Extremal Bounded Holomorphic Functions and an Embedding Theorem for Arithmetic Varieties of Rank ≥ 2

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Let Ω be a bounded symmetric domain of rank ≥ 2, and $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible cocompact lattice, $X := \Omega/\Gamma$. On the projective manifold X there is the canonical Kähler-Einstein metric, which is of nonpositive holomorphic bisectional curvature. In Mok [M1,2] we established a Hermitian metric rigidity theorem for such projective manifolds X, which in the case when Ω is irreducible says that any Hermitian metric of nonpositive curvature in the sense of Griffiths is necessarily a constant multiple of the Kähler-Einstein metric. As a consequence, we proved in the latter case that any nontrivial holomorphic mapping $f : X \to Z$ into a Hermitian manifold Z of nonpositive curvature in the sense of Griffiths is necessarily an isometric immersion totally geodesic with respect to the Hermitian connection on Z. The Hermitian metric rigidity theorem can be taken as a tool in establishing the statement that any $f : X \to Z$ is necessarily a holomorphic immersion, a statement which only concerns the complex structure of X and Z. The Hermitian metric rigidity theorem and its consequences were generalized by To [To] to be applicable to any torsion-free lattice $\Gamma \subset Aut(\Omega)$, where $X := \Omega/\Gamma$ may be noncompact.

For holomorphic mappings into complex manifolds defined on X, we expect that rigidity theorems should hold under much weaker conditions of nonpositivity of the target space. One natural hypothesis, when the target space N is nonsingular, is the existence on N of a continuous complex Finsler metric of nonpositive curvature. This is the case, *e.g.*, when X is uniformized by a bounded domain in a Stein manifold, where the Carathéodory metric on the universal cover, which descends to N, is of nonpositive curvature as a continuous complex Finsler metric. With this and other examples in mind we established in Mok [M3] in the compact case a Hermitian metric rigidity theorem for continuous complex Finsler metrics, where for obvious reasons the conclusion has to be weaker, by proving in the locally irreducible case that any continuous complex Finsler metric on X has to agree with a constant multiple of the canonical metric when we restrict to the characteristic bundle on X, *i.e.*, to vectors of type (1,0) tangent in local liftings to minimal disks. With this "partial" Finsler metric rigidity we showed that, for large classes of irreducible bounded symmetric domains Ω of rank ≥ 2, any nonconstant holomorphic mapping from X into a complex manifold admitting a continuous Finsler metric of nonpositive curvature must be an immersion at some point.

In this article we study specifically the case where the continuous complex Finsler metric arises from the Carathéodory pseudometric. In this case, in addition to the Finsler metric rigidity theorem, we have the additional tool of bounded holomorphic functions, which are at the origin of the Carathéodory pseudometric. The Carathéodory length of any given nonzero tangent vector η of type (1, 0) on X is realized by some Carathéodory extremal function, not necessarily unique. By means of Finsler metric rigidity and a study of Carathéodory extremal functions we prove that any nontrivial holomorphic mapping f of X into a complex manifold uniformized by a bounded domain is necessarily a holomorphic immersion. Moreover, we show that the lifting of $F : \Omega \to \tilde{N}$ to universal covers is a holomorphic *embedding*. The latter, to be referred to as the Embedding Theorem, is especially unexpected, since the corresponding statement is unknown even when the Hermitian metric rigidity theorem is applicable. Our method of proof yields a number of surprising consequences. It shows that the image of X under any proper holomorphic mapping f must have finite fundamental group unless f is an unramified covering map. In particular, for any arithmetic lattice $\Gamma^* \subset Aut(\Omega)$ which is *not* torsion-free, $\pi_1(\Omega/\Gamma^*)$ is finite. In another direction, in the locally irreducible case we show that the universal cover of any *singular* complex space normalized by X admits no nonconstant bounded holomorphic functions.

To illustrate the philosophy of our proofs we will treat first of all the special case where Ω is the polydisk Δ^n and $\Gamma \subset Aut(\Omega)$ is cocompact. In this case using Finsler metric rigidity we showed that the "flats" of a continuous complex Finsler metric on N is compatible with the local foliations on N induced by the canonical foliations on Δ^n, and the crux of the argument is to deduce from there the existence on N of special Carathéodory extremal functions which are constant on leaves of some canonical foliations. We call this the Splitting Phenomenon. This is done by taking "boundary values" of initial Carathéodory extremal functions. The proof that $F : \Delta^n \to \tilde{N}$ separates points will be completed by invoking a density lemma for canonical projections of irreducible lattices into direct factors.

In the cocompact case using the Polydisk Theorem on a bounded symmetric domain the preceding arguments can be easily adapted to show in general that $F : \Omega \to \tilde{N}$ is an immersion, but the proof that F separates points in general requires new ideas. For the separation of points if an analogue of the Splitting Phenomenon can be established for the general case, then in place of a density lemma for polydisks we may use Moore's Ergodicity Theorem applied to a certain moduli space $\mathcal{M}$ of maximal polydisks with additional structures. However, a straightforward generalization of the Splitting Phenomenon breaks down because the orbit of a point $p \in \mathcal{M}$ may fail to be dense for p belonging to some exceptional null subset $E \subset \mathcal{M}$.

To circumvent the difficulty we introduce a new extremal problem adapted to $F : \Omega \to \tilde{N}$ such that for *every* maximal polydisk $P \subset \Omega$ any extremal function adapted to P will automatically be dependent on only one of the direct factors of P. The extremal problem only makes sense on Ω, not on $\tilde{N}$, but is applied to the space $\mathcal{F}$ of holomorphic functions $s : \Omega \to \Delta$ which are pull-backs of bounded holomorphic functions on $\tilde{N}$ by F. Our extremal problem corresponds to defining a continuous Hermitian metric $e(\mathcal{F})$ on the restriction of the tautological line bundle L to the characteristic bundle $\mathcal{S}_\Omega \subset \mathbb{P}T_\Omega$ consisting of projectivizations of vectors tangent to minimal disks. For any characteristic vector $\eta \in \mathcal{S}_x$ and $s \in \mathcal{F}$, the length $\|\alpha\|_s$ is measured by averaging the Poincaré lengths $\|ds(\alpha')\|_{ds^2_\Delta}$ of translates α' of α on some geodesic circle of the minimal disk determined by x and α, and the continuous Hermitian metric $e(\mathcal{F})$ on $L|_{\mathcal{S}_\Omega}$ is defined by taking suprema as s ranges over $\mathcal{F}$. By making use of $e(\mathcal{F})$ and the idea of proof of metric rigidity theorems on $X = \Omega/\Gamma$ we show that appropriate extremal functions adapted to a maximal polydisk P depend only on one of the direct factors to give a proof of the Embedding Theorem.

Although the argument using $e(\mathcal{F})$-extremal functions applies both to the locally irreducible and the locally reducible cases, we have included a separate and more elementary treatment of the case of the polydisk, with an aim to identifying the principal problem in the general case and giving a motivation for introducing the new extremal problem. The latter can be readily adapted to the general case of $X := \Omega/\Gamma$ where $\Gamma \subset \Omega$ is a torsion-free irreducible lattice. The adaptation consists of a justification of a special form of metric rigidity, and will be given after a complete proof of the Embedding Theorem for the cocompact case. While extremal bounded holomorphic functions are well studied in one complex variable, they are seldom understood and exploited in higher dimensions. In this context the present article represents a novel application of extremal bounded holomorphic functions to rigidity problems in Several Complex Variables.

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§1 Preliminaries and statements of results.

(1.1) We consider a continuous complex Finsler metric h on a complex manifold M as equivalently a continuous Hermitian metric on the tautological line bundle L of $\mathbb{P}T(M)$. h is said to be of nonpositive curvature if and only if it is of nonpositive curvature in the sense of currents when regarded as a Hermitian metric on L. In other words, the curvature (1,1) current is a (closed) positive current. As a starting point, we generalized the Hermitian metric rigidity theorem of Mok [1,2] to a "partial" Finsler metric rigidity theorem. Such a theorem was implicit in Mok [3, Proposition 3] for the case of compact complex manifolds X uniformized by irreducible bounded symmetric domains Ω of *rank* ≥ 2. For its formulation, write G for the identity component of the group of biholomorphic automorphisms of Ω and $K \subset G$ for the isotropy subgroup at some base point $o \in \Omega$. K has a one-dimensional centre. Write $\mathfrak{k}$ for the Lie algebra of K and $\mathfrak{k}_s$ for $[\mathfrak{k}, \mathfrak{k}]$. The $\mathfrak{k}_s$ is the Lie algebra of a real semisimple compact Lie group $K_s \subset K$, which we call the semisimple part of K. Then, K_s acts irreducibly on the holomorphic tangent space $T_o(\Omega)$. A nonzero vector $\eta \in T_o(\Omega)$ is called a characteristic vector if and only if it is a highest weight vector of $T_o(\Omega)$ as a K_s-representation space, with respect to some choice of Cartan subalgebra $\mathfrak{h}_s \subset \mathfrak{k}_s$. Denote by $\mathcal{S}'_o \subset T_o(\Omega)$ the set of characteristic vectors at 0, and $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$ its projectivization. Then, $\mathcal{S}_o$ is the highest weight orbit of the isotropy representation of K_s and is thus a homogenenous complex projective submanifold. The characteristic bundle $\mathcal{S}_\Omega \subset \mathbb{P}T(\Omega)$ is the orbit of some $[\eta] \in \mathcal{S}_o$ under G. It descends to $\mathcal{S} \subset \mathbb{P}T_X$ on X. The Hermitian metric rigidity theorem in Mok [1,2] was obtained by an integral formula of Chern forms over $\mathcal{S}$. The same integral formula works single direct factor Δ_k of the polydisk. In the general case we use the Polydisk Theorem and apply a classical ergodicity result for semisimple real Lie groups, as follows.

Polydisk Theorem (cf. Wolf [Wo, p.280]). *Let Ω be a bounded symmetric domain of rank r, equipped with the Kähler-Einstein metric g. Then, there exists an r-dimensional totally-geodesic complex submanifold P biholomorphic to the polydisk Δ^r. Moreover, the identity component $Aut_o(\Omega)$ of $Aut(\Omega)$ acts transitively on the space of all such polydisks.*

Moore's Ergodicity Theorem (cf. Zimmer [Zi, Thm.(2.2.6), p.19]). *Let G be a semisimple real Lie group and Γ be an irreducible lattice on G, i.e., $\Gamma\backslash G$ is of finite volume in the left invariant Haar measure. Suppose $H \subset G$ is a closed subgroup. Consider the action of H on $\Gamma\backslash G$ by multiplication on the right. Then, H acts ergodically if and only if H is noncompact.*

(1.3) Let $X := \Omega/\Gamma$ be as in Proposition 1 and $f : X \to N$ be a nonconstant holomorphic mapping into a complex manifold N admitting a continuous complex Finsler metric of nonpositive curvature. From Finsler metric rigidity, we deduce readily that $df(\eta) \neq 0$ for any nonzero vector η corresponding to a characteristic vector of some local de Rham factor of X. However, it does not rule out the possibility the f is ramified. When we consider Carathéodory metrics and their analogues, we have the additional tool of extremal bounded holomorphic functions. Together with a version of Finsler metric rigidity applicable also to $X := \Omega/\Gamma$ with $\Gamma \subset Aut(\Omega)$ only of finite covolume, we will prove

