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An inequality between the diameter and the inverse dual degree of a tree

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Abstract

Let \( R(T), D(T) \) be respectively the radius and diameter of a nontrivial tree \( T \) and \( I(T) = \sum_{u \in V(T)} 1/\overline{d}(u) \) be the inverse dual degree, where \( \overline{d}(u) = (\sum_{v \in N(u)} d(v))/d(u) \) for each \( u \in V(T) \). In this note we prove that

\[
I(T) \geq \begin{cases} 
R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\
R(T) + 5/6, & \text{if } D(T) \text{ is even},
\end{cases}
\]

with equality if and only if \( T \) is a path of at least 4 vertices. This inequality strengthens a conjecture of Graffiti.

Let \( G = (V(G), E(G)) \) be a simple, connected graph. The distance \( d(u, v) \) between two vertices \( u, v \) of \( G \) is the minimal length of a path from \( u \) to \( v \) in \( G \). The diameter \( D(G) \) of \( G \) is the largest distance between any two vertices of \( G \). The radius \( R(G) \) of \( G \) is \( \min_{u \in V(G)} \max_{v \in V(G)} d(u, v) \). If \( S \subseteq V(G) \) and \( u \in V(G) \setminus S \), then we denote \( d(u, S) = \min_{v \in S} d(u, v) \). The neighbours of \( u \in V(G) \) are vertices adjacent to \( u \) in \( G \) and the neighbourhood \( N(u) \) of \( u \) in \( G \) is the set of
neighbours of \( u \). Since \( G \) is simple, the degree of \( u \) is \( d(u) = |N(u)| \). The **dual degree** of \( u \) and the **inverse dual degree** of \( G \) are respectively \( \bar{d}(u) = (\sum_{v \in N(u)} d(v))/d(u) \) and \( I(G) = \sum_{u \in V(G)} 1/\bar{d}(u) \) [2]. When ambiguity arises we use \( d_G(u), \bar{d}_G(u) \), etc., to emphasize that the underlying graph is \( G \).

The main purpose of this note is to prove an inequality between \( D(T) \) and \( I(T) \). As a consequence we get an inequality involving \( R(T) \) and \( I(T) \) which strengthens the following conjecture (see [1, 3, 4, 5] for results relating to Graffiti conjectures).

**Graffiti Conjecture 577** For any (nontrivial) tree \( T \), \( I(T) \geq R(T) \).

(There are examples of \( G \) which are not trees such that \( I(G) < R(G) \).)

In the following we suppose \( T \) is a (nontrivial) tree and \( P = v_0v_1 \ldots v_D \) is a path of maximal length in \( T \), where \( D = D(T) \). Then \( d(v_0) = d(v_D) = 1 \). Let

\[
a = a(P) = |\{v \in V(T) \setminus V(P) : d(v) \geq 2\}|
\]

\[
b = b(P) = \begin{cases} |\{i : d(v_i) \geq 3, 2 \leq i \leq D - 2\}|, & \text{if } D \geq 4, \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
c = c(P) = \begin{cases} |\{i : d(v_i) \geq 3, i = 1, D - 1\}|, & \text{if } D \geq 2, \\ 0, & \text{if } D = 1. \end{cases}
\]

**Theorem** \( I(T) \geq D(T)/2 + a/3 + b/10 + c/12 + 5/6 \).

A **caterpillar** is a tree with the property that the removal of all degree-one vertices yields a path, called the spine. Note that if \( T \) is a caterpillar, then \( v_1 \ldots v_{D-1} \) is the spine. To prove the theorem we need the following lemmas.

**Lemma 1** Suppose \( T \) is not a caterpillar (so in particular \( D(T) \geq 4 \)) and \( u \) is a vertex not in \( P \) such that \( d(u) \geq 2 \) and \( d(u, V(P)) \) is as large as possible. Let \( T' \) be the subtree obtained from \( T \) by deleting all degree-one vertices adjacent to \( u \). Then \( D(T) = D(T') \) and \( I(T) \geq I(T') + 1/3 \).

