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An inequality between the diameter and the inverse dual degree of a tree

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Abstract

Let $R(T), D(T)$ be respectively the radius and diameter of a nontrivial tree $T$ and $I(T) = \sum_{u \in V(T)} 1/\overline{d}(u)$ be the inverse dual degree, where $\overline{d}(u) = (\sum_{v \in N(u)} d(v))/d(u)$ for each $u \in V(T)$. In this note we prove that

$$I(T) \geq \begin{cases} R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\ R(T) + 5/6, & \text{if } D(T) \text{ is even}, \end{cases}$$

with equality if and only if $T$ is a path of at least 4 vertices. This inequality strengthens a conjecture of Graffiti.

Let $G = (V(G), E(G))$ be a simple, connected graph. The distance $d(u, v)$ between two vertices $u, v$ of $G$ is the minimal length of a path from $u$ to $v$ in $G$. The diameter $D(G)$ of $G$ is the largest distance between any two vertices of $G$. The radius $R(G)$ of $G$ is $\min_{u \in V(G)} \max_{v \in V(G)} d(u, v)$. If $S \subseteq V(G)$ and $u \in V(G) \setminus S$, then we denote $d(u, S) = \min_{v \in S} d(u, v)$. The neighbours of $u \in V(G)$ are vertices adjacent to $u$ in $G$ and the neighbourhood $N(u)$ of $u$ in $G$ is the set of
neighbours of \( u \). Since \( G \) is simple, the degree of \( u \) is \( d(u) = |N(u)| \). The dual degree of \( u \) and the inverse dual degree of \( G \) are respectively \( \overline{d}(u) = (\sum_{v \in N(u)} d(v))/d(u) \) and \( I(G) = \sum_{u \in V(G)} 1/\overline{d}(u) \) [2]. When ambiguity arises we use \( d_G(u), \overline{d}_G(u), \) etc., to emphasize that the underlying graph is \( G \).

The main purpose of this note is to prove an inequality between \( D(T) \) and \( I(T) \). As a consequence we get an inequality involving \( R(T) \) and \( I(T) \) which strengthens the following conjecture (see [1, 3, 4, 5] for results relating to Graffiti conjectures).

**Graffiti Conjecture 577** For any (nontrivial) tree \( T \), \( I(T) \geq R(T) \).

(There are examples of \( G \) which are not trees such that \( I(G) < R(G) \).)

In the following we suppose \( T \) is a (nontrivial) tree and \( P = v_0v_1 \ldots v_D \) is a path of maximal length in \( T \), where \( D = D(T) \). Then \( d(v_0) = d(v_D) = 1 \). Let

\[
\begin{align*}
a &= a(P) = |\{v \in V(T) \setminus V(P) : d(v) \geq 2\}|, \\
b &= b(P) = \begin{cases} |\{i : d(v_i) \geq 3, 2 \leq i \leq D - 2\}|, & \text{if } D \geq 4, \\
0, & \text{otherwise}, \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
c &= c(P) = \begin{cases} |\{i : d(v_i) \geq 3, i = 1, D - 1\}|, & \text{if } D \geq 2, \\
0, & \text{if } D = 1. \end{cases}
\end{align*}
\]

**Theorem** \( I(T) \geq D(T)/2 + a/3 + +b/10 + c/12 + 5/6 \).

A caterpillar is a tree with the property that the removal of all degree-one vertices yields a path, called the spine. Note that if \( T \) is a caterpillar, then \( v_1 \ldots v_{D-1} \) is the spine. To prove the theorem we need the following lemmas.

**Lemma 1** Suppose \( T \) is not a caterpillar (so in particular \( D(T) \geq 4 \)) and \( u \) is a vertex not in \( P \) such that \( d(u) \geq 2 \) and \( d(u, V(P)) \) is as large as possible. Let \( T' \) be the subtree obtained from \( T \) by deleting all degree-one vertices adjacent to \( u \). Then \( D(T) = D(T') \) and \( I(T) \geq I(T') + 1/3 \).

**Proof** We first note that all but one neighbours of \( u \) have degree one, for otherwise there would be a neighbour \( w \) of \( u \) not in \( P \) with \( d(w) \geq 2 \) and \( d(w, V(P)) > d(u, V(P)) \), violating the choice of \( u \). Suppose \( N(u) = \{u_1, \ldots, u_m, v\} \) where \( d(u_i) = 1 \), for \( i \in \{1, \ldots, m\} \) and \( d(v) = r \). Then

