honeycomb lattices (such as electrons in the graphene [7,8] and cold atoms in the optical lattices [9,10]), ultracold atoms in a light-induced gauge field [11–15], and trapped ions [16] may be described by the relativistic Dirac equation. However, Anderson localization in the relativistic region has been less studied in literature [17]. Apart from Klein's finding [18] that the transmission of Dirac particles is essentially different from that of nonrelativistic particles, it is also fundamentally important and interesting to study the aforementioned localization issue of relativistic particles and to work out how to simulate the predicted results with currently available techniques.

In this Letter we study Anderson localization of relativistic particles in disordered one-dimension systems by using the finite scaling method and the transfer-matrix technique. The finite-size scaling analysis reveals that all the states of the massive relativistic Dirac particles are localized in the systems, while the localization length of relativistic particles is longer than that of nonrelativistic particles with the same disorder strength. More intriguingly, the states of the massless Dirac particles are entirely delocalized for arbitrary disorder strength due to the chiral symmetry, providing a distinct example that breaks down the well-known result for nonrelativistic particles that all states are localized in disordered one-dimensional systems. Furthermore, we propose a method to simulate the wanted Dirac particles in disordered one-dimensional chains and the related (de)localization properties with recently or newly developed techniques in the cold atomic systems [19–24].

Let us consider a particle transmitting through a one-dimensional chain with N rectangular barriers as shown in Fig. 1(a), where the potential

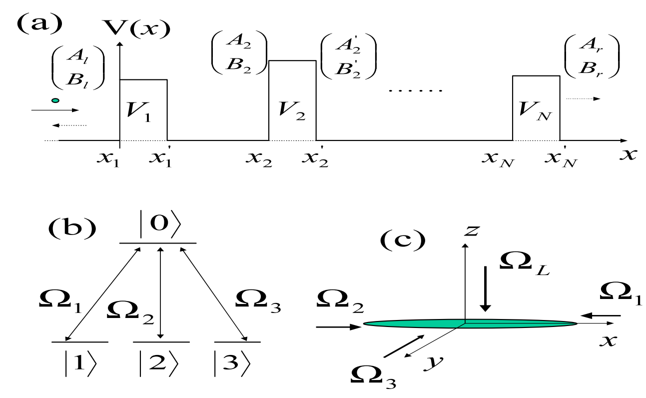

FIG. 1 (color online). Schematic representation of the system. (a) N rectangular potentials. (b) Atom with tripod-level structure interacting with three laser beams. (c) The configuration of the laser beams to realize a Dirac-like equation with the lasers Ω_j and a disordered potential with the laser Ω_L. The atoms are confined in a one-dimensional waveguide by a harmonic trap.

$$V(x) = \begin{cases} V_n & x_n \le x \le x'_n \\ 0 & \text{others,} \end{cases} \qquad (n = 1, 2, \ldots, N), \quad (1)$$

with V_n being a constant randomly distributed in the range $[-\delta, \delta]$. Here δ represents the disorder strength. For simplicity, we assume that $x'_n - x_n \equiv a$ and $x_{n+1} - x'_n \equiv d$ for any n. A relativistic particle with the mass m and energy E is inserted into the N barriers from the left, which is described by the Dirac equation

$$\left[-i\hbar c\sigma_x \frac{d}{dx} + mc^2\sigma_z + V(x) - E\right]\psi(x) = 0, \quad (2)$$

where c denotes the velocity of light, $\sigma_{x,z}$ are the Pauli matrices, and $\psi(x)$ represents a two-component spinor. A general solution of Eq. (2) for any region with a constant potential, e.g., V_n in Fig. 1(a), is given by

$$\psi(x) = A\begin{pmatrix} 1 \\ \kappa_n \end{pmatrix} e^{(i/\hbar)p_n x} + B\begin{pmatrix} 1 \\ -\kappa_n \end{pmatrix} e^{-(i/\hbar)p_n x}, \quad (3)$$

where p_n represents the momentum of the particle, $\kappa_n = (E - mc^2 - V_n)/(cp_n)$, and $(E - V_n)^2 = m^2c^4 + c^2p_n^2$. If E and V_n are fixed, then p_n can in principle be either positive or negative. Here we choose the positive one, and thus the coefficients A and B denote the amplitudes of the spinors moving along the positive x axis and its opposite direction, respectively.

