Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard Number 1

Jun-Muk Hwang [1] and **Ngaiming Mok**

In the study of manifolds having the geometric structure modeled on Hermitian symmetric spaces ([HM1]) and the deformation rigidity of irreducible Hermitian symmetric spaces of the compact type ([HM2]), the following result of Ochiai ([Oc]) played an essential role.

Theorem (Ochiai) *Let X be an irreducible Hermitian symmetric space of the compact type of rank ≥ 2. X has a natural G-structure where G is the reductive Levi factor of the isotropy subgroup of a base point of X. Let $U_1, U_2 \subset X$ be two connected open sets and $\varphi : U_1 \to U_2$ be a biholomorphism preserving the G-structure. Then φ can be extended to a biholomorphic automorphism of X.*

This result was generalized to other rational homogeneous spaces by Yamaguchi ([Ya]), where the statement holds with 'G-structure' replaced by a natural geometric structure on the homogeneous space. Their proof relies on the vanishing of certain Lie algebra cohomology groups. Since this result was very useful in the study of many geometric problems on the rational homogeneous spaces, one may ask whether a more geometric proof can be given using only rational curves, so that it can be generalized to some non-homogeneous projective manifolds. This was partially achieved in secions 3 and 4 of [HM4], where the authors were able to give a proof of the above result of Ochiai and Yamaguchi, via the deformation theory of rational curves and basic theory of differential systems, without using Lie algebra cohomology. Still, it was unsatisfactory in the sense that one has to use group actions to analytically continue φ to the whole X, so the proof works only for the homogeneous manifolds.

In this paper, we overcome this by introducing analytic continuations along special families of rational curves and give a proof which can work for a large class of Fano manifolds of Picard number 1.

To state our result, it is necessary to define a natural 'geometric structure' on a Fano manifold of Picard number 1. This is given by tangent vectors to standard rational curves. Roughly speaking, a standard rational curve is an immersed $\mathbf{P}_1$ in the Fano manifold X whose normal bundle contains only $\mathcal{O}(1)$ and $\mathcal{O}$ factors. Such curves exist by a result of Mori ([Mo]). Choosing a maximal irreducible family $\mathcal{H}$ of standard rational curves, we define the variety of $\mathcal{H}$-tangent $\mathcal{C} \subset \mathbf{P}T(X)$ as the collection of tangent vectors to standard rational curves belonging to $\mathcal{H}$ (see section 1 for details). This corresponds to our geometric structure on X. In the case of a rational homogeneous space X of Picard number 1, the lines on X under the minimal projective embedding of X are standard rational curves and the associated $\mathcal{C}$ corresponds to the natural geometric structure on X. In other words, the condition on φ "preserving the G-structure" in the above Theorem of Ochiai can be replaced by "whose differential sends $\mathcal{C}|_U$ to $\mathcal{C}|_{U'}$". Our main theorem is a generalization of Ochiai's theorem in this sense. We can give a rough outline of the statement of the main theorem as follows. See Theorem 1.2 for the precise statement.

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Main Theorem *Let X be a Fano manifold of Picard number 1. Suppose there exists a family of standard rational curves $\mathcal{H}$ such that the associated $\mathcal{C} \subset \mathbf{P}T(X)$ satisfies certain conditions which hold for many examples as given in section 1. Let X' be any Fano manifold of Picard number 1 and $\mathcal{H}'$ be a family of standard rational curves on X'. Given any connected open subsets $U \subset X, U' \subset X'$ with a biholomorphic map $\varphi : U \to U'$ such that the differential $\varphi_* : \mathbf{P}T_x(X) \to \mathbf{P}T_{\varphi(x)}(X')$ sends each irreducible component of $\mathcal{C}|_U$ to an irreducible component of $\mathcal{C}|_{U'}$ biholomrophically, there exists a biholomorphic map $\Phi : X \to X'$ such that φ is the restriction of Φ to U.*

This result is stronger than Ochiai's even for the irreducible Hermitian symmetric space X in the sense that we need not assume that X' is a priori biholomorphic to X.