Theorem 1. *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be a torsion-free lattice, $X := \Omega/\Gamma$. Let N be a complex manifold and denote by $\tilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic mapping and $F : \Omega \to \tilde{N}$ be its lifting to universal covers. Assume that there exists a bounded holomorphic function h on $\tilde{N}$ such that h is nonconstant on the image $F(\Omega)$. Then, $F : \Omega \to \tilde{N}$ is a holomorphic embedding.*

The analogue of Theorem 1 remains true in the locally reducible case, under a slightly stronger hypothesis. For a reducible bounded symmetric domain Ω, let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be the decomposition of Ω into irreducible factors. A subdomain such as $\Omega_1' = \Omega_1 \times \{(x_2, \ldots, x_m)\}$ will be called an irreducible factor subdomain. We have

Theorem 1'. *Let Ω be a reducible bounded symmetric domain, $\Omega = \Omega_1 \times \cdots \times \Omega_m$ its decomposition into irreducible factors. Let $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a complex manifold and denote by $\tilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic mapping and $F : \Omega \to \tilde{N}$ be its lifting to universal covers. Assume that, for each k, $1 \leq k \leq m$, there exists a bounded holomorphic function h_k on $\tilde{N}$ and an irreducible factor subdomain $\Omega_k' \subset \Omega$ such that h_k is nonconstant on $F(\Omega_k')$. Then, $F : \Omega \to \tilde{N}$ is a holomorphic embedding.*

We will refer to Theorem 1 and Theorem 1' as the Embedding Theorem. When N is uniformized by a bounded domain D in some Stein manifold, the Carathéodory pseudometric is a metric, and we have

Theorem 2. *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let D be a bounded domain in some Stein manifold, $G \subset Aut(D)$ be a torsion-free discrete group of automorphisms, $N := D/G$. Let $f : X \to N$ be a nonconstant holomorphic mapping and $F : \Omega \to D$ be its lifting to universal covering spaces. Then, $F : \Omega \to D$ is a holomorphic embedding.*

The Embedding Theorems can be formulated for complex spaces N. A special case of the generalized theorem is the case when $f : X \to N$ is a local biholomorphism at a generic point. Especially, we have the following consequence which shows that the existence of bounded holomorphic functions on universal covers can be very sensitive to perturbations which introduce singularities.

Theorem 3. *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be a torsion-free lattice, $X := \Omega/\Gamma$. Let N be a singular complex space whose normalization is given by $f : X \to N$, and denoted by $\tilde{N}$ the universal cover of N. Then, there exist no nonconstant bounded holomorphic functions on the universal cover $\tilde{N}$.*

We note that in the special case when N is locally irreducible as a complex space, $f : X \to N$ is bijective and hence a homeomorphism, so that it lifts to a bijective holomorphic map $F : \Omega \to \tilde{N}$. Even in this case $\tilde{N}$ admits no nonconstant bounded holomorphic function (whenever N is actually *singular*). This is in constrast with the situation when $\Omega = \Delta$, the unit disk. In fact, for any irreducible algebraic curve C on a projective manifold uniformized by a bounded domain, there are plenty of bounded holomorphic functions on the universal cover of C, and the normalization of C is uniformized by the unit disk. We note also that the strict analogue of Theorem 3 fails in the locally reducible case.

In conjunction with a result of Margulis' [Ma] regarding normal subgroups of irreducible lattices, we establish the following on fundamental groups of normal complex spaces dominated by X.

Theorem 4. *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be any torsion-free irreducible lattice. Let Z be a normal complex space and $f : X \to Z$ be a proper holomorphic mapping onto Z. Then, either $f : X \to Z$ is an unramified covering map, or $\pi_1(Z)$ is finite.*

A corollary of Theorem 3 of particular interest is the following result concerning the fundamental group of arithmetic varieties.

Corollary 1. *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma^* \subset Aut(\Omega)$ be any irreducible lattice which is not torsion-free. Write $Z = \Omega/\Gamma^*$. Then, $\pi_1(Z)$ is finite.*

From the proof of Theorem 1 we will deduce the following stronger result on equivariant holomorphic mappings on $X = \Omega/\Gamma$.

Theorem 5. *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let M be a complex manifold, $\Phi : \Gamma \to$*

$Aut(M)$ be a homomorphism, and $F : \Omega \to M$ be a nonconstant Φ-equivariant holomorphic map in the sense that $F(\gamma x) = \Phi(\gamma)(F(x))$ for any $x \in \Omega$, $\gamma \in \Gamma$. Assume that there exists a bounded holomorphic function h on M such that $h|_{F(\Omega)}$ is nonconstant. Then, F is a holomorphic embedding.

§2 Irreducible compact quotients of the polydisk

(2.1) For the proof of the Embedding Theorem, we will first deal with the case of irreducible torsion-free cocompact lattices $\Gamma \subset Aut(\Omega)$. From now on $X := \Omega/\Gamma$ will be compact until §4. In this section we consider the case of the polydisk $\Delta^n, n \geq 2$, presenting the arguments for the bidisk Δ^2. For our purpose irreducible lattices in the case of the polydisk are particularly simple, in view of classical density results for canonical projections. Understanding of the polydisk is also important for the general case, since an irreducible bounded symmetric domain is swept out by its maximal polydisks, by the Polydisk Theorem. For irreducible lattices we have the following density lemma which is sufficient for our study of Carathéodory extremal functions by taking boundary values.

Density Lemma (special case of Raghunathan [Ra, Cor.(5.21), p.86]). *Let Ω be a reducible bounded symmetric domain, $\Omega = \Omega \times \cdots \times \Omega_k$ be the decomposition of Ω into irreducible factors. Let $I = (i(1), \ldots, i(p))$, $1 \leq i(1) < \cdots < i(p) \leq k$, be a multi-index and $pr_I : Aut_0(\Omega) \to Aut_0(\Omega_{i(1)}) \times \cdots \times Aut_0(\Omega_{i(p)})$ be the canonical projection. Let $\Gamma \subset Aut_o(\Omega)$ be an irreducible lattice. Then, $pr_I(\Gamma)$ is dense in $Aut_0(\Omega_{i(1)}) \times \cdots \times Aut_0(\Omega_{i(p)})$ whenever $p < k$.*

(2.2) We proceed now to prove Theorem 1' for irreducible compact quotients of the polydisk, presenting the argument in the case of the bidisk for simplicity. Without loss of generality let $\Gamma \subset Aut(\Delta)^2$ be a torsion-free irreducible cocompact lattice, $X := \Delta^2/\Gamma$. Write $z = (z_1, z_2)$ for Euclidean coordinates on $\Delta^2 \subset \mathbb{C}^2$. By the i-th canonical foliation on Δ^2 we will mean the one with leaves $\{x_1\} \times \Delta$ for $i = 1$ resp. $\Delta \times \{x_2\}$ for $i = 2$. Passing to quotients we have the i-th canonical foliation on $X = \Delta^2/\Gamma$. Let $f : X \to N$ be a holomorphic mapping into a complex manifold N, $F : \Delta^2 \to \tilde{N}$ its lifting to universal covers. Assume for the time being that f is an immersion at a generic point. Let $\varepsilon > 0$ be small enough so that for $U_\varepsilon(x) := \Delta(x_1, \varepsilon) \times \Delta(x_2, \varepsilon)$, $F|_{U_\varepsilon(x)} : U_\varepsilon(x) \to \tilde{N}$ is a biholomorphism onto an open neighborhood $V_\varepsilon(p)$ of $p = F(x)$ in $\tilde{N}$. Denote the leaves of the i-th canonical foliation on $U_\varepsilon(x)$ by Λ_i; $i = 1, 2$; and those induced on $V_\varepsilon(p)$ by L_i. For $q \in V_\varepsilon(p)$, $L_i(q)$ will denote the leaf L_i passing through q. Recall that $\mathcal{H}$ is the set of all holomorphic mappings $h : \tilde{N} \to \Delta$. Let now $\eta \in T_p(\tilde{N})$, $\eta \neq 0$. Recall that any $h \in \mathcal{H}$ such that $\|\eta\|_\kappa = \|dh(\eta)\|_{ds^2_\Delta}$, is said to be a Carathéodory extremal function at p for, or adapted to, $\eta \in T_p(\tilde{N})$. Let now $i = 1$ or 2, and $\eta \in T_p(\tilde{N})$ be $dF(\frac{\partial}{\partial z_i})$. Pick any Carathéodory extremal function h adapted to η. Write $T_{\Delta^2} = T_1 \oplus T_2$ as in the Finsler Metric Rigidity Theorem. By the latter theorem $F^*\kappa|_{T_1} \equiv c_1 g|_{T_1}$, $F^*\kappa|_{T_2} \equiv c_2 g|_{T_2}$ for the canonical Kähler-Einstein metric on Δ^2. From the hypothesis of Theorem 1'; $c_1, c_2 \neq 0$. We are going to derive the existence of special extremal functions adapted to η, as follows.

Proposition 1. *Let h be any Carathéodory extremal function $h \in \mathcal{H}$ for $\eta = dF_x(\frac{\partial}{\partial z_i})$, $\eta \in T_p(\tilde{N})$. Let $q \in L_i(p)$, $q = F(y)$, $y \in U_\varepsilon(x)$, and $\eta_q := dF_y(\frac{\partial}{\partial z_i})$, $\eta_p = \eta$. Then, h is a Carathéodory extremal function for η_q at q, and h is constant on the leaf $L_i(p)$.*

Proof. Without loss of generality we may assume that $h(p) = 0$. Consider the holomorphic function $\theta(q) = dh(\eta_q)$ on $L_i(p)$. Then

$$\|\eta_q\|_\kappa \geq \|dh(\eta_q)\|_{ds^2_\Delta} = \frac{|\theta(q)|}{1 - |h(q)|^2} \geq |\theta(q)| \ ,$$
$$\|\eta_p\|_\kappa = \frac{|\theta(p)|}{1 - |h(p)|^2} = |\theta(p)| \ .$$

By Finsler metric rigidity, $\|\eta_q\|_\kappa$ is a positive constant λ, since $\|\frac{\partial}{\partial z_i}\|$ is of constant length on Δ^2 on each leaf $\Lambda_i(x)$. Thus,

$$\log|\theta(q)| \leq \|\eta_q\|_\kappa = \log\lambda \ ;$$
$$\log|\theta(p)| = \log\lambda \ .$$

It follows from the harmonicity of $\log|\theta(q)|$ in q that $\log|\theta(q)|$ is a constant. As a consequence,

$$\|\eta_q\|_\kappa = \|dh(\eta_q)\|_{ds^2_\Delta} = \lambda \ ; \quad h(q) = 0 \ .$$

In particular, h is a Carathéodory extremal function for η_q at any $q \in L_i(p)$ and h is constant on $L_i(p)$, as desired. □

Proposition 1 can be strengthened to give Carathéodory extremal functions compatible with the local foliations on $F(\Delta^2)$ induced by the canonical foliations on Δ^2. In other words, we have

Proposition 2. *There exists a Carathéodory extremal function $h \in \mathcal{H}$ for $\eta = dF(\frac{\partial}{\partial z_i})$ such that $h|_{V_\varepsilon(p)}$ is constant on each leaf $L_i(q)$, $q \in V_\varepsilon(p)$.*

Take $i = 1$. For the Carathéodory extremal functions h on $\tilde{N}$ adapted to $\eta = dF\left(\frac{\partial}{\partial z_1}\right)$, $F^*h = s(z_1, z_2)$ enjoys the special properties that (i) $s(x_1, z_2)$ is independent of z_2, and (ii) $\left|\frac{\partial s}{\partial z_1}(x_1, z_2)\right| = \frac{c}{1-|x_1|^2}$ for the constant $c = c_1$ appearing in the Finsler Metric Rigidity Theorem. Replacing h by $\varphi \circ h$ for some Möbius transformation φ if necessary, we may assume in (i) that $s(x_1, z_2) = 0$. From (ii) we deduce that $\frac{\partial s}{\partial z_1}(x_1, z_2)$ is a constant. To prove Proposition 2 we will modify h and hence s so that $s(z_1, z_2)$ is independent of z_2.