**Proof** We first note that all but one neighbours of \( u \) have degree one, for otherwise there would be a neighbour \( w \) of \( u \) not in \( P \) with \( d(w) \geq 2 \) and \( d(w, V(P)) > d(u, V(P)) \), violating the choice of \( u \). Suppose \( N(u) = \{u_1, \ldots, u_m, v\} \) where \( d(u_i) = 1 \), for \( i \in \{1, \ldots, m\} \) and \( d(v) = r \). Then
$D(T) = D(T')$. Let $\sigma = \sum_{v \in N(v) \setminus \{u\}} d(x)$. Since $\sigma + 1 \geq r$, we have

$$I(T) - I(T') = \sum_{i=1}^{m} \left( \frac{1}{d_T(u_i)} + \frac{1}{d_T(v)} - \frac{1}{d_T(v)} \right) \geq 1 + \frac{1}{m + r} - \frac{1}{r} + \frac{1}{r} + \frac{1}{r + s} - \frac{1}{r + s} - \frac{1}{r + s + 1} \geq 1 + \frac{1}{m + r} - \frac{1}{r} - \frac{1}{m + 1}.$$ 

Note that $1/(m + x) - 1/x$ is an increasing function of $x$ and $r \geq 2, m \geq 1$. We have from the inequality above that

$$I(T) - I(T') \geq 1 + \frac{1}{m + 2} - \frac{1}{2} - \frac{1}{m + 1} = \frac{1}{2} - \frac{1}{(m+1)(m+2)} \geq \frac{1}{3}.$$ 

Q.E.D.

**Lemma 2** Suppose $T$ is a caterpillar but not a path and $D = D(T) \geq 4$. If $d(v_1) \geq 3$ (respectively $d(v_{D-1}) \geq 3$) and let $T'$ be the subtree obtained from $T$ by deleting all degree-one vertices adjacent to $v_1$ (respectively $v_{D-1}$) excepting $v_0$ (respectively $v_D$). Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/12$.

**Proof** Suppose $d(v_1) = m + 2 \geq 3, d(v_2) = r, d(v_3) = s$. Then $r, s \geq 2$ and $D(T) = D(T')$. We have

$$I(T) - I(T') = \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=0}^{2} \left( \frac{1}{d_T(v_i)} - \frac{1}{d_T(v_i)} \right) = m + \frac{1}{m + 2} - \frac{1}{2} + \frac{1}{m + r} - \frac{1}{2} + \frac{1}{m + r + s} - \frac{1}{2} \geq m \left( \frac{1}{2(m + 2)} - \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)} - \frac{1}{(r+1)(r+2)} \right) \geq m \left( \frac{1}{2(m + 2)} - \frac{1}{(r+1)(r+2)} \right) \geq m \left( \frac{1}{2(m + 2)} - \frac{1}{m(m+5)} \right) = \frac{1}{12}.$$ 

Q.E.D.

**Lemma 3** Suppose $T$ is a caterpillar but not a path and $D = D(T) \geq 4$. If $d(v_1) = d(v_{D-1}) = 2$ and let $T'$ be the subtree obtained by deleting all degree-one neighbours of $v_\alpha$, where $v_\alpha$ is the vertex nearest to one terminal vertex of $P$ such that $d(v_\alpha) \geq 3$. Then $D(T) = D(T')$ and $I(T) \geq I(T') + 1/10.$
Proof Without loss of generality we may assume \( \alpha \leq \lfloor \frac{D}{2} \rfloor \). Let \( u_1, \ldots, u_m \) be all the degree-one neighbours of \( v_\alpha \). We have \( d(v_{\alpha-1}) = 2 \). Let \( d(v_{\alpha-2}) = r \) (\( r = 1 \) if \( \alpha = 2 \) and \( r = 2 \) otherwise), \( d(v_{\alpha+1}) = s, d(v_{\alpha+2}) = t \). Clearly we have \( D(T) = D(T') \). If \( D \geq 5 \), then \( t \geq 2 \), hence we have

\[
I(T) - I(T') = \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=\alpha-1}^{\alpha+1} \left( \frac{1}{d_T(v_i)} - \frac{1}{d_{T'}(v_i)} \right) \\
= \frac{m}{m+2} + \left( \frac{2}{m+3} - \frac{2}{r+2} \right) + \left( \frac{m+2}{m+s+2} - \frac{2}{s+2} \right) \\
+ \frac{s}{m+s+t} - \frac{s}{s+2} \\
= \frac{3(m+2)(m+3)}{m(m+5)} \\
\geq \frac{1}{6} \geq \frac{1}{10}.
\]

If \( D = 4 \), then a straightforward calculation shows that

\[
I(T) - I(T') = \frac{1}{6} + \frac{4}{m+3} - \frac{2}{m+2} - \frac{2}{m+4} \\
= \frac{1}{6} - \frac{2}{(m+2)(m+3)(m+4)} \\
\geq \frac{1}{10}.
\]

Q.E.D.