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\[ D(T) = D(T'). \] Let \( \sigma = \sum_{x \in N(v) \setminus \{u\}} d(x). \] Since \( \sigma + 1 \geq r \), we have
\[
I(T) - I(T') = \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \frac{1}{d_T'(u_i)} + \frac{1}{d_T'(v)} + \frac{r}{\sigma + 1}
\]
\[
= \frac{m}{m+1} + \frac{m+1}{m+r} - \frac{1}{r} + \frac{1}{m+1} + \frac{1}{m+r} + \frac{r}{\sigma + 1}
\]
\[
\geq 1 + \frac{1}{m+1} - \frac{1}{r} - \frac{1}{m+1}.
\]

Note that \( 1/(m + x) - 1/x \) is an increasing function of \( x \) and \( r \geq 2, m \geq 1. \) We have from the inequality above that
\[
I(T) - I(T') \geq 1 + \frac{1}{m+2} - \frac{1}{m+1} - \frac{1}{m+1}
\]
\[
= \frac{1}{2} - \frac{1}{m+1}(m+2)
\]
\[
\geq \frac{1}{3}.
\]

Q.E.D.

**Lemma 2** Suppose \( T \) is a caterpillar but not a path and \( D = D(T) \geq 4. \) If \( d(v_1) \geq 3 \) (respectively \( d(v_{D-1}) \geq 3 \)) and let \( T' \) be the subtree obtained from \( T \) by deleting all degree-one vertices adjacent to \( v_1 \) (respectively \( v_{D-1} \)) excepting \( v_0 \) (respectively \( v_D \)). Then \( D(T) = D(T') \) and \( I(T) \geq I(T') + 1/12. \)

**Proof** Suppose \( d(v_1) = m + 2 \geq 3, d(v_2) = r, d(v_3) = s. \) Then \( r, s \geq 2 \) and \( D(T) = D(T'). \) We have
\[
I(T) - I(T') = \sum_{i=1}^{m} \frac{1}{d_T(u_i)} + \sum_{i=0}^{2} \frac{1}{d_T(v_i)} - \frac{1}{d_T'(v)}
\]
\[
= \frac{m}{m+2} + \frac{1}{m+2} - \frac{1}{2} + \frac{m+1}{m+r+1} - \frac{1}{2} + \frac{1}{m+r+1} + \frac{r}{s} + \frac{1}{m+r+s} - \frac{r}{s+1}
\]
\[
\geq m\left( \frac{1}{2(m+2)} - \frac{1}{(r+1)(m+r+1)} \right) + r\left( \frac{1}{m+r+s} - \frac{1}{s+1} \right)
\]
\[
\geq m\left( \frac{1}{2(m+2)} - \frac{1}{(r+1)(m+r+1)} \right)
\]
\[
= \frac{1}{2(m+2)} - \frac{1}{3(m+3)}
\]
\[
\geq \frac{6(m+2)(m+3)}{12(m+5)}
\]

Q.E.D.

**Lemma 3** Suppose \( T \) is a caterpillar but not a path and \( D = D(T) \geq 4. \) If \( d(v_1) = d(v_{D-1}) = 2 \) and let \( T' \) be the subtree obtained by deleting all degree-one neighbours of \( v_\alpha, \) where \( v_\alpha \) is the vertex nearest to one terminal vertex of \( P \) such that \( d(v_i) \geq 3. \) Then \( D(T) = D(T') \) and \( I(T) \geq I(T') + 1/10. \)
Proof Without loss of generality we may assume $\alpha \leq \lfloor \frac{D}{2} \rfloor$. Let $u_1, \ldots, u_m$ be all the degree-one neighbours of $v_\alpha$. We have $d(v_{\alpha - 1}) = 2$. Let $d(v_{\alpha - 2}) = r$ ($r = 1$ if $\alpha = 2$ and $r = 2$ otherwise), $d(v_{\alpha + 1}) = s, d(v_{\alpha + 2}) = t$. Clearly we have $D(T) = D(T')$. If $D \geq 5$, then $t \geq 2$, hence we have

$$I(T) - I(T') = \sum_{i=1}^{m} \left( \frac{1}{d_P(u_i)} \right)_{\alpha} - \sum_{i=\alpha}^{\alpha+1} \left( \frac{1}{d_P(v_i)} - \frac{1}{d_T(v_i)} \right)$$

$$= \frac{m+2}{m+2} + \left( \frac{m+2}{m+3} \right) + \left( \frac{m+2}{m+3} \right)$$

$$= \frac{m+2}{m+2} + \left( \frac{m+2}{m+3} \right) + \left( \frac{m+2}{m+3} \right)$$

$$\geq \frac{1}{6} \geq \frac{1}{10}.$$ 

If $D = 4$, then a straightforward calculation shows that

$$I(T) - I(T') = \frac{1}{6} + \frac{4}{m+3} - \frac{2}{m+2} - \frac{2}{m+2}$$

$$= \frac{1}{6} - \frac{2}{m+3} + \frac{1}{m+2} - \frac{1}{m+2}$$

$$\geq \frac{1}{10}.$$ 

Q.E.D.