We now look into the transmission for N potentials as shown in Fig. 1(a). Denoting the amplitudes of the spinor at a position approaching x_n (x'_n) with an infinitesimal amount from the left (right) as $\{A_n, B_n\}$ ($\{A'_n, B'_n\}$), we may obtain a relation between the amplitudes based on the continuity of the wave function,

$$\begin{pmatrix} A'_n \\ B'_n \end{pmatrix} = M_n^D \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \quad (4)$$

where M_n^D denotes the transfer matrix of the nth barrier, and its elements are given by

$$(M_n^D)_{11} = \left(\cos\frac{p_n a}{\hbar} + i\frac{\kappa^2 + \kappa_n^2}{2\kappa\kappa_n}\sin\frac{p_n a}{\hbar}\right)e^{-(i/\hbar)pa},$$
$$(M_n^D)_{12} = \left(i\frac{\kappa_n^2 - \kappa^2}{2\kappa\kappa_n}\sin\frac{p_n a}{\hbar}\right)e^{-(i/\hbar)p(x_n + x'_n)}, \quad (5)$$
$$(M_n^D)_{21} = (M_n^D)_{12}^*, \qquad (M_n^D)_{22} = (M_n^D)_{11}^*,$$

with $\kappa = (E - mc^2)/(cp)$ and $E^2 = c^2p^2 + m^2c^4$.

For comparison, we recall the results for the nonrelativistic case which is described by the Schrödinger equation $[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E_k]\Phi(x) = 0$, where E_k is nonrelativistic kinetic energy. In this case, we have a relation similar to Eq. (4), but M_n^D is replaced by M_n^S which is the transfer matrix for the nth barrier calculated by the Schrödinger equation. It is straightforward to derive that the elements of the matrix M_n^S have the same form as Eq. (5), but κ and p_n (and κ_n) are replaced by the nonrelativistic counterparts $p^S = \sqrt{2mE_k}$ and $p_n^S = \sqrt{2m(E_k - V_n)}$, respectively.

We consider an incoming relativistic particle with energy E (E_k for nonrelativistic kinetic energy) from the left, then the amplitude of the outgoing at the right side of the N barriers is related to the amplitude of the incoming by the relation

$$\begin{pmatrix} A_r \\ B_r \end{pmatrix} = M^J \begin{pmatrix} A_l \\ B_l \end{pmatrix},$$

where the total transfer matrix M^J (hereafter we use the superscripts $J = D$ and S to denote the relativistic and nonrelativistic cases, respectively) for the N barriers reads

$$M^J = M_N^J D^J M_{N-1}^J \cdots D^J M_2^J D^J M_1^J. \quad (6)$$

Here $D^J = \mathrm{diag}\{\exp(-ip^J d/\hbar), \exp(ip^J d/\hbar)\}$ represents the displacement matrix between two nearest neighbor barriers, with $p^D = \sqrt{E^2/c^2 - m^2c^2}$.

The transport properties can be extracted from the transfer matrices for both relativistic and nonrelativistic cases. At zero temperature, the conductance through the N barriers is given by the Landauer formula [25] $G^J = \frac{e^2}{h}g^J$, where $g^J = 1/|(M^J)_{11}|^2$ is the dimensionless conductance. It is noted that the localization length ξ^J or the Lyapunov exponent γ^J is defined as $\gamma^J \equiv 1/\xi^J = -\lim_{L\to\infty}\langle \ln g^J\rangle/L$, where L is the total length of the chain $L = Nb = N$ (we choose $b = d + a$ as the unit of length), and $\langle\cdots\rangle$ denotes the averaging over the disorders. The ξ^J is a function of the energy E (E_k) and can be used to characterize a localized state: a state is a localized state if ξ^J is finite and is a delocalized (extended) state if ξ^J is divergent.

