Ruin Problems for a Discrete Time Risk Model with Random Interest Rate*

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Abstract

In this paper, we study a discrete time risk model with random interest rate. The convergence of the discounted surplus process is proved by using martingale techniques, an expression of ruin probability is obtained, and bounds for ruin probability are included. In the second part of the paper, the distribution of surplus immediately after ruin, the distribution of surplus just before ruin, the joint distribution of the surplus immediately before and after ruin, and the distribution of ruin time are discussed.

Keywords: Martingale, interest income, convergence of the discounted surplus process, new better than used distribution, new worse than used distribution, recursive formula.

1 Introduction

In actuarial science, particularly in risk theory, ruin probability is a main research area. Mathematically, ruin theory is closely related to queuing theory; they share the same

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methodology and, very often, the same result can be used in both areas with different interpretations. The notions of new better than used (NBU) distribution and new worse than used (NWU) distribution used in this paper play an important role in reliability theory.

The ruin theory in a model with stochastic interest rate has received increasingly large amounts of attention recently. The ruin probability in a discrete model with random interest rate was considered in two interesting papers Cai (2002a) and Cai (2002b), by assuming that the interest rate forms an independent and identically distributed (i.i.d.) sequence in the former, and an autoregressive time series model in the latter, Lundberg type inequalities for the ruin probability were obtained. Cai and Dickson (2003) obtained exponential type upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest. Paulsen [1998] considered a diffusion model with stochastic interest incomes. Kalashnikov and Norberg [2000] assumed that the surplus of an insurance business is invested in a risky asset, and obtained upper and lower bounds for the ruin probability. Paulsen and Gjessing [1997] provided some results for a model with stochastic investment incomes.

Recently, people in actuarial science have been paying increasing attention to the severity of ruin and related problems. Gerber, Goovaert and Kass (1987) discussed the distribution of surplus immediately after ruin. Dufresne and Gerber (1988a, b) introduced the distribution of surplus prior to ruin, and Dickson and Reis (1994) studied the joint distribution of surplus immediately before and after ruin. Gerber and Shiu (1997, 1998) considered the joint distribution of surplus before and after ruin and the time of ruin. Vylder and Goovaerts (1988) presented some recursive formulas for calculating finite time ruin probabilities, and Dickson and Waters (1991) provided a recursive calculation formula for the survival probability,

In this paper, we first derive the convergence result for the discounted surplus process by using martingale techniques. From this convergence result, an expression for the ruin probability can be derived. Bounds for the ruin probability are also obtained. We then use a similar method to that in Vylder and Goovaerts (1988), Cai (2002b), and Sun and Yang (2003) to obtain some recursive formulas and equations for the above-mentioned ruin functions.

2 The model and some assumptions

Let the surplus of an insurance company at the end of nth time period be denoted by U_n, suppose X_n is the premium of insurance company received at the beginning of the nth time period, Y_n is the claim paid at the end of the nth time period, R_n is the short term interest rate in the nth time period. The dynamic of the surplus is given by:

$$U_n = (U_{n-1} + X_n)(1 + R_n) - Y_n, \tag{2.1}$$

where $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. non negative random variables, so are sequences $Y_1, Y_2, \cdots, Y_n, \cdots$ and $R_1, R_2, \cdots, R_n, \cdots$. Assuming that the net profit condition holds, i.e.,

$$E((1 + R)^{-1}Y) \leq E(X) < +\infty,$$

where Y has the same distribution as that of Y_i, X has the same distribution as that of X_i. Let u be the initial surplus of the insurance company, then the model which we described above can be rewritten as

$$U_n = \frac{1}{H_n}[u + \sum_{k=1}^{n}(X_k - Y_k(1 + R_k)^{-1})H_{k-1}], \tag{2.2}$$

where $H_n = \prod_{i=1}^{n}(1 + R_i)^{-1}$ is the discount factor, and $H_0 = 1$.

Using the standard notation, the ruin probability of insurance company can be defined as follows:

$$\psi(u) = P(\inf_{n\geq 0}(U_n < 0)|U_0 = u) = P(T < \infty|U_0 = u),$$

where $T = \inf\{n; U_n < 0\}$ denotes the ruin time.