Now let us prove the main theorem. If $T = P_n$, the path with $n$ vertices, then

$$I(P_n) - D(P_n)/2 = \begin{cases} 
3/2, & n = 2 \\
1, & n = 3 \\
5/6, & n \geq 4.
\end{cases}$$

If $D(T) = 2$, then $T$ is a star with $a = b = 0, c = 1$ and $I(T) - D(T)/2 = 1 \geq c/12 + 5/6$. If $D(T) = 3, T \neq P_3$, then $a = b = 0$ and $T$ has exactly two vertices with degree $\geq 2$. Suppose the degrees of them are $l + 1, m + 1$. Then $\max\{l, m\} \geq 2$ and

$$I(T) - D(T)/2 = \frac{l+m+2}{l+m+2} + \frac{l}{l+1} + \frac{m}{m+1} - 3/2$$

$$= \frac{l+m+1}{l+1} - \frac{m+1}{m+1} + 3/2$$

$$\geq \frac{5}{12} + \frac{5}{6}.$$ 

In the following we suppose $T$ is not a path and $D = D(T) \geq 4$. If $T$ is not a caterpillar, let $u$ be the vertex not in $P$ such that $d(u) \geq 2$ and $d(u, V(P))$ is as large as possible. Then all but one neighbours of $u$ have degree one. Removing from $T$ all the degree-one neighbours of
we get a subtree $T_1$ with $D(T) = D(T_1), I(T) \geq I(T_1) + 1/3$, according to Lemma 1. If $T_1$
not a caterpillar, then repeat this procedure until a caterpillar is obtained. It is clear that
after $a$ steps we get a sequence $T = T_0, T_1, \ldots, T_a$ such that each $T_{i+1}$ is a subtree of $T_i$
and $D(T_i) = D(T_{i+1})$ and $I(T_i) \geq I(T_{i+1}) + 1/3$. So we have $D(T) = D(T_a)$ and $I(T) \geq I(T_a) + a/3$.

If $d(v_1) \geq 3$ in $T_a$, then delete all the degree-one neighbours of $v_1$ except $v_0$. We get $T_{a+1}$
with the same diameter as $T$ such that $I(T_a) \geq I(T_{a+1}) + 1/12$, according to Lemma 2. If
$d(v_{D-1}) \geq 3$, we do the same thing. In this way $c$ subtrees are added to the sequence above and
we get $T = T_0, T_1, \ldots, T_a, T_{a+c}$ with $D(T) = D(T_{a+c})$ and $I(T) = I(T_{a+c}) + a/3 + c/12$.

Now we have $d_{T_{a+c}}(v_1) = d_{T_{a+c}}(v_{D-1}) = 2$ and $d_{T_{a+c}}(v_i) = d(v_i), i \notin \{1, D-1\}$. If $T_{a+c}$ is not
a path, then according to Lemma 3 we can delete all degree-one neighbours of some $v_a$ and obtain
a subtree $T_{a+c+1}$ with $I(T_{a+c+1}) \geq I(T_{a+c+1}) + 1/10$. Repeat the procedure until we obtain a path
$P$. When the process stops we get a sequence $T = T_0, T_1, \ldots, T_a, \ldots, T_{a+c}, \ldots, T_{a+c+b} = P$ with
$I(T) \geq I(T_{a+c}) + a/3 + c/12 \geq I(P) + a/3 + b/10 + c/12$. Since $I(P) = D(P)/2 + 5/6$, as we have
just proved it for paths, and since $D(P) = D(T)$, we get $I(T) \geq D(T)/2 + a/3 + b/10 + c/12 + 5/6$.
This completes the proof.

Note that $R(T) = \lfloor D(T)/2 \rfloor$ for any tree $T$ and $a, b, c$ are non-negative integers. Hence we
have the following corollary.

**Corollary** For any (nontrivial) tree $T$

$$I(T) \geq \begin{cases} 
R(T) + 1/3, & \text{if } D(T) \text{ is odd} \\
R(T) + 5/6, & \text{if } D(T) \text{ is even}
\end{cases}$$

with equality if and only if $T$ is a path of at least four vertices.

This corollary tells us that $I(T) - R(T)$ is bounded below. We point out that it is unbounded
above. In fact, for the full binary tree $T$ of height $h \geq 3$ we have $I(T) - R(T) = 2^{h+2}/5 - h - 1/4$,
which can be arbitrarily large as $h$ tends to infinity.

**References**

[2] S. Fajtlowicz, Written on the wall, a list of conjectures of Graffiti, preprint, University of Houston, USA.