Proof of Proposition 2. Denote by $\Phi : \Gamma \to \pi_1(N)$ the homomorphism induced by $f : X \to N$. We are going to modify h and s by composing with elements of $\Phi(\Gamma)$ and taking limits, in such a way that it amounts to taking boundary values of s. For $x = (x_1, x_2) \in \Delta^2$, let $\mathcal{E}_x$ denote the set of pullbacks $s = F^*h$ of Carathéodory extremal functions h at $F(x) = p$ such that $h(p) = 0$. We note that

(i) for any $s \in \mathcal{E}_x$, $\frac{\partial s}{\partial z_1}(x_1, z_2)$ is independent of z_2;

(ii) for any $\gamma \in \Gamma$, $s \in \mathcal{E}_x$ implies that $s \circ \gamma \in \mathcal{E}_{\gamma^{-1}(x)}$.

Write $\mathcal{E}$ for the union of all $\mathcal{E}_x$, as x ranges over Δ^2. For $x \in \Delta^2$ and positive integers k we define inductively $\mathcal{E}_{x,k}$; $\mathcal{E}_{x,1} = \mathcal{E}_x$; to be the set of all elements in $\mathcal{E}_x$ such that $\frac{\partial^i s}{\partial z_1^i}(x_1, z_2)$ is independent of z_2 for any nonnegative integer $i \leq k$. Write $\mathcal{E}_k$ for the union of all $\mathcal{E}_{x,k}$ as x ranges over Δ^2. Clearly $\mathcal{E}_k \subset \mathcal{E}_\ell$ whenever $k \geq \ell$. We define $\mathcal{E}_{x,\infty} := \bigcap_{k \geq 1} \mathcal{E}_{x,k}$, $\mathcal{E}_\infty = \bigcup_{x \in \Delta^2} \mathcal{E}_{x,\infty}$. For each positive integer k we are going to establish:

$$(*)_k \qquad\qquad \mathcal{E}_k \neq \emptyset.$$

Denote by $(*)$ the statement that $(*)_k$ holds for all positive integers k. Proposition 2 consists of the statement that $\mathcal{E}_\infty \neq \emptyset$. We assert that this follows from $(*)$. To see this, let $s_k \in \mathcal{E}_k$. Let $K \subset \Delta^2$ be a compact subset which contains a fundamental domain of Δ^2 with respect to Γ. Suppose $s_k \in \mathcal{E}_{x,k}$. Let $\gamma_k \in \Gamma$ be an element such that $\gamma_k^{-1}(x) \in K$. Then, $s_k \circ \gamma_k \in \mathcal{E}_{\gamma_k^{-1}}(x)$. In what follows we replace s_k by $s_k \circ \gamma_k$. Then, for some $x_k \in K$; $x_k = (x_{k,1}, x_{k,2})$; $s_k(x_k) = 0$ and $\frac{\partial s}{\partial z_1}(x_{k,1}, z_1) = a_k$ for some constant $a_k \neq 0$. Since $|a_k| = |\frac{\partial s}{\partial z_1}(x_{k,1}, z_2)| = \frac{c}{1-|x_{k,1}|^2}$, we conclude from $x_k \in K$ that $|a_k|$ is uniformly bounded from below by some positive number b. (In this passage the suffix in x_k carries two different meanings but the context should make it clear.) Since $s_k : \Delta^2 \to \Delta$, $\{s_k\}$ constitutes a normal family. Passing to a subsequence we may assume that x_k converges to $x_\infty \in K$, $x_\infty = (x_{\infty,1}, x_{\infty,2})$, and that s_k converges uniformly on compact sets to some holomorphic function $s : \Delta^2 \to \Delta$ such that $s(x_\infty) = 0$, and $\frac{\partial s}{\partial z_1}(x_{\infty,1}, z_2) = a_\infty$ such that $|a_\infty| \geq b > 0$. Write now $s_k = F^* h_k$ for some Carathéodory extremal function h_k on $\tilde{N}$. Then $h_k : \tilde{N} \to \Delta$ forms a normal family, and we may assume without loss of generality that h_k converges to some bounded holomorphic function h_∞ such that $s_\infty = F^* h_\infty$. From this we conclude that s_∞ is the pull-back of a Carathéodory extremal function. It follows that $s_\infty \in \mathcal{E}_{x_\infty,k}$ for every positive integer k, i.e., $s_\infty \in \mathcal{E}_\infty$, as asserted.

It remains to establish $(*)$, which we will do by induction. For $k = 1$ we know actually that $\mathcal{E}_{x,1} \neq \emptyset$ for any $x \in \Delta^2$, by Finsler metric rigidity and the resulting Proposition 1. Suppose $(*)_k$ is established. Let $s \in \mathcal{E}_{k,x}$ for some $x \in \Delta^2$, $x = (x_1, x_2)$. For $a \in \Delta$ let $\psi_a \in Aut(\Delta)$ be such that $\psi_a(0) = a$. Writing $t = s \circ (\psi_{x_1}, id)$ we have $t(0, z_2) = 0$ and $t(z_1, z_2) = c_1 z_1 + c_2 z_1^2 + \cdots + c_k z_1^k + c_{k+1}(z_2) z_1^{k+1} + \cdots$. t is uniquely determined up to a rotation of Δ, so that $|c_{k+1}(z_2)| := \lambda_{k+1}(s; z_2)$ is completely determined by s and z_2. Note that for any $\gamma \in \Gamma$, $\gamma = (\gamma_1, \gamma_2)$, $\lambda_{k+1}(s; z_2) = \lambda_{k+1}(s \circ \gamma^{-1}; \gamma_2(z_2))$. Furthermore, $\lambda_{k+1}(s; z_2)$ is uniformly bounded from above independent of z_2 and of s, by Cauchy estimates. Let μ be the supremum of all $\lambda_{k+1}(s; z_2)$ as s ranges over all of $\mathcal{E}_k$ and z_2 runs over Δ. Then, there exist $x_m = (x_{m,1}, x_{m,2}) \in \Delta^2$, $s_m \in \mathcal{E}_{x_m,k}$, and $y_m = (x_{m,1}, y_{m,2}) \in \Delta$ such that, writing $\lambda_{k+1}(s_m; y_{m,2}) := \mu_k$, μ_k increases and converges to μ. Let $\gamma_m \in \Gamma$ be such that $\gamma_m(x_{m,1}, y_{m,2}) = u_m = (u_{m,1}, u_{m,2}) \in K$. Write $\sigma_m := s_m \circ \gamma_m^{-1}$. Then, $\sigma_m \in \mathcal{E}_{u_m,k}$, and $\lambda_{k+1}(\sigma_m; u_{m,2}) = \mu_k$. Passing to a subsequence if necessary, $u_m \in K$ converges to some $u = (u_1, u_2) \in K$ and the normal family $\{\sigma_m\}$ converges uniformly on compact subsets of Δ^2 to some holomorphic function $\sigma \in \mathcal{E}_{u,k}$. Furthermore, $\lambda_{k+1}(\sigma; u_2) = \mu$, while $\lambda_{k+1}(\sigma; z_2) \leq \mu$ for any $z_2 \in \Delta$. By the Maximum Principle we conclude that $c_{k+1}(z_2)$ must be a constant, so that $\sigma \in \mathcal{E}_{k+1}$, as desired. This proves $(*)$ by induction, and the proof of Proposition 2 is complete. $\square$

Remarks. Regarding ($*$) from the Density Lemma we deduce readily that at any $x \in \Delta^2$ and for any positive integer k, $\mathcal{E}_{x,\infty} \neq \emptyset$. We will refer to the existence of extremal functions h in Proposition 2, where F^*h depends only on one direct factor, as the Splitting Phenomenon.

Proof of Theorem 1' for Δ^n in the cocompact case. Consider the case of the bidisk. The assumptions in Proposition 1 and 2 that $f : X \to N$ is generically an immersion was just for linguistic convenience. As is apparent from the proofs there the hypothesis of Theorem 1' already ensures the applicability of analogues of Propositions 1 and 2 since we can work with the foliations on Δ^2 by pulling back extremal functions. We claim that $f : X \to N$ is unramified, i.e., equivalently, $F : \Delta^2 \to \tilde{N}$ is unramified. To this end let $x \in \Delta^2$ and $\xi \in T_x(\Delta^2)$, $x = (x_1, x_2)$, such that $dF(\xi) = 0$. By the proof of Proposition 1 there exist Carathéodory extremal functions h_1 and h_2 on $\tilde{N}$ such that for $s_i = h_i \circ F$; $i = 1, 2$; we have $s_1(x_1, z_2) = s_1(x_1, 0)$, $s_2(z_1, x_2) = s_2(0, x_2)$. Write $H = (h_1, h_2)$. From $dF(\xi) = 0$ it follows that $0 = dH(dF(\xi)) = (ds_1(\xi_1), ds_2(\xi_2)) = \big(\xi_1 \frac{\partial s_1}{\partial z_1}(x_1), \xi_2 \frac{\partial s_2}{\partial z_2}(x_2)\big)$, so that $\xi_1 = \xi_2 = 0$, *i.e.*, $\xi = 0$. In other words, F is unramified, as claimed. To prove Theorem 1' for the bidisk it remains to show that F separates points.

Suppose $x, y \in \Delta^2$, $x \neq y$, are such that $F(x) = F(y)$. Let now h be a Carathéodory extremal function as obtained in Proposition 2 such that $h \circ F(z_1, z_2) = s(z_i)$ for $i = 1$ or 2. In what follows take $i = 1$. Write $\Phi : \Gamma \to \pi_1(N)$ for the homomorphism f_* induced by f, identifying $\pi_1(N)$ with the group of Deck transformations of the covering map $\pi : \tilde{N} \to N$. Then, for any $\gamma \in \Gamma$, $z \in \Delta^2$, we have $F(\gamma z) = \Phi(\gamma)(F(z))$, so that

$$F(\gamma x) = \Phi(\gamma)(F(x)) = \Phi(\gamma)(F(y)) = F(\gamma y)\ ;$$
$$s(\gamma_1 x_1) = h(F(\gamma x)) = h(F(\gamma y)) = s(\gamma_1 y_1)\ ,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\gamma = (\gamma_1, \gamma_2)$. By the Density Lemma, as γ ranges over Γ, γ_1 ranges over a dense subset of $Aut(\Delta)$, with respect to the complex topology. We may take $x_1 = 0$. In particular, given any $\theta \in \mathbb{R}$ we can choose $\gamma = (\gamma_{n,1}, \gamma_{n,2})$ such that $\gamma_{n,1}(z_1)$ converges to $e^{i\theta} z_1$. It follows that for any $\theta \in \mathbb{R}$, $s(e^{i\theta} y_1) = \lim_{n\to\infty} s(\gamma_{n,1} y_1) = \lim_{n\to\infty} s(\gamma_{n,1}(0)) = s(0)$, so that s is constant on the circle of radius $|y_1|$. Since $y_1 \neq 0$, s must be constant, a contradiction. Thus $F(x) = F(y)$ implies $x_1 = y_1$. Similarly $F(x) = F(y)$ implies $x_2 = y_2$, so that $x = y$, i.e., F is an embedding, as desired.