It is hard to obtain an analytical expression for the localization length ξ^J in a general case; however, it has been shown that $\lim_{L\to\infty}(\langle \ln g^S\rangle/L)$ always exists for any energy of a nonrelativistic particle in an arbitrary weak one-dimensional disordered system [2]. In a similar way, we can find that $\lim_{L\to\infty}(\langle \ln g^D\rangle/L)$ also exists for relativistic massive particles. The numerical procedure for both relativistic and nonrelativistic cases is as follows: one can define $\alpha_N^J = \frac{1}{N_c}\sum_{i=1}^{N_c}\frac{1}{N}\ln g_i^J(N)$, where $g_i^J(N)$ is the conductance for a specific configuration of fixed N barriers, and then α_N^J is an averaged quantity for the number of N_c configurations. We find that, for both nonrelativistic and relativistic massive particles, a convergent value α_N^J can always be derived for sufficient large N, which could be considered as the localization length ξ^J of the system. The result implies that the state for the massive Dirac particles is also a localized state for arbitrarily weak disorders, as in the nonrelativistic case. The localization length (Lyapunov exponent) as a function of the potential width a is plotted in Fig. 2. It is seen that the localization length of the relativistic particles is always longer than that of the nonrelativistic particles.

We now turn to examine the validity of the single-parameter scaling equation $\beta^J \equiv \frac{d\ln g^J}{d\ln L}$ in the relativistic region. The scaling quantities β^J as a function of $\langle \ln g^J\rangle$ for

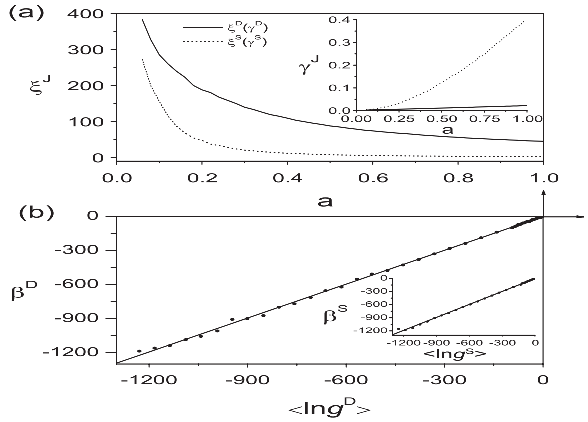

FIG. 2. (a) The localization length ξ^J as a function of the potential width a. The inset is the corresponding Lyapunov exponent γ^J. (b) The scaling quantity β^D (β^S in the inset) as a function of $\langle \ln g^D \rangle$. $\langle \ln g^J \rangle$ is the average of 1000 configurations. The other parameters are $E = 1.05mc^2$ and $\delta \in [-2, 2]$ with the units of mc^2.

$m = 2.5 \times 10^{-4} m_0$ with m_0 the mass of a nuclear in both relativistic and nonrelativistic cases are plotted in Fig. 2(b). The nonrelativistic case was studied in Ref. [6], and our results are essentially the same as those presented there. It is seen clearly from Fig. 2(b) that the assumption of the single-parameter scaling theory is still valid for the massive Dirac particles in the disordered systems.

It is interesting to note that an analytical result for a massless particle ($m = 0$) can be derived, from which one is able to find unexpectedly that the particle is entirely delocalized for arbitrary energy. For massless particles, the transfer matrix of total N barriers is obtained as $M = \mathrm{diag}\{e^{i\varphi/\hbar}, e^{-i\varphi/\hbar}\}$ with $\varphi = -Npb + \sum_{n=1}^{N} p_n a$. The transmission amplitude $t = \exp(i\varphi/\hbar)$ is a pure phase factor, and the dimensionless conductance $g^D \equiv 1$. In this case, the localization length ξ^D for the massless particles approaches infinity, and thus breaks down the famous conclusion that the particles are always localized for any weak disorder in one-dimensional systems.