Now let us prove the main theorem. If \( T = P_n \), the path with \( n \) vertices, then

\[
I(P_n) - D(P_n)/2 = \begin{cases} 
3/2, & n = 2 \\
1, & n = 3 \\
5/6, & n \geq 4.
\end{cases}
\]

If \( D(T) = 2 \), then \( T \) is a star with \( a = b = 0 \), \( c = 1 \) and \( I(T) - D(T)/2 = 1 \geq c/12 + 5/6 \). If \( D(T) = 3, T \neq P_4 \), then \( a = b = 0 \) and \( T \) has exactly two vertices with degree \( \geq 2 \). Suppose the degrees of them are \( l + 1, m + 1 \). Then \( \max\{l, m\} \geq 2 \) and

\[
I(T) - D(T)/2 = \frac{l+m+2}{l+m+1} + \frac{l}{l+1} + \frac{m}{m+1} - \frac{3}{2} \\
= \frac{l+m+1}{l+1} - \frac{1}{m+1} + \frac{3}{2} \\
\geq \frac{1}{12} + \frac{5}{6}.
\]

In the following we suppose \( T \) is not a path and \( D = D(T) \geq 4 \). If \( T \) is not a caterpillar, let \( u \) be the vertex not in \( P \) such that \( d(u) \geq 2 \) and \( d(u, V(P)) \) is as large as possible. Then all but one neighbours of \( u \) have degree one. Removing from \( T \) all the degree-one neighbours of
we get a subtree $T_1$ with $D(T) = D(T_1), I(T) ≥ I(T_1) + 1/3$, according to Lemma 1. If $T_1$ is not a caterpillar, then repeat this procedure until a caterpillar is obtained. It is clear that after $a$ steps we get a sequence $T = T_0, T_1, \ldots, T_a$ such that each $T_{i+1}$ is a subtree of $T_i$ and $D(T_i) = D(T_{i+1})$ and $I(T_i) ≥ I(T_{i+1}) + 1/3$. So we have $D(T) = D(T_a)$ and $I(T) ≥ I(T_a) + a/3$.

If $d(v_1) ≥ 3$ in $T_a$, then delete all the degree-one neighbours of $v_1$ except $v_0$. We get $T_{a+1}$ with the same diameter as $T$ such that $I(T_a) ≥ I(T_{a+1}) + 1/12$, according to Lemma 2. If $d(v_{D-1}) ≥ 3$, we do the same thing. In this way $c$ subtrees are added to the sequence above and we get $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}$ with $D(T) = D(T_{a+c})$ and $I(T) = I(T_{a+c}) + a/3 + c/12$.

Now we have $d_{T_{a+c}}(v_1) = d_{T_{a+c}}(v_{D-1}) = 2$ and $d_{T_{a+c}}(v_i) = d(v_i), i \notin \{1, D-1\}$. If $T_{a+c}$ is not a path, then according to Lemma 3 we can delete all degree-one neighbours of some $v_a$ and obtain a subtree $T_{a+c+1}$ with $I(T_{a+c}) ≥ I(T_{a+c+1}) + 1/10$. Repeat the procedure until we obtain a path $P$. When the process stops we get a sequence $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}, \ldots, T_{a+c+b} = P$ with $I(T) ≥ I(T_{a+c}) + a/3 + c/12 ≥ I(P) + a/3 + b/10 + c/12$. Since $I(P) = D(P)/2 + 5/6$, as we have just proved it for paths, and since $D(P) = D(T)$, we get $I(T) ≥ D(T)/2 + a/3 + b/10 + c/12 + 5/6$.

This completes the proof.

Note that $R(T) = \lceil D(T)/2 \rceil$ for any tree $T$ and $a, b, c$ are non-negative integers. Hence we have the following corollary.

**Corollary** For any (nontrivial) tree $T$

$$I(T) ≥ \begin{cases} R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\ R(T) + 5/6, & \text{if } D(T) \text{ is even} \end{cases}$$

with equality if and only if $T$ is a path of at least four vertices.

This corollary tells us that $I(T) - R(T)$ is bounded below. We point out that it is unbounded above. In fact, for the full binary tree $T$ of height $h ≥ 3$ we have $I(T) - R(T) = 2^{h+2}/5 - h - 1/4$, which can be arbitrarily large as $h$ tends to infinity.

**References**

[2] S. Fajtlowicz, Written on the wall, a list of conjectures of Graffiti, preprint, University of Houston, USA.