The proof of Theorem 1' for the polydisk Δ^n, follows *verbatim*. □

Proposition 1, which has led to the proof that $f : X \to N$ is an immersion, will be adapted to give the same statement of the general cocompact case. For the proof of separation of points in the general cocompact case, a straightforward adaptation of Proposition 2 fails completely. We will in its place formulate and prove a variant of Proposition 2 involving a new extremal problem. The proof necessitates new ideas.

§3 The Embedding Theorem in the locally irreducible case via a new extremal problem

(3.1) For the proof of the Embedding Theorem in the cocompact case we will need to prove analogues of Propositions 1 and 2. For this purpose it will be necessary to study the action of automorphism groups on spaces of totally-geodesic complex submanifolds using Moore's Ergodicity Theorem. We start with some preliminary discussion on implications of the latter theorem in our context.

For a connected real Lie group G and for any closed subgroup $S \subset G$, the left (*resp.* right) coset space G/S (resp. $S \setminus G$) inherits the canonical structure of a smooth manifold, on which G acts as diffeomorphisms. Although G/S resp. $S \setminus G$ may not carry a G-invariant measure the notion of a null subset is well defined, *viz.*, a set $E \subset G/S$ (resp. $S \setminus G$) is said to be a null subset if it is of measure zero with respect to any choice of a Riemannian metric on G/S (resp. $S \setminus G$). Given two closed subgroups $S_1, S_2 \subset G$ it makes sense therefore to talk about ergodicity of the left action of S_1 on G/S_2 (resp. the right action of S_2 on $S_1 \setminus G$. We have the following special case of [Zi, Corollary 2.2.3, p.18].

Lemma 1. *Let G be a connected real Lie group and $S_1, S_2 \subset G$ be closed subgroups. Then S_1 acts ergodically on the left on G/S_2 if and only if S_2 acts ergodically on the right on $S_1 \setminus G$.*

From now on Ω denotes an irreducible bounded symmetric domain of rank $r \geq 2$, $G = Aut_o(\Omega)$, $\Omega = G/K$. We apply Lemma 1 to the simple Lie group G, $S_1 = \Gamma \subset G$ a lattice, and $S_2 = H$ some noncompact closed subgroup G to be determined. From Moore's Ergodicity Theorem we conclude therefore that Γ acts ergodically on the left on G/H.

Since G is paracompact, Zimmer [Zi, Proposition 2.1.7, p.10] applies to give the following density result.

Lemma 2. *Let $H \subset G$ be a closed subgroup. Then there exists a null subset $E \subset G/H$ such that for any point $gH \in G/H - E$, the orbit $\Gamma(gH)$ is dense in G/H, in the metric topology on G/H defined by the canonical smooth structure on G/H.*

Remarks. In general the subset $E \subset G$ may be nonempty. An example close to what we are considering where $E \subset G/H$ is nonempty is the following. Let Ω be any bounded symmetric domain and $D \subset \Omega$ be a totally-geodesic complex submanifold. Then, the subgroup H of $G := Aut_o(\Omega)$ which fixes D as a subset is a noncompact closed Lie subgroup. The coset space G/H parametrizes the space of all totally-geodesic complex submanfolds in Ω congruent to D under the action of G. Denote by $o \in G/H$ the point corresponding to D itself. The inclusion $D \subset \Omega$ gives an inclusion $Aut_o(D) \subset G$. Let $\Gamma \subset G$ be a torsion-free irreducible lattice. If $\Gamma \cap Aut_o(D)$ is a lattice in $Aut_o(D)$, then the Γ-orbit of o is discrete in G/H. In fact, denoting by $\pi : \Omega \to X$ the canonical projection and write $Z := \pi(D)$, which is a subvariety of X from the assumptions, then $\pi^{-1}(Z)$ is closed and is the union of $\gamma(D)$ as γ ranges over Γ. This means that the Γ-orbit of o is a discrete subset of G/H.

By the Polydisk Theorem (stated in (1.3)) there is a totally-geodesic r-dimensional polydisk $P \subset \Omega$, and G acts transitively on the space of such polydisks. We give here a brief description of maximal polydisks, and refer the reader to Wolf [Wo, p.280] and Mok [M3, Chapter 3, p.89ff.] for details. Write $\Omega = G/K$ as in the above and use notations as in (1.1).

Write $\mathfrak{g}$ for the Lie algebra of real Lie group G, and denote by $\mathfrak{g}^{\mathbb{C}}$, etc. the complexification of the real Lie algebra $\mathfrak{g}$, etc. Let $\mathfrak{z} \subset \mathfrak{k}$ be the one-dimensional centre of $\mathfrak{k}$. We have $\mathfrak{k} = \mathfrak{k}_s + \mathfrak{z}$. Write $\mathfrak{h} = \mathfrak{h}_s + \mathfrak{z}$. Then, $\mathfrak{h} \subset \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$. Write $\mathfrak{h}^* = Hom(\mathfrak{h}, \mathbb{R})$ and $\mathfrak{h}^*_{\mathbb{R}} = \sqrt{-1}\mathfrak{h}^*$. Let $\Phi \subset \mathfrak{h}^*_{\mathbb{R}}$ be the space of $\mathfrak{h}^{\mathbb{C}}$-roots ρ of $\mathfrak{g}^{\mathbb{C}}$. The root space belonging to $\rho \in \Phi$ is one-dimensional, generated by $E_\rho \in \mathfrak{g}^{\mathbb{C}}$ satsifying $[h, E_\rho] = \rho(h)E_\rho$ for any $h \in \mathfrak{h}^{\mathbb{C}}$. Choosing in an appropriate way a positive Weyl chamber on $\mathfrak{h}^*_{\mathbb{R}}$ we have the notion of positive *resp.* negative roots. Let θ be the Lie algebra involution on $\mathfrak{g}$ corresponding to the symmetry of G/K at $o = eK$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ for the Cartan decomposition with respect to the involution θ. The space of roots decompose into $\Phi_K \cup \Phi_M$ consisting of compact *resp.* noncompact roots according to whether $E_\rho \in \mathfrak{k}^{\mathbb{C}}$ or $E_\rho \in \mathfrak{m}^{\mathbb{C}}$. Then, $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$, where $\mathfrak{m}^+$ is spanned by root spaces belonging to Φ^+_M, the set of positive noncompact roots, and $\mathfrak{m}^- = \overline{\mathfrak{m}^+}$. $\mathfrak{m}^+$ *resp.* $\mathfrak{m}^-$ can be canonically identified with $T_o(\Omega)$ resp. $\overline{T_o(\Omega)}$. We say that two roots φ_1, φ_2 are strongly orthogonal if and only if neither $\varphi + \psi$ nor $\varphi - \psi$ is a root. Let $\Psi \subset \Phi^+_M$ be a maximal subset of roots which are mutually orthogonal. Then $card(\Psi)$ agrees with the rank r of $\Omega = G/K$. Let $\mathfrak{a}^+ \subset \mathfrak{m}^+ \cong T_o(\Omega)$ be the complex vector subspace spanned by $\{E_\psi : \psi \in \Psi\}$, and write $\mathfrak{a}^- = \overline{\mathfrak{a}^+}$. Then $\mathfrak{a}^+ + \mathfrak{a}^- = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ for some maximal abelian subspace $\mathfrak{a} \subset m$. All $\mathfrak{a}^+ \subset \mathfrak{m}^+$ are conjugate under the isotropy action of K. Furthermore, for each $\mathfrak{a}^+$ there is a unique totally-geodesic polydisk $P \subset \Omega$, $P \cong \Delta^r$, passing through o such that $T_o(P) = \mathfrak{a}^+$. P will be called a maximal polydisk. From the general theory of Riemannian symmetric spaces we have readily (cf. Helgason [He, Lemma 6.3, pp.247-8].)

Lemma 3. *Fix a maximal polydisk $P \subset \Omega$ passing through o and write $T_o(P) := \mathfrak{a}^+$. Then, $T_o(\Omega) = \bigcup_{k \in K} k(a^+)$, where k acts on $T_o(\Omega)$ by the isotropy action. As a consequence, given any (nonzero) $\xi \in T_o(\Omega)$ there exists some maximal polydisk $Q \subset \Omega$ passing through o such that $\xi \in T_o(Q)$.*

For the proof of Theorem 1 we have the following immediate analogue of Proposition 1.

Proposition 3. *Let Ω be an irreducible bounded symmetric domain of rank $r \geq 2$. Let $P \subset \Omega$ be a totally-geodesic polydisk of dimension r, as given by the Polydisk Theorem. Write $P \cong \Delta^r \cong \Delta \times \Delta^{r-1}$. Denote by $z = (z_1, \dots, z_r)$ Euclidean coordinates on $P \cong \Delta^r \subset \mathbb{C}^r$. Let $\Gamma \subset Aut_o(\Omega) := G$ be a torsion-free cocompact lattice; $X := \Omega/\Gamma$. Let N be a complex manifold, $f : X \to N$ be a holomorphic mapping and $F : \Omega \to \tilde{N}$ its lifting to universal covering spaces. Assume that there exists a bounded holomorphic function on $\tilde{N}$ which is nonconstant on $F(\Omega)$. Let $x = (x_1, \dots, x_r) \in P$ be an arbitrary point. Then, there exists a Carathéodory extremal function h on $\tilde{N}$, $s := h \circ F : \Omega \to \tilde{N}$, such that on the polydisk $P \cong \Delta^r \subset \mathbb{C}^r$ we have $s(x_1; z_2, \cdots z_r) = 0$ in terms of Euclidean coordinates, and, at any point of $F(\{x_1\} \times \Delta^{r-1})$, h is a Carathéodory extremal function on $\tilde{N}$ adapted to the vector $dF(\frac{\partial}{\partial z_1}) \neq 0$.*

From Proposition 3 we have readily

Proof that $f : X \to N$ is an immersion in Theorem 1 in the cocompact case. Equivalently we need to prove that $F : \Omega \to \tilde{N}$ is an immersion. For any $s \in \Omega$ and any nonzero vector $\xi \in T_x(\Omega)$ we have to prove that $dF(\xi) \neq 0$. By Lemma 3 there exists a maximal polydisk

$P \subset \Omega$ passing through x such that $\xi \in T_x(P)$. The argument for the proof that $F : \Omega \to \tilde{N}$ is an immersion in the case of the polydisk given in (2.2) in conjunction with Proposition 3, applies to show here that $F : \Omega \to \tilde{N}$ is an immersion. $\square$

As an analogue to the Splitting Phenomenon given by Proposition 2 for the bidisk, we have

Proposition 4. *With the same assumptions as in the statement of Proposition 3, there exists bounded holomorphic function $h : \tilde{N} \to \Delta$, $s := h \circ F : \Omega \to \tilde{N}$, such that on the maximal polydisk $P \cong \Delta^r \subset \mathbb{C}^n$ we have $s(z_1; z_2, \cdots, z_r) = s(z_1)$ in terms of Euclidean coordinates.*

There is an essential difference in the formulation of Proposition 4 from that of Proposition 2. The bounded holomorphic function $h : \tilde{N} \to \Delta$ will not be a Carathéodory extremal function. It will rather be an extremal function for an extremal problem on Ω to be defined in the next section. The extremal problem will be constructed in such a way that any extremal function for the problem adapted to a given maximal polydisk will automatically have the property as stated in the conclusion of Proposition 4.