The inherent physics is simply the chiral symmetry. The time-independent Hamiltonian for the Dirac particles is given by $H_D = -i\hbar c \sigma_x \frac{d}{dx} + mc^2 \sigma_z + V(x)$, and the chiral operator for a Dirac spinor is the matrix $\gamma^5 = \sigma_x$ in one-dimensional cases. Under the discrete chiral transformation the spinor is transformed as $\psi_c = \gamma^5 \psi$ and the transformed Hamiltonian

$$H_c = \gamma^5 H_D \gamma^5 = -i\hbar c \sigma_x \frac{d}{dx} - mc^2 \sigma_z + V(x). \qquad (7)$$

Then the chirality is conserved for a massless particle. Noting that $\gamma^5 \phi_\pm = \pm \phi_\pm$ with

$$\phi_\pm = \begin{pmatrix} 1 \\ \pm\kappa \end{pmatrix},$$

the general solution of the massless Dirac equation described in Eq. (3) can be rewritten as $\psi(x) = A\phi_+ e^{(i/\hbar)px} + B\phi_- e^{-(i/\hbar)px}$; i.e., the first (second) term is actually the eigenstate of the chiral operator with positive (negative) chirality. Assuming that the incoming wave function is $\psi_{\mathrm{in}}(x) = A\phi_+ e^{(i/\hbar)px}$, the outgoing wave function $\psi_{\mathrm{out}}(x) = A'\phi_+ e^{(i/\hbar)px} + B'\phi_- e^{-(i/\hbar)px}$ has the same positive chirality; i.e., the reflection rate B' must be zero for massless particles because of the conservation of the chirality. However, the chirality is not conserved for a massive particle, so the reflection B' in principle cannot be always zero. In this case the massive particle should be localized for any weak disorders. Alternatively, an intuitive picture of localization of relativistic particles may be understood with the Klein paradox. For massive particles, when the height of the potential barrier reaches the order of mc^2, the barrier becomes nearly transparent. Since some of the barriers are transparent, the localization length increases. For the massless particles mc^2 is zero and thus every barrier is transparent. Therefore, massless particles are always delocalized.

We now turn to address how to simulate the relativistic particles with cold atoms. We consider the adiabatic motion of atoms having a tripod-level configuration in the field of three laser beams, as shown in Figs. 1(b) and 1(c) [12–14]. The ground states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are coupled to an excited state $|0\rangle$ through spatially varying laser fields, with the corresponding Rabi frequencies Ω_1, Ω_2, and Ω_3, respectively. The full quantum state of the atoms $\Phi(\mathbf{r})$ can be described as $\Phi(\mathbf{r}) = \sum_{j=0}^{3} \phi_j(\mathbf{r})|j\rangle$, where $\mathbf{r}$ is the atomic position. The original Hamiltonian of the atom with the mass m_a takes the form $H = \frac{\mathbf{P}^2}{2m_a} + V_H(\mathbf{r}) + V_L(\mathbf{r}) + H_{\mathrm{int}}$, where $V_H(\mathbf{r}) \equiv \sum_{j=0}^{3} V_j^H(\mathbf{r})|j\rangle\langle j|$ represents an external harmonic trapping potential, and $V_L(\mathbf{r})$ denotes a state-independent random potential. H_{int} is the laser-atom interaction Hamiltonian given by $H_{\mathrm{int}} = -\hbar \sum_{j=1}^{3} (\Omega_j |0\rangle\langle j| + \mathrm{H.c.})$, where the Rabi frequencies are chosen as $\Omega_1 = \Omega \sin\theta e^{-ikx}/\sqrt{2}$, $\Omega_2 = \Omega \sin\theta e^{ikx}/\sqrt{2}$, and $\Omega_3 = \Omega \cos\theta e^{-iky}$. Here $\Omega = \sqrt{|\Omega_1|^2 + |\Omega_2|^2 + |\Omega_3|^2}$, k is the laser wave vector, and the angle θ defines the relative intensity [12–14]. Following Ref. [12], one is able to obtain an effective one-dimensional Dirac-type Hamiltonian as