We will also consider the following related distributions. Let

$$G(u, q) = P(|U_T| \leq q|U_0 = u).$$

That is, $G(u, q)$ denotes the distribution of surplus immediately after ruin. Similarly,

$$F(u, p) = P(U_{T-} \leq p|U_0 = u)$$

denotes the distribution of surplus immediately before ruin, here $T-$ denotes the time just before ruin, and

$$H(u, p, q) = P(U_{T-} \leq p, |U_T| \leq q|U_0 = u),$$

where p, q are positive real numbers, is the joint distribution of surplus before and after ruin. Some recursive formulas for these ruin functions will be developed.

3 Convergence of discounted surplus process and some related results

In this section, we assume that, for all $n \geq 1$, X_n and Y_n are independent of $\{R_1, R_2, \cdots, R_{n-1}\}$.

Let $V_n = H_n U_n - u$, that is, V_n is the difference between discounted surplus and initial surplus.

Theorem 3.1. There exists an integrable random variable V_∞, such that *a.e.*

$$V_n \longrightarrow V_\infty. \tag{3.1}$$

Moreover

$$E[V\infty] = E[X - \frac{Y}{1+R}]\frac{h}{1-h}, \tag{3.2}$$

where $h = E[(1+R)^{-1}] < 1$.

Proof: $V_n = H_n U_n - u = \sum_{k=1}^{n} [(X_k - Y_k(1+R_k)^{-1}) \prod_{i=1}^{k-1}(1+R_i)^{-1}]$
Let $\mathcal{F}_n = \sigma\{X_i, Y_i, R_i, i \leq n\}$, then

$$\begin{aligned} E[V_n|\mathcal{F}_{n-1}] &= V_{n-1} + E[(X_n - Y_n(1+R_n)^{-1}) \prod_{i=1}^{n-1}(1+R_i)^{-1}|\mathcal{F}_{n-1}] \\ &= V_{n-1} + H_{n-1}E[(X_n - Y_n(1+R_n)^{-1})|\mathcal{F}_{n-1}] \\ &= V_{n-1} + H_{n-1}\Big[E(X_n) - E(Y_n(1+R_n)^{-1})\Big]. \end{aligned}$$

Using the assumption of $E((1+R)^{-1}Y) \leq E(X)$, we have

$$E(X_n) - E[Y_n(1+R_n)^{-1}] \geq 0.$$

Then

$$E(V_n|\mathcal{F}_{n-1}) \geq V_{n-1}.$$

So $\{V_n, n \geq 0\}$ is a sub- martingale.

Moreover, we have

$$\sup_n E|V_n| \leq \sup_n \{\sum_{k=1}^{n} E|(X_k - Y_k(1+R_k)^{-1}) \prod_{i=1}^{k-1}(1+R_i)^{-1}|\}$$

$$
\begin{aligned}
&\le \sup_n \sum_{k=1}^{n} E|(X_k + Y_k(1+R_k)^{-1}) \prod_{i=1}^{k-1}(1+R_i)^{-1}| \\
&= \sup_n \sum_{k=1}^{n} E[X_k + Y_k(1+R_k)^{-1}] E(\prod_{i=1}^{k-1}(1+R_i)^{-1}) \\
&= E[X + Y(1+R)^{-1}] \sup_n \sum_{k=1}^{n} (E[(1+R)^{-1}])^{k-1},
\end{aligned}
$$

here we have used the independent assumption.

Let $E[(1+R)^{-1}] = h$, since the random variable R_i is positive, we have $0 < h < 1$. Therefore the right hand side of the expression above can be written as

$$
\begin{aligned}
&E(X + Y(1+R)^{-1}) \sup_n \sum_{k=1}^{n} h^{k-1} \\
&\le E[X + Y(1+R)^{-1}] \frac{h}{1-h} < \infty.
\end{aligned}
$$

By the martingale convergence theorem, there exists an integrable random variable V_∞, such that $V_n \longrightarrow V_\infty$ *a.e.*

Furthermore it is easy to show that

$$
E[V_\infty] = E[X - Y(1+R)^{-1}] \frac{h}{1-h}.
$$

Hence, theorem 3.1 is proved.