(3.2) Recall that $\mathcal{H}$ is the space of holomorphic maps $h : \tilde{N} \to \Delta$. Denote by $\mathcal{F}$ the space of holomorphic maps $s : \Omega \to \Delta$ of the form $s = F^*h$, $h \in \mathcal{H}$. We are going to define an extremal problem on the irreducible bounded symmetric domain Ω of rank ≥ 2 for the space $\mathcal{F}$, as follows. For every $x \in \Omega$, and every characteristic vector $\alpha \in \mathcal{S}'_x$, there is a unique minimal disk Δ_α such that $x \in \Delta_\alpha$ and such that α is tangent to Δ_α. Fix a positive number ε. The ensuing construction depends on the choice of ε. For convenience we will choose ε sufficiently small, in a way to be specified later in (3.4). Denote by $B_\alpha(x, \varepsilon)$ the geodesic ball on $(\Delta_\alpha, g|_{\Delta_\alpha})$ centred at x and of radius ε, and by $S_{\alpha,\varepsilon}$ the geodesic circle $\partial B_\alpha(x, \varepsilon)$ on Δ_α. Let $s \in \mathcal{F}$. We define a length function $\| \cdot \|_s$ on $\mathcal{S}'_\Omega$, as follows. At every point $y \in S_{\alpha,\varepsilon}$ let $\alpha_y \in T_{\Delta_\alpha}$ be a tangent vector of the same length as α with respect to the canonical Kähler-Einstein metric. Define $u(s, \alpha, y) = \|ds(\alpha_y)\|_{ds^2_\Delta}$. Then $u(s, \alpha, y) \geq 0$ is defined independent of the choice of α_y. Let $\|\alpha\|_s$ be the average of $u(s, \alpha, y)$ as y runs over S_α, with respect to a measure of total mass 1 on S_α invariant under the isotropy group of $(\Delta_\alpha, g|_{\Delta_\alpha})$. Define now

$$\|\alpha\|_{e(\mathcal{F})} := sup\{\|\alpha\|_s : s \in \mathcal{F}\} \ .$$

Writing $\|0\| = 0$ we have a length function defined on $\mathcal{S}'_\Omega \cup \{0\}$, which corresponds to a Hermitian metric on the tautological line bundle L over the characteristic bundle $\mathcal{S}_\Omega$. From Cauchy estimates, the length functions $\|\alpha\|_s$ are uniformly Lipschitz on any compact subset of $\mathcal{S}'_\Omega$, so that the suprema define a Lipschitz function on $\mathcal{S}_\Omega$, and $\| \cdot \|_{e(\mathcal{F})}$ is a continuous Hermitian metric on $L|_{\mathcal{S}_\Omega}$. For any $\gamma \in \Gamma$, and $s \in \mathcal{F}$, where $s = F^*h$ for some $h \in \mathcal{H}$, we have $\gamma^* s = F^*(\Phi_\gamma^* h)$, $\Phi_\gamma = f_*\gamma \in \pi_1(N)$, so that $\mathcal{F}$ is invariant under the canonical action of Γ. As a consequence $\| \cdot \|_{e(\mathcal{F})}$ descends to a continuous Hermitian metric on the tautological line bundle over $\mathcal{S}_0$. On the quotient manifold X we will denote the tautological line bundle and the corresponding length function by the same symbols L resp. $\| \cdot \|_{e(\mathcal{F})}$. $e(\mathcal{F})$ will also be used to denote the continuous Hermitian pseudometric on L.

The idea of proof of Proposition 4 is to apply Finsler metric rigidity to $(L|_\mathcal{S}, e(\mathcal{F}))$. We note that the argument of Finsler metric rigidity applies, provided that we have a continuous

Hermitian metric on $L|_{\mathcal{S}}$ of nonpositive curvature in the sense of (1.2). However, it is not clear from the construction of our length function $\|\cdot\|_{e(\mathcal{F})}$ that the curvature is nonpositive. As a matter of fact, the averaging process applies to local log-plurisubharmonic functions defined on some open subsets of $\mathcal{S}$, and the averaging process in general does not give log-plurisubharmonic functions, for the following reason. Denote by $\pi : \mathcal{S} \to X$ the canonical projection. Let φ be a log-plurisubharmonic function on some open subset U of X and consider the log-plurisubharmonic function $\pi^*\varphi$. Let $V \subset U$ be a nonempty relatively compact open subset and choose $\varepsilon > 0$ sufficiently small so that the averaging process makes sense on $\pi^{-1}(V)$. φ is by definition constant on the fiber $\pi^{-1}(x)$ for any $x \in U$. However, the averaging at $[\alpha] \in \mathcal{S}_x$ depends on $[\alpha]$, and obviously one can choose U, φ smooth, $V \subset U$ and $x \in X$ such that the resulting function φ_ε is not constant on $\pi^{-1}(x)$. But then φ_ε cannot be log-plurisubharmonic on the projective variety $\mathcal{S}_x$. By the same reasoning one cannot expect in general to get continuous Hermitian metrics of nonpositive curvature by the averaging process described in the last paragraph.

Although we cannot expect *a priori* that $(L|_{\mathcal{S}}, e(\mathcal{F}))$ is of nonpositive curvature, we know nonetheless that its restriction to certain submanifolds of $\mathcal{S}$ is of nonpositive curvature, and we are going to show that this is enough to establish metric rigidity.

(3.3) Let $x \in \Omega$ and $\alpha \in \mathcal{S}'_x$. Since $\mathcal{F}$ is a normal family there exists a bounded holomorphic function $s \in \mathcal{F}$ such that $\|\alpha\|_{e(\mathcal{F})}$ agrees with $\|\alpha\|_s$. We call s an $e(\mathcal{F})$-extremal function at x adapted to α. In analogy with Proposition 1 on Carathéodory extremal functions on $\tilde{N}$, $e(\mathcal{F})$-extremal functions enjoy special properties when restricted to maximal polydisks. In fact, they are more rigid so that the analogue of Propoition 2 is automatic. We are going to prove the following result concerning $e(\mathcal{F})$-extremal functions which imply Proposition 4.

Proposition 4'. *Let $P \subset \Omega$ be a maximal polydisk, $P \cong \Delta^r$, and use Euclidean coordinates of the latter as coordinates for P. Let $x \in P, x = (x_1; x')$ and denote by $P' \subset P$ the polydisk corresponding to $\{x_1\} \times \Delta^{r-1}$. Let α be a nonzero characteristic vector at x tangent to the minimal disk D corresponding to $\Delta \times \{x'\}$ and denote by s an $e(\mathcal{F})$-extremal function at x adapted to α. Then $s(z_1; z_2, \cdots, z_r) = s(z_1)$.*

For the proof of Proposition 4' we will show that $e(\mathcal{F})$ defines a continuous Hermitian metric on $L|_{\mathcal{S}}$ which is of nonpositive curvature when restricted to certain submanifolds. We will then formulate a version of metric rigidity for continuous Hermitian metrics on $L|_{\mathcal{S}}$ which shows that the partial nonpositivity of curvatures at our disposal is enough. The metric rigidity result will be applied to prove that the $e(\mathcal{F})$-extremal functions have the splitting property when restricted to maximal polydisks. To start with we have the following lemma which is relevant to our averaging process on geodesic circles on minimal disks.

Lemma 4. *Let $U \subset \mathbb{C}^n$ be an open subset, and $a, b \in \mathbb{R}; a < b$. Let $u : [a,b] \times U \to \mathbb{R}$ be a continuous function such that for any $t \in [a,b]$, writing $u_t(z) := u(t,z)$, $u_t : U \to \mathbb{R}$ is plurisubharmonic. Define $\varphi : U \to R$ by $\varphi(z) := \log \int_a^b e^{u_t(z)} dt$. Then, φ is plurisubharmonic. Moreover, $e^\varphi \sqrt{-1}\partial\overline{\partial}\varphi \geq \int_a^b e^{u_t} \sqrt{-1}\partial\overline{\partial} u_t \, dt$ in the sense of currents.*

Proof. The problem being local, it is enough to prove Lemma 4 with U replaced by a Euclidean ball $B \subset U$ relatively compact in U. We may extend u to $I \times B$ for some open interval I containing $[a, b]$ in such a way that $u(z) := u(t, z)$ remains plurisubharmonic for $t \in I$. Applying smoothing convolution operators we get smooth functions $u^\epsilon : [a, b] \times B \to \mathbb{R}$ such that $u_t^\epsilon : B \to \mathbb{R}$ is plurisubharmonic and u_t^ϵ converges uniformly on B to u_t for each $t \in [a, b]$ as $\in$ converges to 0. Defining φ^ϵ using u^ϵ in place of u, we are going to verify the lemma for the function φ^ϵ in place of φ. Since the uniform limit of plurisubharmonic functions is plurisubharmonic, and $e^{\varphi^\epsilon}\sqrt{-1}\partial\overline{\partial}\varphi^\epsilon$ converges to $e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi$ as positive currents, etc., we will have proven Lemma 4 by letting ϵ tend to 0.

From now on we change the meaning of the notations and assume that u is already smooth. The function φ_t is defined as an integral, and can thus be approximated by Riemann sums. Expressing φ as a uniform limit of Riemann sums, it suffices to prove the analogue of the lemma for the sum of a finite number of smooth plurisubharmonic functions in place of an integral over $[a, b]$. Thus, we have $u_i : [a, b] \times U \to \mathbb{R}$ smooth and plurisubharmonic for $1 \le i \le N$, and, defining a new $\varphi(z) = \log(e^{u_1(z)} + \cdots + e^{u_N(z)})$, we have to prove that φ is plurisubharmonic, and that $e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \ge \sum e^{u_k}\sqrt{-1}\partial\overline{\partial}u_k$ as smooth (1,1)-forms. We give a geometric proof, as follows. For each i, $1 \le k \le N$, we can define a Hermitian metric on the trivial line bundle $\mathcal{O}$ on U by writing $\|1\|_{h_k} = e^{u_k}$. Then, $-\sqrt{-1}\partial\overline{\partial}\varphi$ is the curvature form of the Hermitian line bundle $(\mathcal{O}, h_1 + \cdots h_N)$, while $-\sqrt{-1}\partial\overline{\partial}u_k$ is the curvature form of the Hermitian line bundle $(\mathcal{O}, h_k)$, $1 \le k \le N$. The inequality $e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \ge \sum e^{u_k}\sqrt{-1}\partial\overline{\partial}u_k$ and in particular the plurisubharmonicity of φ follow from the Gauss equation on curvatures of Hermitian holomorphic vector subbundles, when we regard $(\mathcal{O}, h_1 + \cdots + h_N)$ as a Hermitian holomorphic vector subbundle of $(\mathcal{O}^n, h_1 \oplus \cdots \oplus h_N)$ by means of the diagonal embedding, as desired. Note that if we introduce a scaling constant $\lambda \in \mathbb{R}$, $\lambda > 0$, and consider instead $\tilde{\varphi} = \log(\lambda(e^{u_1} + \cdots + e^{u_N}))$, it corresponds to replacing u_k by $u_k + \log\lambda$, and the inequality becomes $e^{\tilde{\varphi}}\sqrt{-1}\partial\overline{\partial}\tilde{\varphi} \ge \sum e^{u_k + \log\lambda}\sqrt{-1}\partial\overline{\partial}u_k = \lambda\sum e^{u_k}\sqrt{-1}\partial\overline{\partial}u_k$. This is the case for Riemann sums, which allows us to pass to limits to establish Lemma 4 for u smooth and hence for u continuous.