When both X and X' are hypersurfaces of low degree in the projective space, our result can follow from the work of Jensen and Musso ([JM]) which completed a study initiated by E. Cartan and G. Fubini. Although the method of proof and basic ideas are completely different, we think that it is fair to say that the origin of this type of problem goes back to E. Cartan and G. Fubini, and we name the extension of the above kind as 'Cartan-Fubini type extension'.

We expect that there are many applications of the Cartan-Fubini type extension property. As a matter of fact, our works [HM1], [HM2], [HM3] can be viewed as applications. Another application is the rigidity of generically finite morphisms which we explain at the end of Section 1, after giving precise statement of the main theorem, Theorem 1.2, and some examples.

The proof of Theorem 1.2 will be given in Sections 2-4. Section 2 and Section 3 are the main part of the analytic continuation. Our analytic continuation is different from the classical one in the sense that it should by carried out only along the rational curves involved. For this, we introduce the concept of 'parametrized analytic continuation'. The proof will be finished in Section 4 by proving first that the map can be extended to a bimeromorphic map and then that it cannot have a ramification locus.

A few words on the terminology are in order. When we say an open set, it is in the classical topology, not Zariski topology, unless it is specifically said so. By a generic point of an analytic variety, we mean a point outside the union of countably many proper analytic subvarieties. A variety is not necessarily irrecucible, but has only finitely many components.

1 Statement and examples of the main result

We start with defining some terms that we are going to use throughout the article. We will skip most of the proofs of standard facts, refering 1.1 of [HM4] and II.2 of [Kl] for further details.

A rational curve $h : \mathbf{P}_1 \to X$ on a complex manifold X is called a **standard rational curve**, if $h^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for nonnegative integers p, q. In this case, h is necessarily a holomorphic immersion and birational. From $H^1(\mathbf{P}_1, h^*T(X)) = 0$, the space $Hom(\mathbf{P}_1, X)$ of morphisms from $\mathbf{P}_1$ to X is smooth at the point $[h]$ and the tangent space is $H^0(\mathbf{P}_1, h^*T(X))$. Let $\mathcal{H}$ be an irreducible component of $Hom(\mathbf{P}_1, X)$ containing a standard rational curve. Then a generic point of $\mathcal{H}$ is a standard rational curve. An irreducible component $\mathcal{H}$ of $Hom(\mathbf{P}_1, X)$

will be called a **standard component** if a generic member of $\mathcal{H}$ is a standard rational curve. The following properties of standard rational curves will be useful.

Lemma 1.1 *Let $h : \mathbf{P}_1 \to X$ be a standard rational curve. Then*

(1) The image of deformations of h cover an open neighborhood of $h(\mathbf{P}_1)$ in X.

(2) Let h_t be a deformation of $h = h_0$ parametrized by the disc $\Delta := \{t \in \mathbf{C}, |t| < 1\}$. Suppose the deformation h_t fixes two points, namely, for two distinct points $o, \infty \in \mathbf{P}_1$ and for all t, $h_t(o) = h_0(o)$ and $h_t(\infty) = h_o(\infty)$. Then h_t is a trivial deformation in the sense that $h_t(s) = h_0(s)$ for all $s \in \mathbf{P}_1$.

Proof. (1) follows from $H^1(\mathbf{P}_1, h^*T(X)) = 0$ and the fact $h^*T(X)$ is generated by global sections. (2) follows from the fact that the normal sheaf $h^*T(X)/T(\mathbf{P}_1) \cong [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ cannot have sections vanishing at two distinct points. □