In fact, we can find the characteristic function of V_∞.

Suppose that V_∞ has distribution function $F_\infty(\cdot)$. In the following part of this section we show that the ruin probability can be expressed in terms of $F_\infty(\cdot)$.

Theorem 3.2 Under the above-mentioned assumptions, We have the following expression:

$$
\psi(u) = \frac{F_\infty(-u)}{E[F_\infty(-U_T | T < \infty)]}. \tag{3.3}
$$

Proof: Let $Z_n = \sum_{k=n+1}^{\infty} (X_k - Y_k(1+R_k)^{-1}) H_{k-1}$, then we have

$$
u + V_\infty = u + V_n + Z_n = (H_n^{-1}(u + V_n) + H_n^{-1} Z_n) H_n = (U_n + H_n^{-1} Z_n) H_n,
$$

for all $n > 0$.

It is obvious that the event $[T<\infty]$ contains the event $[u+V_\infty<0]$, then for $T<\infty$, we have

$$P(u+V_\infty<0)=P((U_T+H_T^{-1}Z_T)H_T<0,T<\infty).$$

It is not difficult to see that $H_T^{-1}Z_T\stackrel{d}{=}V_\infty$, therefore

$$\begin{aligned} F_\infty(-u) &= P(u+V_\infty<0)\\ &= P((U_T+H_T^{-1}Z_T)H_T<0|T<\infty)P(T<\infty)\\ &= P(H_T^{-1}Z_T<-U_T|T<\infty)P(T<\infty)\\ &= E[F_\infty(-U_T|T<\infty)]\psi(u). \end{aligned}$$

This completes the proof of theorem 3.2.

Remark. If $F_\infty(0)>0$, because $U_T<0$ and $F_\infty(x)\le 1$, we have

$$F_\infty(-u)\le\psi(u)\le\frac{F_\infty(-u)}{F_\infty(0)}.$$

We say a distribution function $F(x)$ is a new worse than used (NWU) distribution, if $F(x)$ is a d.f. of a non-negative random variable and $\bar F(x)=1-F(x)$, and $\bar F(x)\bar F(y)\le\bar F(x+y)$ for $x\ge0$ and $y\ge0$. We say that $F(x)$ is new better than used (NBU) if $\bar F(x)\bar F(y)\ge\bar F(x+y)$ for $x\ge0$ and $y\ge0$.

Proposition 3.3. The following results hold:
(1). If the distribution function of Y is NBU, then we have

$$\psi(u)\ge\frac{F_\infty(-u)}{E[F_\infty(Y)]}. \tag{3.4}$$

(2). If the distribution function of Y is NWU, then

$$\psi(u)\le\frac{F_\infty(-u)}{E[F_\infty(Y)]}. \tag{3.5}$$

(3). If the distribution function of Y is exponential, then

$$\psi(u)=\frac{F_\infty(-u)}{E[F_\infty(Y)]}. \tag{3.6}$$

Proof : We only prove (1) here, (2) can be proved in the same way and (3) is a direct result of (1) and (2).

$$\begin{aligned} &E[F_\infty(-U_T|T<\infty)]=P(V_\infty<-U_T|T<\infty)\\ =\;&P(V_\infty<-U_{T-}+Y|Y>U_{T-})=P(Y>V_\infty+U_{T-}|Y>U_{T-}), \end{aligned}$$

where U_{T-} denotes the surplus just before ruin. From the definition, we have that $U_{T-} \geq 0$.

If $V_\infty \geq 0$, since the distribution of Y is NBU, we have that

$$P(Y > V_\infty + U_{T-}|Y > U_{T-}) \leq P(Y > V_\infty) = E[F_\infty(Y)].$$

If $V_\infty < 0$, we also have

$$E[F_\infty(-U_T|T < \infty)] \leq P(Y > V_\infty) = E[F_\infty(Y)].$$

From Theorem 3.2, we obtain that

$$\psi(u) \geq \frac{F_\infty(-u)}{E[F_\infty(Y)]}.$$

This is (3.4).