Alternatively, without using geometry, the inequality $e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \ge \sum e^{u_k}\sqrt{-1}\partial\overline{\partial}u_k$ (with the new definition of φ) in case of $N = 2$ follows from the direct computation

$$\begin{aligned}
&(e^{u_1} + e^{u_2})\sqrt{-1}\partial\overline{\partial}\log(e^{u_1} + e^{u_2}) \\
&= e^{u_1}\sqrt{-1}\partial\overline{\partial}u_1 + e^{u_2}\sqrt{-1}\partial\overline{\partial}u_2 + \frac{e^{u_1+u_2}}{e^{u_1} + e^{u_2}}\sqrt{-1}(\partial u_1 - \partial u_2) \wedge \overline{(\partial u_1 - \partial u_2)}\ .
\end{aligned}$$

The case of general N follows by induction. $\square$

The following result is a strengthening of the principle underlying the proof of the Hermitian Metric Rigidity Theorem of Mok [M3] in the compact case. In [M1,2] we worked with Hermitian holomorphic line bundles of nonpositive curvature in the generalized sense, and global nonpositivity was required to justify the integration by parts. Here we observe that for certain problems it is enough that the continuous Hermitian metrics are of nonpositive curvature when restricted to certain complex submanifolds. Suppose L is some holomorphic line bundle, and h_1, h_1 are continuous Hermitian metrics on L. Then $h_2 = e^u h_1$, and e^u can be regarded as a Hermitian metric on the trivial line bundle. With this understanding the principle can be formulated entirely in terms of continuous functions, as follows.

Proposition 5. *Let (Z,ω) be an m-dimensional compact Kähler manifold, and θ be a smooth closed nonnegative (1,1)-form on Z such that $Ker(\theta)$ is of constant rank $q>0$ everywhere on Z. Denote by $\mathcal{K}$ the foliation on Z with holomorphic leaves defined by the distribution $Re(Ker(\theta))$. Let $u : Z \to \mathbb{R}$ be a continuous function whose restriction to every leaf $\mathcal{L}$ of the foliation $\mathcal{K}$ is plurisubharmonic. Then, the restriction of u to every leaf $\mathcal{L}$ is pluriharmonic. If u is Lipschitz, then it is constant on every leaf $\mathcal{L}$. If in addition there is a dense leaf of $\mathcal{K}$, then u is constant on Z.*

Proof. Let U be any small coordinate open set on Z, $(\eta_1,\cdots,\eta_q)$ be a smooth basis of $Ker(\theta)|_U$ and complete it to a basis $(\eta_1,\cdots,\eta_m)$ of T_U. Denote the corresponding dual basis of T_U^* by $(\nu_1,\cdots,\nu_m)$. Consider the currents on U given by

$$T = \sqrt{-1}\,\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1}\ , \quad S = dT\ .$$

Then,

$$\begin{aligned}
&S(\sqrt{-1}\eta_1 \wedge \overline{\eta_1} \wedge \cdots \wedge \sqrt{-1}\eta_m \wedge \overline{\eta_m}) \\
&= (\sqrt{-1}\partial\overline{\partial}u \wedge \omega^{q-1})(\sqrt{-1}\eta_1 \wedge \overline{\eta_1} \wedge \cdots \wedge \sqrt{-1}\eta_q \wedge \overline{\eta_q}) \times \\
&\quad \times \theta^{m-q}(\sqrt{-1}\eta_{q+1} \wedge \overline{\eta_{q+1}} \wedge \cdots \wedge \sqrt{-1}\eta_m \wedge \overline{\eta_m})\ ,
\end{aligned}$$

which follows readily from the fact that $\theta(\eta_k \wedge \overline{\eta_k}) = 0$ whenever $1 \le k \le q$. Since $u|_{\mathcal{L}}$ is plurisubharmonic on every leaf $\mathcal{L}$, the (m,m)-current S is a nonnegative measure. From

$$S(1) = \int_Z \sqrt{-1}\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} = 0$$

it follows readily that $S \equiv 0$ and hence

$$\sqrt{-1}\partial\overline{\partial}u \wedge \omega^{q-1}(\sqrt{-1}\eta_1 \wedge \overline{\eta_1} \wedge \cdots \wedge \sqrt{-1}\eta_q \wedge \overline{\eta_q}) = 0\ ,$$

which implies that

$$\sqrt{-1}\partial\overline{\partial}u \wedge \omega^{q-1}|_{\mathcal{L}_o} \equiv 0\ .$$

for almost every local leaf $\mathcal{L}_o$, by Fubini's Theorem. Since by assumption the restriction of u to every leaf $\mathcal{L}$ is plurisubharmonic, $\omega|_{\mathcal{L}}$ is a Kähler form, and u is continuous, it follows that u is pluriharmonic on each leaf $\mathcal{L}$. When u is Lipschitz we can perform integration by parts to get

$$0 = \int_Z -\sqrt{-1}u\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} = \int_Z \sqrt{-1}\partial u \wedge \overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1},$$

which forces u to be constant on each (complete) leaf $\mathcal{L}$ of $\mathcal{K}$. If there exists a leaf $\mathcal{L}$ dense on Z, obviously the continuous function u is constant on the topological closure Z, so that u is constant, as desired. □

We now apply Proposition 5 to $Z = \mathcal{S}$ and to holomorphic line bundle $L|_{\mathcal{S}}$, equipped with two Hermitian metrics $h_1 = \hat{g}|_{\mathcal{S}}$ induced by the canonical Kähler-Einstein metric g, and $h_2 = e(\mathcal{F}) + \hat{g}$. We are going to prove

Proposition 6. *Let Ω be a an irreducible bounded symmetric of rank ≥ 2, and g be the canonical Kähler-Einstein metric on Ω. Let $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Denote by $\hat{g}$ the canonical Hermitian metric on the tautological line bundle L, and by $\mathcal{S} \subset \mathbb{P}T_X$ the characteristic bundle on X. Let $e(\mathcal{F})$ be the continuous Hermitian metric on $L|_{\mathcal{S}}$ as defined in (3.2). Then, there is some constant $c > 0$ such that $e(\mathcal{F}) \equiv c\hat{g}$ on $L|_{\mathcal{S}}$.*

Proof. Consider the nonnegative closed smooth (1,1)-form $-c_1(L, \hat{g})$ on $\mathbb{P}T_X$ and let α be a characteristic vector on X, $\alpha \in \mathcal{S}'_x$. Then, from [M1], $Ker(-c_1(L, \hat{g})([\alpha]) \subset T_{[\alpha]}(\mathcal{S})$. Write $\theta = -c_1(L, \hat{g})|_{\mathcal{S}}$.

Lift α to a characteristic vector at $o \in \Omega$ and maintain the same notation for the lifted vector. In the notations of (3.1), with respect to some choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we may take α to be a root vector belonging to a root $\psi \in \Psi$. Let $\psi^{\perp}$ be the set of all positive noncompact roots φ such that $\psi - \varphi$ is not a root. Then, the root vectors $\{E_{\varphi} : \varphi \in \psi^{\perp}\}$ span the null space $\mathcal{N}_{\alpha}$ associated to α, i.e., the space of all $\eta \in T_o(\Omega)$ such that $R_{\alpha\bar{\alpha}\eta\bar{\eta}} = 0$. Write $q := \dim \mathcal{N}_{\alpha}$. Then, using Lie triple systems arising from $\psi^{\perp}$, we see that $\mathcal{N}_{\alpha} = T_o(N)$ for some q-dimensional totally-geodesic complex submanfold $N \subset \Omega$. Moreover $\mathbb{C}\alpha + \mathcal{N}_{\alpha} = T_o(M)$ for some totally-geodesic complex submanifold $M \subset \Omega$ which can be canonically identified with $\Delta \times N$. From the construction $M \supset P$ for the maximal polydisk $P \subset \Omega$ determined by Ψ. We may write $N = N_{\alpha}, M = M_{\alpha}$. Because of the product structure $M_{\alpha} \cong \Delta_{\alpha} \times N_{\alpha}$, there is a parallel vector field A on N_{α} which gives the vector α at o. Write $\mathcal{A} \subset \mathcal{S}_{\Omega}|_{N_{\alpha}}$ for its tautological lifting. Denote also by $\pi : \mathcal{S}_{\Omega} \to \mathcal{S}$ the canonical projection. Then, the leaf $\mathcal{L}$ at $[\alpha] \in \mathcal{S}_x$ is precisely $\pi(\mathcal{A})$.

The canonical Kähler-Einstein metric g on Ω restricts to a product metric on $M = \Delta \times N$. It follows that $(L|_{\mathcal{L}}, \hat{g}|_{\mathcal{L}})$ is the trivial Hermitian holomorphic line bundle. On the other hand, $h_2 = e(\mathcal{F})$, when restricted to $\mathcal{L}$, gives a continuous Hermitian metric of nonpositive curvature on $L|_{\mathcal{L}}$, by Lemma 4, again by using the product structure $M = \Delta \times N$ on Ω. Writing $h_2 = e^u h_1$ we conclude that u is plurisubharmonic on each leaf $\mathcal{L}$ of the foliation $\mathcal{K}$ on $Z = \mathcal{S}$ defined by $Re(Ker(\theta))$. By Proposition 5, we conclude that $u|_{\mathcal{L}}$ is pluriharmonic. From Cauchy estimates it follows readily that u is Lipschitz. Furthermore, from Moore's Ergodicity Theorem and Lemma 2 there exists a dense leaf $\mathcal{L}$ of $\mathcal{K}$. It follows that u is constant on $\mathcal{S}$. In other words, $e(\mathcal{F})$ agrees with some constant multiple of $\hat{g}$ on $L|_{\mathcal{S}}$, as desired. $\square$

Remarks. We note that for our application to study $e(\mathcal{F})$-extremal functions on Ω the weaker statement that $u|_{\mathcal{L}}$ is pluriharmonic for every leaf $\mathcal{L}$ of $\mathcal{K}$ is already sufficient.

Proof of Proposition 4'. Let $x \in \Omega$, $P \subset \Omega$ be a maximal polydisk, and $\alpha \in T_x(X)$ be a characteristic vector. Let $s \in \mathcal{F}$ be an $e(\mathcal{F})$-extremal function adapted to α at x, i.e., $\|\alpha\|_s = \|\alpha\|_{e(\mathcal{F})}$. Write $P \cong \Delta^r = \Delta \times \Delta^{r-1}$. In the notations of the proof of Proposition 6 we have totally-geodesic complex submanifolds $N_{\alpha}, M_{\alpha} \subset \Omega$, $M_{\alpha} \cong \Delta_{\alpha} \times N_{\alpha}$. (From now on we drop the subscript α.) Recall that $P \subset M$. We may assume x to be the origin and use Euclidean coordinates $(z_1; z_2, \cdots, z_r)$ of Δ^r as coordinates for $P \cong \Delta^r$. We may take α to be $\frac{\partial}{\partial z_1}$ at 0. By Proposition 5, $\|\frac{\partial}{\partial z_1}\|_{e(\mathcal{F})}$ is constant on $N \supset \{0\} \times \Delta^{r-1} := P'$. For any $z \in P$

write α_z for $\frac{\partial}{\partial z_1}$ at z and $\nu(z)$ for $ds(\alpha_z)$. Let $\delta > 0$ be such that $S_{\alpha,\varepsilon} = \{(\delta e^{i\theta}; o) : \theta \in \mathbb{R}\}$ for the geodesic circle $S_{\alpha,\varepsilon} \subset \Delta_\alpha$. Then, for $y \in P' \subset N$, $y = (0, y')$, we have

$$\|\alpha_y\|_{e(\mathcal{F})} \geq \|\alpha_y\|_s = \frac{a}{2\pi}\int_0^{2\pi} \|ds(\delta e^{i\theta}; y')\|_{ds^2_\Delta} d\theta = \frac{a}{2\pi}\int_0^{2\pi} \frac{|\nu(\delta e^{i\theta}, y')|}{1 - |s(\delta e^{i\theta}, y')|^2} d\theta := e^{\varphi(y)} ,$$