Given a standard component $\mathcal{H}$, the natural action of the automorphism group of $\mathbf{P}_1$ gives $\mathcal{H}$ a structure of $\mathbf{PGL}_2$-principal bundle over an analytic space $\mathcal{K}$. The graphs of the elements of $Hom(\mathbf{P}_1, X)$ induces a $\mathbf{P}_1$-bundle $\mathcal{U}$ over $\mathcal{K}$, with natural universal family morphisms $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$. Let $\mathcal{K}^o \subset \mathcal{K}$ be the Zariski-open subset consisting of standard rational curves and $\mathcal{U}^o := \rho^{-1}(\mathcal{K}^o)$ be the universal family over $\mathcal{K}^o$. Then $\mathcal{K}^o$ is a complex manifold of dimension $n + p - 1$. By associating the tangent vectors to standard rational curves, we define the tangent morphism $\tau : \mathcal{U}^o \to \mathbf{P}T(X)$, which is a holomorphic immersion. Let $\mathcal{C} \subset \mathbf{P}T(X)$ be the closure of the image of τ. $\mathcal{C}$ will be called the **variety of $\mathcal{H}$-tangents**, or **variety of rational tangents** if the choice of $\mathcal{H}$ is clear. For a point $x \in X$, we call $\mathcal{C}_x := \mathcal{C} \cap \mathbf{P}T_x(X)$ **the variety of $\mathcal{H}$-tangents at** x. We define $\mathcal{U}_x := \mu^{-1}(x)$ and $\mathcal{U}_x^o = \mathcal{U}_x \cap \mathcal{U}^o$. Let $\tau_x : \mathcal{U}_x^o \to \mathbf{P}T_x(X)$ be the restriction of τ. For a generic point $x \in X$, $\mathcal{C}_x$ is equal to the closure of the image of τ_x.

The foliation on $\mathcal{U}^o$ defined by the fibers of ρ induces a multi-valued foliation $\mathcal{F}$ on a Zariski-open set of $\mathcal{C}$ by the immersion $\tau : \mathcal{U}^o \to \mathcal{C}$. $\mathcal{F}$ will be called the **tautological foliation** on $\mathcal{C}$. This name is not precise in the sense that $\mathcal{F}$ may be multi-valued. However, in the case we deal with in this article, it will be a univalent foliation.

When X is a projective manifold, $\mathcal{K}$ is a quasi-projective scheme which is the semi-normalization of the subvariety of the Chow variety corresponding to the images of elements of $\mathcal{H}$ and $\rho : \mathcal{U} \to \mathcal{K}$ is induced by the universal family over the Chow variety. See II.2 of [Kl] for details. It follows that we can naturally compactify $\mathcal{K}$ and $\mathcal{U}$ to projective varieties and the universal family morphisms ρ and μ can be extended. For projective X, we will use the same symbols $\mathcal{K}, \mathcal{U}$ to denote these projective varieties. $\rho : \mathcal{U} \to \mathcal{K}$ is no longer a $\mathbf{P}_1$-bundle, but just its generic fiber is $\mathbf{P}_1$. Mostly, we will work with $\mathcal{K}$ instead of $\mathcal{H}$, because we only use the property of the image of $h : \mathbf{P}_1 \to X$. For simplicity, we will call the image curve $C = h(\mathbf{P}_1)$ simply as a standard rational curve.

Now let X and X' be a Fano manifold of Picard number 1. By Mori's bend-and-break trick ([Mo]), X and X' contains a standard rational curve. Let $\mathcal{H}$ (resp. $\mathcal{H}'$) be a standard component and $\mathcal{C}$ (resp. $\mathcal{C}'$) be the variety of $\mathcal{H}$-tangents (resp. $\mathcal{H}'$-tangents) which has fiber dimension p (resp. p'). We say that **Cartan-Fubini type extension** holds for the pair $(X, \mathcal{H})$, if for any choice of $X', \mathcal{H}'$ with $p = p'$ and any connected open subsets $U \subset X, U' \subset X'$ with a biholomorphic map $\varphi : U \to U'$ such that the differential $\varphi_* : \mathbf{P}T_x(X) \to \mathbf{P}T_{\varphi(x)}(X')$ sends each irreducible component of $\mathcal{C}_x$ to an irreducible component of $\mathcal{C}'_{\varphi(x)}$ for all generic $x \in U$, there exists a biholomorphic map $\Phi : X \to X'$ such that φ is the restriction of Φ to U. In other words, a local holomorphic map preserving varieties of rational tangents extends to a global holomorphic map. Our main result is the following.