4 Recursive formulas or equations for ruin functions

In this section, we assume that the three random variable sequences $\{X_n, n = 1, 2, \cdots\}$, $\{Y_n, n = 1, 2, \cdots\}$ and $\{R_n, n = 1, 2, \cdots\}$ are independent of each other.

Let $W_n = -[X_n - Y_n(1 + R_n)^{-1}]$, then the model we considered in this paper can be written as

$$U_n = \frac{1}{H_n}\{u - \sum_{k=1}^{n} W_k H_{k-1}\}, \tag{4.1}$$

where $\{W_n, n = 1, 2, \cdots\}$ is an $i.i.d$ random variable sequence. Notice that for any $k \geq 1$, W_k is independent of H_{k-1}.

In order to discuss the ruin functions, we should obtain the distribution of W_n first. Let $A(z)$ be the distribution function of W (W is the generic random variable of W_n), then

$$\begin{aligned} A(z) &= P(W \leq z) = P(Y(1 + R)^{-1} - X \leq z) = E[F_Y((1 + R)(z + X))] \\ &= \int_0^{+\infty}\int_0^{+\infty} F_Y((1 + r)(z + x))dF_X(x)dF_R(r). \end{aligned}$$

We will also need the joint distribution function of W and R, denoted as $B(s,t)$,

$$\begin{aligned} B(s,t) &= P(W \le s, R \le t) = P(Y(1+R)^{-1} - X \le s, R \le t) \\ &= \int_0^t E\Big[F_Y((s+X)(1+r))\Big] dF_R(r) \\ &= \int_0^t \int_0^{+\infty} F_Y((s+x)(1+r)) dF_X(x) dF_R(r), \end{aligned}$$

where $F_Y(\cdot)$, $F_X(\cdot)$ and $F_R(\cdot)$ denote the distributions of random variables Y, X and R respectively. The integral here means the Lebesgue Stieltjes integral.

Let T denote the ruin time, then

$$T = \inf\{n > 0 : U_n < 0\} = \inf\{n > 0 : \sum_{k=1}^{n} W_k H_{k-1} < u\}.$$

The equations obtained below are useful in calculating the numerical values of the ruin functions.

4.1 The joint distribution of surplus just before ruin and surplus immediately after ruin

Recall that

$$H(u,p,q) = P\Big(U_{T-} \le p, |U_T| \le q, T < \infty | U_0 = u\Big),$$

where p and q are positive real numbers.

We consider the following function,

$$H_1(u,p,q) = P\Big(U_T \le -q, U_{T-} > p, T < \infty | U_0 = u\Big).$$

Note that $H_1(u,p,q)$ is the joint distribution of surplus immediately before and after ruin.

$$\begin{aligned} H_1(u,p,q) &= \sum_{n=1}^{\infty} P\Big(U_T \le -q, U_{T-1} > p, T = n | U_0 = u\Big) \\ &= \sum_{n=1}^{\infty} P\Big(S_n \ge u + \frac{q}{(1+R_1)\cdots(1+R_n)}, S_{n-1} < u - \frac{p}{(1+R_1)\cdots(1+R_{n-1})}, \\ &\quad S_{n-2} \le u, \cdots, S_1 \le u\Big) \\ &= \sum_{n=1}^{\infty} h_n(u,p,q), \end{aligned}$$

where $S_n = \sum_{i=1}^n W_i H_{i-1}$, $S_0 = 0$ and

$$\begin{aligned}
& h_n(u,p,q) = P\Big(U_T \le -q, U_{T-1} > p, T = n | U_0 = u\Big) \\
= \quad & P\Big(S_n \ge u + \frac{q}{(1+R_1)\cdots(1+R_n)}, S_{n-1} < u - \frac{p}{(1+R_1)\cdots(1+R_{n-1})}, \\
S_{n-2} \quad & \le u, \cdots, S_1 \le u\Big).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
h_1(u,p,q) &= P\Big(S_1 \ge u + \frac{q}{1+R_1}, S_0 < u - p\Big) = P\Big(W_1 \ge u + \frac{q}{1+R_1}, 0 < u - p\Big) \\
&= \begin{cases} \int_0^{+\infty} \Big(1 - A(u + \frac{q}{1+r} -)\Big) dF_R(r) & p < u \\ 0 & p \ge u \end{cases}
\end{aligned}$$