$$H_k \approx c_\star \sigma_x p_x + \gamma_z \sigma_z + V_1^H(x) + V_L(x), \qquad (8)$$

up to an irrelevant constant, provided that the wave vector of the atoms $p_x/\hbar \ll k\cos\theta$ [26]. Here $\gamma_z \equiv \frac{\hbar^2 k^2}{2m_a} \sin^4\theta$, $c_\star = \frac{\hbar k}{m_a} \cos\theta$ is the effective "speed of light." In the derivation, we have assumed that the trapping potential $V^H(\mathbf{r})$ is independent of the internal states. Comparing the original Dirac Eq. (2) with the Dirac-like Eq. (8) achieved in cold atoms, the effective speed of light in cold atoms is $c_\star$ and the effective mass $m = \frac{m_a}{2} \tan^2\theta \sin^2\theta$. Note that the mass m of the simulated Dirac particle is not the mass m_a of the cold atom itself and it is a remarkable feature that the mass m in the simulated Dirac-like equation can be controlled by the laser beams. Thus both the massive and massless Dirac equations can be realized with cold atoms [27].

Finally, we discuss briefly the detection of the localization length. For concreteness, we assume that the tripod-level configuration in Fig. 1(b) is provided by the atoms of ^{87}Rb, where the ground states $\{|1\rangle, |2\rangle, |3\rangle\}$ are the hyperfine states $5^2S_{1/2}(F = 1, m_F = -1, 0, 1)$ and the excited state $|0\rangle$ is given by either the state of $5^2P_{3/2}(F = 0)$ or $5^2P_{3/2}(F = 2, m_F = 0)$. In this case, the harmonic potential $V_H(\mathbf{r})$ has been experimentally realized by the far-off resonant laser beams in the implementation of the spinor condensates of ^{87}Rb [24]. In addition, a feasible approach to detect the localization length of the relativistic particles can follow the case in nonrelativistic particles implemented in Ref. [20], except that three additional laser beams represented by Ω_j are required. The experiment starts with an elongated cluster of ultracold ^{87}Rb atoms trapped by a harmonic potential V_H. A far-off-resonance laser beam (such as wavelength 1.06 μm used in Ref. [20]) creates an optical waveguide along the horizontal x axis, and a loose longitudinal trap is also realized by such laser beam. Three laser beams with the resonant wavelength of rubidium (wavelength 0.78 μm, near the resonant also works) but different polarizations, as shown in Fig. 1(c), are shined on the atoms to create an atomic gas that could be described by the Dirac equation. The longitudinal confinement is switched off at time $t = 0$, and the atomic gas starts to expand in the guide along the x direction. A disordered potential V_L is applied to the expanding atomic gas using an optical speckle field produced by passing a laser beam (with the wavelength about 0.514 μm [20]) through a diffusing plate. One then detects the spatial distribution of the atoms at increasing evolution time using absorption imaging. As in the experiments [19,20], one can directly measure the localization length of the particles by the density profiles. Therefore, the comparison of Anderson localization between relativistic and nonrelativistic cases can be made for the conditions with and without the additional laser beams Ω_j. Considering that the two main ingredients to observe (de)localization of Dirac particles, the disordered potentials [20] and light-induced gauge field [23], have been achieved in the recent experiments on the atoms of ^{87}Rb, it is expected that the cold atoms may offer a novel platform for the study of Anderson localization in the relativistic region.

In summary, we have found that the relativistic particles tend to delocalization and revealed that the massless ones are entirely delocalized in disordered one-dimensional systems. The predicted features may be tested by future experiments with ultracold atoms. On the other hand, the (de)localization of the relativistic particles may also be observed in a disordered graphene, where Dirac electrons are confined to move in one dimension and the impurities are small enough such that the scattering does not occur between the two Dirac points.