Theorem 1.2 *Let X be a Fano manifold with Picard number 1. Suppose there exists a standard component $\mathcal{H}$ with $p, q > 0$ such that for a generic point $x \in X$, the Gauss map for each irreducible component of $\mathcal{C}_x$ at x as a projective subvariety of $\mathbf{P}T_x(X)$ is generically finite. Then Cartan-Fubini type extension holds for $(X, \mathcal{H})$.*

There are many examples of Fano manifolds where the conditions for Theorem 1.2 hold. The condition on the Gauss map holds, if it holds for some component of $\mathcal{C}_x$ at generic $x \in X$ by the irreducibility of $\mathcal{C}$. By Zak's result ([Za]) or its weaker version ([GH]), this condition is satisfied if $\mathcal{C}_x$ is smooth and not linear. Suppose $\mathcal{H}$-curves are lines under a projective embedding of X. Then the smoothness of $\mathcal{C}_x$ at generic $x \in X$ is well-known and the condition $p, q > 0$ is equivalent to $3 \leq c_1(X) \leq \dim(X)$. So Theorem 1.2 works in the following two cases:

(1) Rational homogeneous space G/P of Picard number 1 different from the projective space. $\mathcal{K}$ is the set of lines under the minimal projective embedding. $\mathcal{C}_x$ is smooth and not linear.

(2) Smooth linearly nondegenerate complete intersections $X \subset \mathbf{P}_N$ of dimension ≥ 2 and of multi-degree $(d_1, \ldots, d_l)$ with $1 < d_1 + \cdots + d_l \leq N - 2$. $\mathcal{K}$ is the set of lines of $\mathbf{P}_N$ lying on X. $\mathcal{C}_x$ is a smooth complete intersection for generic x. Defining equations of $\mathcal{C}_x$ can be obtained by differentiating the defining equations of X.

The following is an example where the standard rational curves are not lines under a projective embedding:

(3) Let X be the moduli space of stable bundles of rank 2 with a fixed determinant of odd degree over a smooth projective curve of genus ≥ 5. Through a generic point of X, there exists a standard rational curve arising from Hecke correspondence, called a Hecke curve. For the corresponding standard component, $\mathcal{C}_x$ is a ruled surface which is nondegenerate and smooth in $\mathbf{P}T_x(X)$ for generic $x \in X$. See [Hw] for details.

In the statement of Theorem 1.2, the condition that $q > 0$ is necessary. In fact, if $q = 0$, which is the case for the projective space, the condition on φ of preserving varieties of rational tangents is void and φ can be just any local biholomorphic map.

On the other hand, the condition $p > 0$ is restrictive. There are many Fano manifolds with Picard number 1 such that $p = 0$ for all standard components with $q > 0$. Most notably, smooth hypersurfaces of degree n in $\mathbf{P}_{n+1}$ belong to this case. But we do not know whether there exists an example with $p = 0$ for which the Cartan-Fubini type extension property does not hold.

Our proof heavily depends on the condition $p > 0$. The condition on the Gauss map will be used only for the following result proved in 3.1 of [HM4].

Proposition 1.3 *Assume that $(X, \mathcal{H})$ satisfies the assumptions of Theorem 1.2. Then the tangent morphism $\tau : \mathcal{U}^o \to \mathcal{C}$ is birational. Furthermore, for any choice of Fano manifold X' with Picard number 1, a standard component $\mathcal{H}'$ with $\mathcal{C}' \subset \mathbf{P}T(X')$ having fiber dimension p over X', and any connected open subsets $U \subset X, U' \subset X'$, if there exists a biholomorphic map $\varphi : U \to U'$ satisfying $\varphi_*(\mathcal{C}_x) \subset \mathcal{C}'_{\varphi(x)}$ for all generic $x \in U$, then for any member C of $\mathcal{K}$,*

$\varphi(C \cap U)$ is contained in $C' \cap U'$ for some member C' of $\mathcal{K}'$. In other words, φ sends local pieces of $\mathcal{H}$-curves to local pieces of $\mathcal{H}'$-curves.