$$\|\alpha_o\|_{e(\mathcal{F})} = \frac{a}{2\pi}\int_0^{2\pi} \frac{|\nu(\delta e^{i\theta}, o)|}{1 - |s(\delta e^{i\theta}, o)|^2} = e^{\varphi(o)} ,$$

where a is the constant such that $\|\alpha_z\|_g = \frac{1}{a}$ for z belonging to the geodesic circle $S_{\alpha,\varepsilon}$, $\|\alpha_o\|_g = 1$. By Proposition $\|\alpha_y\|_{e(\mathcal{F})}$ is constant on P'. By Lemma 4, $\varphi(y)$ is plurisubharmonic in $y \in P'$ and attains its maximum at the origin. It follows that $\varphi \equiv C$ on P' for some constant $C > 0$. Write $e^{u_\theta(y)}$ for the integrand in the definition of $e^{\varphi(y)}$. Again by Lemma 4

$$e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq \frac{a}{2\pi}\int_0^{2\pi} e^{u_\theta}\sqrt{-1}\partial\overline{\partial}u_\theta \; d\theta$$

in the sense of currents. It follows that for almost all $\theta \in [0, 2\pi]$, u_θ is pluriharmonic. However, $\sqrt{-1}\partial\overline{\partial}u_\theta$ is the pull-back of the curvature form of (Δ, ds^2_Δ) by $\sigma_\theta : P' \to \Delta$, given by $\sigma_\theta(y') = s(\delta e^{i\theta}; y')$, so that σ_θ must be constant for almost all $\theta \in [0, 2\pi]$, hence for all θ by continuity. It follows that the $e(\mathcal{F})$-extremal function s must be of the form $s(z_1; z_2, \cdots, z_r) = s(z_1)$ when restricted to the polydisk P. The proof of Proposition 4' is complete, from which also Proposition 4 follows. □

(3.4) Using Proposition 4 we are readily to complete the proof of Theorem 1 in the cocompact case.

Proof of Theorem 1 in the cocompact case. Recall that the $e(\mathcal{F})$-extremal function $s : \Omega \to \Delta$ adapted to (P, P') is of the form $s(z_1, z_2, \ldots, z_r) = s(z_1)$ when restricted to $P \cong \Delta^r$. For any $\theta \in \mathbb{R}$ define $s_\theta : \Delta^r \to \Delta$ by $s_\theta(z_1; z_2, \cdots, z_r) = s(e^{i\theta}z_1; z_2, \cdots, z_r)$. We are going to show that for almost every maximal polydisk $P \subset \Omega$, and for any $\theta \in R$ we can write $s_\theta = F^*h_\theta|_P$ for some $h_\theta : \tilde{N} \to \Delta$. Consider the problem of classifying triples (P, P', α), where (P, P') is as in the above, and α is some characteristic vector of unit length with respect to the canonical Kähler-Einstein metric at a point $x \in P'$, $x = (x_1; x')$, tangent to the minimal disk $D \cong \Delta \times \{x'\}$. We declare (P, P', α) and (P, P', β) to be equivalent if $\rho_*(\alpha) = \rho_*(\beta)$ for the canonical projection $\rho : P \cong \Delta^r \to \Delta$ onto the first factor. The inclusion $P \subset \Omega$ gives an inclusion $Aut_o(P) \subset Aut_o(\Omega) = G$. Let $H \subset G$ be the closed subgroup which preserves the triple (P, P', α) up to equivalence. Then, H contains $\{id\} \times Aut(\Delta^{r-1})$, and must therefore be noncompact. We may assume that $\Gamma \subset Aut_o(\Omega) = G$. By Moore's Ergodicity Theorem and Lemma 1, Γ acts ergodically on G/H. Thus, by Lemma 2, there exists a null subset $E \subset G/H$ such that for every $p \in G/H - E$, the Γ-orbit of p is dense in G/H. We will say that (P, P', α) is generic if the corresponding point $p \in G/H$ lies outside E. Let (P, P', α) be a generic triple. For any fixed $\theta \in \mathbb{R}$, $(P, P', e^{i\theta}\alpha)$ is in the closure of the Γ-orbit of (P, P', α). Let $\gamma_i \in \Gamma$ be such that $\gamma_i(P, P', \alpha)$ converges to $(P, P', e^{i\theta}\alpha)$ as elements in G/H. Then, $\gamma_i^* s$ converges to the holomorphic function $s(e^{-i\theta}z_1; z_2, \cdots, z_r)$. It follows that for any $\theta \in \mathbb{R}$ the function s_θ as defined above actually lies in $\mathcal{F}$. Averaging $e^{-i\theta}s_\theta$ over $[0, 2\pi]$ we obtain a holomorphic function $\sigma : \Omega \to \Delta$, $\sigma = F^*g$ for some $g : \tilde{N} \to \Delta$. Here g is obtained by a normal family argument by interpreting the integral as a limit of finite sums, and may not be uniquely determined by $\{h_\theta\}$. From Taylor expansions we see that, when restricted to the maximal polydisk P, we have $\sigma(z_1; z_2, \cdots, z_r) = s'(0)z_1$ and P' corresponds to $\{0\} \times \Delta^{r-1}$. We call such a function σ a special function. The existence of a special function is a property of the pair (P, P') alone, and we say that the special function σ is adapted to (P, P'). We say that a pair (P, P') is generic if and only if (P, P', α) is generic for some α (and hence for any α). What we have proven is that for a generic pair (P, P') there exists a special function adapted to (P, P').

$s'(0)$ can be related to the constant c in the identity $F^*\kappa \equiv c\hat{g}$ for the Carathéodory metric as given in the Finsler Metric Rigidity Theorem. Here all metrics will be applied to the covering domain Ω. We claim that, if $\varepsilon > 0$ is sufficiently small, then there exists a constant $a > 0$ such that for any choice of generic pair (P, P'), we have $|s'(0)| \geq a$. (Obviously $|s'(0)|$ is bounded from above independent of (P, P'), from Cauchy estimates.) Let θ be an extremal function for $\mathcal{F} = F^*\mathcal{H}$ at $o \in P'$ adapted to $\alpha = \frac{\partial}{\partial z_1}$. Then $\|\alpha\|_{e(\mathcal{F})} \geq \|\alpha\|_\theta$ by definition. On the other hand, by the sub-mean value inequality, $\|\alpha\|_\theta \geq \|\alpha\|_{F^*\kappa}$ since θ is an extremal function adapted to α. By Cauchy estimates for second derivatives we can choose $\varepsilon > 0$ sufficiently small such that $|t'(z_1) - t'(0)| < \frac{c}{2}$ for any holomorphic function $t : \Delta_\alpha \to \Delta$ and any $(z_1; o) \in S_{\alpha,\varepsilon}$. If $|s'(0)| < \frac{c}{2}$ we will have $|s'(z_1)| < c$, whenever $(z_1; o) \in S_{\alpha,\varepsilon}$, contradicting the estimate $\|\alpha\|_{e(\mathcal{F})} \geq \|\alpha\|_{F^*\kappa} = c$, as claimed.

From the lower bound $|s'(0)| > a$ and taking limits we conclude that for every pair (P, P') there is a special function $\sigma \in \mathcal{F}$. When P and a base point $x \in P$ are given, the latter statement is valid for P' being any of the $(r-1)$-dimensional polydisk $P_k \subset P$ corresponding to setting $z_k = 0$ for some some k, $1 \leq k \leq r$. As in the proof of Theorem 1' in the cocompact case for polydisks we conclude that $F|_P$ separates points for any maximal polydisk $P \subset \Omega$. Hence $F : \Omega \to \tilde{N}$ separates points. Since $f : X \to N$ is an immersion, we have proven that $F : \Omega \to \tilde{N}$ is an embedding. The proof of Theorem 1 for the cocompact case is complete. □

Remarks. The last part of the proof reveals a subtle role played by metric rigidity. For instance, there may exist a maximal polydisk $P \subset \Omega$ such that $\pi(P) \subset X$ is a *reducible* quotient of the polydisk, in which the argument that F separates points on P cannot be applied directly. Metric rigidity enables us to work with generic maximal polydisks and pass to limits.

Proof of Theorem 1' in the cocompact case. Since Moore's Ergodicity Theorem applies to the case of irreducible lattices and the Polydisk Theorem also applies to reducible domains, the proof of Theorem 1 actually applies to give a proof of Theorem 1'.

§4 The Embedding Theorem for arithmetic varieties of rank ≥ 2 and proofs of other results

(4.1) We are ready to complete the proof of the Embedding Theorem by extending the argument for all torsion-free irreducible lattices $\Gamma \subset Aut(\Omega)$, *i.e.,* to include the case where $X := \Omega/\Gamma$ is noncompact and of finite volume with respect to the Kähler-Einstein metric. The Hermitian metric rigidity for quotient manifolds of finite volume was established in Mok [M1] under a boundedness assumption and in To [To] in full generality. Here we will need an adaptation of Mok [M1] to the case of continuous complex Finsler metrics defined on $\mathcal{S}$ satisfying the boundedness assumption which are however only partially of nonpositive curvature as in Proposition 6. The proof of Finsler metric rigidity as stated in (1.1) in the cocompact case relied on nonpositivity of curvatures. In our case, the use of full nonpositivity of curvature can be replaced by the property that our complex Finsler metrics on $L|_{\mathcal{S}}$ are actually uniformly Lipschitz. More precisely, we have

Lemma 5. *Let Ω be a bounded symmetric domain of rank ≥ 2, $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$, $f : X \to N$ be a holomorphic mapping into a complex manifold N, and $F : \Omega \to \tilde{N}$ be its lifting to universal covers. Let κ be the Carathéodory pseudometric on $\tilde{N}$, and $e(\mathcal{F})$ be the continuous Hermitian pseudometric on Ω defined on $L|_{\mathcal{S}_\Omega}$ as in (3.2), and denote by the same symbols the Hermitian metrics obtained by descending to $X = \Omega/\Gamma$. Define $v, u : \mathcal{S} \to \mathbb{R}$, $v, u \geq 0$, by $f^*\kappa + \hat{g} = e^v\hat{g}$, $e(\mathcal{F}) + \hat{g} = e^u\hat{g}$. Denote by ω_{KE} the Kähler form of the canonical Kähler-Einstein metric g on X, $\pi : \mathcal{S} \to X$ the canonical projection, and by ν the Kähler form on $\mathcal{S}$ given by $\nu = \pi^*\omega_{KE} - c_1(L, \hat{g})$. Then, $v, u : \mathcal{S} \to \mathbb{R}$; $v, u \geq 0$; are bounded and uniformly Lipschitz, i.e., dv and du are locally bounded 1-forms such that for some positive constant C; $\|dv\|_{\hat{g}}$, $\|du\|_{\hat{g}} \leq C$ almost everywhere on $\mathcal{S}$.*

Proof. Obvious from Cauchy estimates on second derivatives for holomorphic functions $s : \Omega \to \Delta$, $s \in \mathcal{F} = F^*\mathcal{H}$. $\square$

From Lemma 5 we have the following partial analogue of Proposition 5 for complete Kähler manifolds of finite volume.