$$\begin{aligned}
& h_2(u,p,q) = P\Big(S_2 \ge u + \frac{q}{(1+R_1)(1+R_2)}, S_1 < u - \frac{p}{1+R_1}\Big) \\
= \;& P\Big(W_1 + \frac{W_2}{1+R_1} \ge u + \frac{q}{(1+R_1)(1+R_2)}, W_1 < u - \frac{p}{1+R_1}\Big) \\
= \;& \int_0^\infty \int_0^\infty \int_{-x}^{u-\frac{p}{1+r}} P\Big(W_2 \ge (u-y)(1+r) + \frac{q}{1+R_2}\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r).
\end{aligned}$$

Using the same method, we can obtain that for $n \ge 3$,

$$h_n(u,p,q) = \int_0^\infty \int_0^\infty \int_{-x}^{u} h_{n-1}((u-y)(1+r), p, q) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r).$$

Therefore, we have the following results:

(1) When $p < u$,

$$\begin{aligned}
& H_1(u,p,q) = \sum_{n=1}^\infty h_n(u,p,q) \\
= \;& \int_0^\infty \Big(1 - A(u + \frac{q}{1+r} -)\Big) dF_R(r) \\
+ \;& \int_0^\infty \int_0^\infty \int_{-x}^{u-\frac{p}{1+r}} P\Big(W_2 \ge (u-y)(1+r) + \frac{q}{1+R_2}\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r) \\
+ \;& \sum_{n=3}^\infty \int_0^{+\infty} \int_0^\infty \int_{-x}^{u} h_{n-1}((u-y)(1+r), p, q) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r) \\
= \;& \int_0^\infty \Big(1 - A(u + \frac{q}{1+r} -)\Big) dF_R(r) \\
+ \;& \int_0^\infty \int_0^\infty \int_{-x}^{u-\frac{p}{1+r}} h_1((u-y)(1+r), p, q) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{n=3}^{\infty}\int_0^{+\infty}\int_0^{\infty}\int_{-x}^{u} h_{n-1}((u-y)(1+r),p,q)dF_Y((y+x)(1+r))dF_X(x)dF_R(r)\\
&= \int_0^{\infty}\Big(1-A(u+\frac{q}{1+r}-)\Big)dF_R(r)\\
&+ \int_0^{+\infty}\int_0^{\infty}\int_{-x}^{u}\sum_{n=1}^{\infty} h_n((u-y)(1+r),p,q)dF_Y((y+x)(1+r))dF_X(x)dF_R(r),
\end{aligned}
$$

so $H_1(u,p,q)$ satisfies

$$
\begin{aligned}
H_1(u,p,q) &= \int_0^{\infty}\Big(1-A(u+\frac{q}{1+r}-)\Big)dF_R(r)\\
&+ \int_0^{+\infty}\int_0^{\infty}\int_{-x}^{u} H_1\Big((u-y)(1+r),p,q\Big)dF_Y((y+x)(1+r))dF_X(x)dF_R(r).
\end{aligned}
\tag{4.2}
$$

(2). When $p \geq u$,

$$
\begin{aligned}
&H_1(u,p,q) = h_1(u,p,q)+\sum_{n=2}^{\infty}h_n(u,p,q)\\
&= \int_0^{\infty}\int_0^{+\infty}\int_{-x}^{u-\frac{p}{1+r}} P\Big(W_2 \geq (u-y)(1+r)+\frac{q}{1+R_2}\Big)dF_Y((y+x)(1+r))dF_X(x)dF_R(r)\\
&+ \int_0^{+\infty}\int_0^{+\infty}\int_{-x}^{u}\sum_{n=2}^{\infty} h_n((u-y)(1+r),p,q)dF_Y((y+x)(1+r))dF_X(x)dF_R(r),
\end{aligned}
$$

so

$$
H_1(u,p,q) = \int_0^{+\infty}\int_0^{+\infty}\int_{-x}^{u} H_1((u-y)(1+r),p,q)dF_Y((y+x)(1+r))dF_X(x)dF_R(r) \tag{4.3}
$$