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[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] For a review, see P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[3] D. J. Thouless, Phys. Rep. **13**, 93 (1974).
[4] E. Abrahams *et al.*, Phys. Rev. Lett. **42**, 673 (1979).
[5] P. W. Anderson *et al.*, Phys. Rev. B **22**, 3519 (1980).
[6] E. N. Economou and C. M. Soukoulis, Phys. Rev. Lett. **47**, 973 (1981).
[7] G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).
[8] K. S. Novoselov *et al.*, Nature (London) **438**, 197 (2005); Y. Zhang *et al.*, *ibid.* **438**, 201 (2005).
[9] S. L. Zhu, B. Wang, and L. M. Duan, Phys. Rev. Lett. **98**, 260402 (2007); C. Chamon *et al.*, Phys. Rev. B **77**, 235431 (2008); C. Wu and S. Das Sarma, *ibid.* **77**, 235107 (2008).
[10] I. I. Satija *et al.*, Phys. Rev. A **77**, 043410 (2008); J. Ruostekoski, J. Javanainen, and G. V. Dunne, *ibid.* **77**, 013603 (2008); J. Ruostekoski, G. V. Dunne, and J. Javanainen, Phys. Rev. Lett. **88**, 180401 (2002).
[11] J. Ruseckas *et al.*, Phys. Rev. Lett. **95**, 010404 (2005).
[12] G. Juzeliunas *et al.*, Phys. Rev. A **77**, 011802(R) (2008).
[13] T. D. Stanescu, C. Zhang, and V. Galitski, Phys. Rev. Lett. **99**, 110403 (2007).
[14] J. Y. Vaishnav and C. W. Clark, Phys. Rev. Lett. **100**, 153002 (2008).
[15] S. L. Zhu *et al.*, Phys. Rev. Lett. **97**, 240401 (2006); Y. Li, C. Bruder, and C. P. Sun, *ibid.* **99**, 130403 (2007); X.-J. Liu *et al.*, *ibid.* **98**, 026602 (2007).
[16] L. Lamata *et al.*, Phys. Rev. Lett. **98**, 253005 (2007).
[17] C. L. Roy and C. Basu, Phys. Rev. B **45**, 14 293 (1992); C. R. de Oliveira and R. Prado, J. Phys. A **38**, L115 (2005).
[18] O. Klein, Z. Phys. **53**, 157 (1929).
[19] G. Roati *et al.*, Nature (London) **453**, 895 (2008).
[20] J. Billy *et al.*, Nature (London) **453**, 891 (2008).
[21] L. Fallani *et al.*, Phys. Rev. Lett. **98**, 130404 (2007).
[22] J. E. Lye *et al.*, Phys. Rev. Lett. **95**, 070401 (2005).
[23] Y.-J. Lin *et al.*, Phys. Rev. Lett. **102**, 130401 (2009).
[24] M. S. Chang *et al.*, Phys. Rev. Lett. **92**, 140403 (2004).
[25] D. Landauer, Philos. Mag. **21**, 863 (1970).
[26] We focus here on the case with an extremely anisotropic potential, which is sufficiently strong along the transverse direction of the x axis, such that the original massive atoms are actually, or at least in principle, confined in a one-dimensional guide along the x axis. A crucial condition of this Dirac equation is $p_\alpha/\hbar \sim 1/l_\alpha \ll k\cos\theta$ ($\alpha = x, \rho$), where l_x (l_ρ) is the longitudinal (transverse) confinement and $k \sim 8.0 \times 10^6$ m^{-1} for the atoms of ^{87}Rb. If we simply take $\theta = \arccos(1/8)$, this condition implies that the dynamics of the atoms along the transverse direction is nonrelativistic for $l_\rho \leq 1.0$ μm and thus the atoms may still be confined by the original trap, while it is relativistic along the x axis for $l_x \gg 1.0$ μm.
[27] If we choose a specific potential given by $V_1^H - V_3^H = \hbar^2k^2\sin^2\theta/2m_a$, we are able to obtain a massless one-dimensional Dirac equation as the γ_z in Eq. (8) vanishes.