Proposition 5'. *Let (Z, ω) be an m-dimensional complete Kähler manifold of finite volume, and θ be a smooth closed nonnegative (1,1)-form on Z, bounded with respect to ω, such that $Ker(\theta)$ is of constant rank $q > 0$ everywhere on Z. Denote by $\mathcal{K}$ the foliation on Z with holomorphic leaves defined by the distribution $Re(Ker(\theta))$. Let $u : Z \to \mathbb{R}$ be a uniformly Lipschitz function whose restriction to every leaf $\mathcal{L}$ of the foliation $\mathcal{K}$ is plurisubharmonic. Then, u is constant on each leaf $\mathcal{L}$. If in addition there is a dense leaf of $\mathcal{K}$, then u is constant on Z.*

Proof. Fix a base point $z_0 \in Z$ and denote by B_R the geodesic ball centred at z_0 of radius R. For $R > 0$ there exists a smooth function ρ_R such that $\rho_R \equiv 1$ on B_R, $\rho_R \equiv 0$ on B_{R+1} and such that $\|d\rho_R\| \leq 2$, where norms are measured in terms of the Kähler form ω on Z. Define on Z the currents

$$T_R = \sqrt{-1}\rho_R\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} , \quad S_R = dT_R .$$

Then,

$$S_R = \sqrt{-1}\partial\rho_R \wedge \overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} + \sqrt{-1}\rho_R\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} .$$

S_R is a d-closed (m,m)-current with compact support. Since u is Lipschitz, and u is plurisubharmonic on leaves $\mathcal{L}$ of $\mathcal{K}$, coefficients of S_R are complex-valued measures. We have

$$(*)\qquad 0 = S_R(1) = \int_Z \sqrt{-1}\partial\rho_R \wedge \overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} + \int_Z \rho_R \sqrt{-1}\partial\overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} ,$$

where $\sqrt{-1}\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} := S$ is a nonnegative measure. Thus, by $(*)$

$$\begin{aligned} S(B_R) &\leq \int_Z \rho_R \sqrt{-1}\partial\overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} \\ &\leq Const. \int_{Supp(\overline{\partial}\rho_R)} \|\overline{\partial} u\| \\ &\leq Const. \times Volume(Z - B_R) . \end{aligned}$$

Since (Z,ω) is of finite volume, letting $R \to \infty$ we conclude that

$$\lim_{R\to\infty} S(B_R) = 0 \ ; \quad \text{hence} \quad S \equiv 0 ,$$

since S is a nonnegative measure. Integrating by parts we have

$$\begin{aligned} 0 &= \int_Z -\sqrt{-1}\rho_R u \partial\overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} \\ &= \int_Z \rho_R \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} + \int_Z \sqrt{-1} u \partial\rho_R \wedge \overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} , \end{aligned}$$

so that

$$\int_{B_R} \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} \leq Const. \times Volume(Supp(\partial\rho_R)) ,$$

as both u and $\|\overline{\partial}u\|$ are bounded. Since $Volume(Z,\omega) < \infty$, we conclude by letting $R \to \infty$ that

$$\sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \theta^{m-q} \wedge \omega^{q-1} \equiv 0 ,$$

which means precisely that u is constant on almost every local leaf $\mathcal{L}_o$ of $\mathcal{K}$. Since the Lipschitz function u is continuous, u is constant on all leaves $\mathcal{L}$ of $\mathcal{K}$. Obviously u is constant on Z if there exists a dense leaf $\mathcal{L}$ of $\mathcal{K}$. □

In the same notations as in Proposition 5', we have

Proposition 6'. *There exists some constants $c_1, c_2 > 0$ such that*

$$f^*\kappa \equiv c_1 \hat{g} \quad \textit{and} \quad e(\mathcal{F}) \equiv c_2 \hat{g} \quad \textit{on} \quad L|_S .$$

Proof. The proof of Proposition 6 applies ***verbatim*** **as a consequence of the metric rigidity** result Proposition 5'. □

Finally, we have

Proof of the Embedding Theorem (Theorems 1 and 1'). Given Proposition 5' and 6', the analogues of Proposition 3 and 4 in (3.1) follow. The rest of the proof of Theorem 1 is identical to the cocompact case. Theorem 1' follows from an obvious reformulation of the metric rigidity result Proposition 5'. □

(4.2) It remains to complete the proofs of Theorems 2-5 and Corollary 1.

Proof of Theorem 2. Since D is a bounded domain in a Stein manifold, and $F : \Omega \to D$ is nonconstant, there exists some bounded holomorphic function h on D such that $h|_{F(\Omega)}$ is nonconstant. Thus, for Ω irreducible, $F : \Omega \to D$ is a holomorphic embedding, by Theorem 1. In the locally reducible case, we can conclude that $F : \Omega \to D$ is an embedding by Theorem 1', provided that f has been shown to be an immersion at a generic point. By the proof of Theorem 1', if $f : X \to N$ is not generically an immersion, then there exists some nontrivial canonical projection $\rho_I : \Omega \to \Omega_{i(1)} \times \cdots \times \Omega_{i(p)}$ such that F is constant on the fibers $\mathcal{L}$ of π_I. It follows that $f : X \to N$ is constant on $\pi(\mathcal{L})$ for the canonical projection $\pi : \Omega \to X$. By the Density Lemma in (2.1), there exists some fiber $\mathcal{L} \subset \Omega$ such that $\pi(\mathcal{L})$ is dense in X, so that f is constant on X, a plain contradiction. Thus, Theorem 1' applies to give a proof of Theorem 2 in the locally reducible case, as desired. □

Proof of Theorem 3. In the proof of Theorem 1 we were working on $\mathcal{F} = F^*\mathcal{H}$ on the irreducible bounded symmetric domain Ω. For instance, Finsler metric rigidity applies even when N is singular to show that $df(\alpha) \neq 0$ for any characteristic vector α on X. Here N can be locally embedded as a complex-analytic subvariety of a domain in some $\mathbb{C}^m$, so that f can be locally regarded as a vector-valued holomorphic function, and the statement that $df(\alpha) \neq 0$ carries a meaning independent of the choice of local holomorphic embeddings. The assumption that N is nonsingular is inessential in the proof of Theorem 1 and Theorem 3 follows as a special case of the generalization of Theorem 1 to possibly singular target complex spaces N. □

A counter-example to the analogue of Theorem 3 in the locally reducible case. For the locally reducible case the proof of Theorem 1' only implies that at any point $p \in \tilde{N}$ bounded holomorphic functions on $\tilde{N}$ cannot give local coordinates at p. It is in fact possible that there exist nontrivial bounded holomorphic functions on $\tilde{N}$. We give here an example in the case of $X = \Delta^2/\Gamma$, with $\Gamma \subset Aut(\Delta)^2$ a torsion-free irreducible lattice. Pick any $x \in X$. Lifting x to the origin $o \in \Delta^2$ we identify some open neighborhood U of $x \in X$ with an open neighborhood of $o \in \Delta^2$, to be denoted also by U. Let $V \subset U$ be a relatively open subset, $o \in V$. We have $X = U \cup (X - \overline{V})$. Consider now the holomorphic map $\rho : \mathbb{C}^2 \to \mathbb{C}^4$ defined by $\rho(z_1, z_2) = (z_1, z_2^2, z_2^3, z_1 z_2)$. Clearly ρ maps U onto an open subset $U' = \rho(U)$ of an irreducible affine subvariety of $\mathbb{C}^4$, such that $\rho|_U : U \to U'$ is a normalization and a homeomorphism, and $o \in U'$ is the unique singular point of U', which is non-normal. Define a new complex space N by piecing together U' and $X - \overline{V}$, where $y \in U'$ and $w \in X - \overline{V}$ are identified if and only if $y = \rho(w)$. We have then naturally a complex-analytic homeomorphism $\nu : X \to N$, which is a normalization. Since $\nu : X \to N$ is a homeomorphism, we have canonically $\nu_* : \Gamma = \pi_1(X) \cong \pi_1(N)$. Clearly any bounded holomorphic function h on the universal cover Δ^2 of X of the form $h(z_1, z_2) = h(z_1)$ descends to a bounded holomorphic function on the universal cover $\tilde{N}$ of N. (We note that the proof of Theorem 1' implies that these are the only possible bounded holomorphic functions on $\tilde{N}$.) This furnishes a counter-example to the analogue of Theorem 4 for the locally reducible case. □

Proof of Theorem 4. Since f is surjective, $dim(Z) \leq dim(X)$. We claim that f is finite. Otherwise there exists a positive-dimensional irreducible subvariety $S \subset X$ such that f maps S to a point. Then the image of $\pi_1(S)$ in $\pi_1(X) = \Gamma$ is trivial, and S lifts to a compact subvariety of the covering bounded domain Ω, which is impossible. We note furthermore that $f_*(\Gamma) \subset \pi_1(Z)$ is a subgroup of finite index. Suppose otherwise and write $\tau : Z' \rightarrow Z$ for the intermediate covering of Z corresponding to the subgroup $f_*(\Gamma)$. Then, the holomorphic mapping $f : X \rightarrow Z$ lifts to $f' : X \rightarrow Z'$. Since f and hence f' are proper, $f'(X) \subset Z'$ is a subvariety, and must therefore agree with Z as f' is finite and $dim(Z) = dim(X)$. This contradicts with the fact that $\tau : Z' \rightarrow Z$ has infinite fibers.

We will use an algebraic result of Margulis [Ma, Chapter VIII, Theorem A, p.258ff.], according to which any normal subgroup of Γ is either finite or of finite index in Γ. To prove Theorem 4 suppose $f_*(\Gamma) \subset \pi_1(Z)$ is not finite. Then, $Ker(f_*) \subset \Gamma$ must be finite, by Margulis [Ma, *loc. cit.*]. Let $F : \Omega \rightarrow \tilde{Z}$ be the lifting of f to universal covering spaces. Since $f : X \rightarrow Z$ is finite, proper and surjective, and $Ker(f_*) \subset \Gamma$ is also finite, $F : \Omega \rightarrow \tilde{Z}$ is a finite proper surjective map. It follows that, given any bounded holomorphic function θ on Ω, one can form symmetric polynomials over the fibers of $F : \Omega \rightarrow \tilde{Z}$ to get bounded holomorphic functions on Ω which are constant on fibers of F. Since Z is assumed normal, these bounded holomorphic functions descend to bounded holomorphic functions on the normal complex space $\tilde{Z}$. Obviously one can choose θ to obtain this way nontrivial holomorphic functions on $\tilde{Z}$. By the generalization of Theorem 1 to possibly singular target complex spaces N (cf. Proof of Theorem 3), it follows that $F : \Omega \rightarrow \tilde{Z}$ is a biholomorphism. In particular, $f : X \rightarrow Z$ is an unramified covering map, as desired. □

Proof of Corollary 1. For the irreducible lattice $\Gamma^* \subset Aut(\Omega)$ with nontrivial torsion elements, there exists a torsion-free subgroup $\Gamma \subset \Gamma^*$ of finite index. Consider the canonical map $f : X = \Omega/\Gamma^* \rightarrow \Omega/\Gamma = Z$, which is finite, proper and surjective. We give Z the structure of a normal complex space but note that the statement of Corollary 1 is independent of the choice of complex structure on Z, which is uniquely determined as a topological space. By Theorem 3, either $\pi_1(Z)$ is finite, or $f : X \rightarrow Z$ is an unramified covering map. But the latter cannot occur, since the lifting $F : \Omega \rightarrow \tilde{Z}$ to universal covering spaces fails to be a local embedding at any point $x \in \Omega$ fixed by some nontrivial torsion element of Γ. In other words, $\pi_1(Z)$ is finite, as desired. □

Finally, Theorem 5 follows immediately from the proof of Theorem 1. We note also that there is an obvious analogue of Theorem 5 for the locally reducible case, which we omit.

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