4.2 The distribution of surplus immediately before ruin

From the definition of function $F(u,p)$, we know that

$$
F(u,p) = \psi(u) - P\Big(U_{T-} > p, T<\infty | U_0 = u\Big) = \psi(u) - F_1(u,p).
$$

Letting $q=0$ in (4.3), when $p \geq u$, we have that

$$
F_1(u,p) = \int_0^{\infty}\int_0^{\infty}\int_{-x}^{u} F_1\Big((u-y)(1+r),p\Big)dF_Y((y+x)(1+r))dF_X(x)dF_R(r).
$$

Letting $q=0$ in (4.2), when $p<u$, we have that

$$
F_1(u,p) = 1-A(u)+\int_0^{\infty}\int_0^{\infty}\int_{-x}^{u} F_1\Big((u-y)(1+r),p\Big)dF_Y((y+x)(1+r))dF_X(x)dF_R(r).
$$

4.3 The distribution of surplus immediately after ruin

Recall that

$$\begin{aligned} G(u,q) &= P\Big(|U_T| \leq q | U_0 = u\Big) = P\Big(-q \leq U_T < 0 | U_0 = u\Big) \\ &= \psi(u) - P\Big(U_T < -q | U_0 = u\Big) = \psi(u) - G_1(u,q). \end{aligned}$$

Letting $p = 0$ in (4.2), we have

$$\begin{aligned} G_1(u,q) &= \Big(1 - A(u)\Big) \\ &+ \int_0^\infty \int_0^\infty \int_{-x}^u G_1\Big((u-y)(1+r), q\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r). \end{aligned}$$

This is the equation satisfied by the distribution of surplus immediately after ruin.

4.4 Recursive formula for finite time ruin probability

Define the finite time ruin probability as follows:

$$\psi_n(u) = P(T \leq n),$$

then the non ruin probability before or at time n is

$$\varphi_n(u) = 1 - \psi_n(u) = P(T > n).$$

Similar to Cai (2002,a, b) and Sun and Yang (2003), we have

$$\psi_1(u) = 1 - \varphi_1(u) = 1 - A(u),$$

$$\psi_2(u) = 1 - \varphi_2(u) = 1 - \int_0^\infty \int_0^\infty \int_{-x}^u \psi_1\Big((1+r)(u-y)\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r),$$

$\ldots\ldots$

$$\psi_n(u) = 1 - \varphi_n(u) = 1 - \int_0^\infty \int_0^\infty \int_{-x}^u \psi_{n-1}\Big((1+r)(u-y)\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r). \tag{4.4}$$

Let $Q_n(u) = P(T = n)$ be the ruin time distribution, then from recursive formula above we obtain

$$\begin{aligned} Q_1(u) &= P(T = 1) = P(T > 0) - P(T > 1) = 1 - A(u), \\ Q_2(u) &= P(T = 2) = P(T > 1) - P(T > 2) = P\Big(W_1 \leq u, W_1 + \frac{W_2}{(1+R_1)} > u\Big) \\ &= \int_0^\infty \int_0^\infty \int_{-x}^u Q_1\Big((1+r)(u-y)\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r), \end{aligned}$$

......

$$Q_n(u) = \int_0^\infty \int_0^\infty \int_{-x}^u Q_{n-1}\Big((1+r)(u-y)\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r), \quad (4.5)$$

for all $n \geq 2$.

4.5 Equation for ultimate ruin probability

Theorem 4.1 The ultimate ruin probability satisfies the following equation:

$$\psi(u) = 1 - A(u) + \int_0^\infty \int_0^\infty \int_{-x}^u \psi\Big((1+r)(u-y)\Big) dF_Y((y+x)(1+r)) dF_X(x) dF_R(r). \quad (4.6)$$

Proof: The result can be obtained by letting $p = 0$ and $q = 0$ in (4.2).

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