Proof. The birationality of τ is stated in Corollary 3.1.5 of [HM4], where it is proved that the tautological foliation is uniquely determined by the variety of minimal rational tangents if the Gauss map condition is satisfied. The second statement is an immediate consequence of this. □

The proof of Theorem 1.2 will be given in Sections 2-4. We want to finish this section with an application. The Cartan-Fubini type extension property implies the rigidity of generically finite morphisms in the following sense.

Theorem 1.4 *Let $(X_0, \mathcal{H}_0)$ be a Fano manifold of Picard number 1 with the Cartan-Fubini type extension property. Let Y be any complete variety and $\pi : \mathcal{X} \to \Delta := \{t \in \mathbf{C}, |t| < 1\}$ be a regular family of Fano manifolds of Picard number 1 such that $X_0 = \pi^{-1}(0)$. Then for any surjective morphism $f : Y \times \Delta \to \mathcal{X}$ over Δ such that the restriction $f_t : Y \to X_t = \pi^{-1}(t)$ is generically finite for each $t \in \Delta$, there exists $\epsilon > 0$ and a unique family of biholomorphic morphisms $g_t : X_0 \to X_t$ for $|t| < \epsilon$ satisfying $f_t = g_t \circ f_0$.*

Corollary 1.5 *Given any complete varieties X and Y of the same dimension, let $Hol(Y, X)$ be the set of surjective holomorphic maps from Y to X. Then for any fixed Y and any Fano manifold X of Picard number 1 having the Cartan-Fubini type extension property with respect to some choice of a standard component, $Hol(Y, X)$ is countable up to automorphisms of X. Furthermore there exist only countably many such Fano manifolds X, for which $Hol(Y, X) \neq \emptyset$.*

For the proof of Theorem 1.3, we need to recall some results from [HM3]. Let Y be a projective manifold and $y \in Y$ be a point. In Section 1 of [HM3], we define the notion of a variety of distinguished tangents. Roughly speaking, an irreducible subvariety of $\mathbf{P}T_y(Y)$ is a variety of distinguished tangents if it is the closure of tangent vectors to a family of curves passing through y which corresponds to a stratum of a natural stratification of the Hilbert scheme of curves through y. We refer to [HM3] for precise definitions. What we need here is the fact that there are only countably many varieties of distinguished tangents in $\mathbf{P}T_y(Y)$, which is an immediate consequence of the definition. We also need the following proposition.

Proposition 1.6 (Proposition 3 in [HM3]) *Let $f : Y \to X$ be a generically finite surjective morphism from a projective manifold Y to a Fano manifold X of Picard number 1. Choose a standard component on X and let $\mathcal{C} \subset \mathbf{P}T(X)$ be the variety of rational tangents. Then for any generic point $y \in Y$, each irreducible component of the subvariety $df_y^{-1}(\mathcal{C}_{f(y)}) \subset \mathbf{P}T_y(Y)$ is a variety of distinguished tangents.*

In [HM3], this was stated for a finite morphism f and varieties of "minimal rational tangents" on X. But the proof works equally well for the general case stated above.

Proof of Theorem 1.4. A standard rational curve $h : \mathbf{P}_1 \to X_0$ can be viewed as a standard rational curve of $\mathcal{X}$. By Lemma 1.1 (1), there exists a family $h_t : \mathbf{P}_1 \to X_t$ for $|t| < \epsilon$ for some $\epsilon > 0$, which is a standard rational curve in each X_t. Let $\mathcal{H}_t$ be the standard component of $Hom(\mathbf{P}_1, X_t)$ containing h_t. Let $\mathcal{X}_\epsilon = \pi^{-1}(\{|t| < \epsilon\})$.

Let $\mathcal{H}$ be the standard component of $Hom(\mathbf{P}_1, \mathcal{X}_\epsilon)$ containing h_t's and let $\mathcal{C} \to \mathbf{P}T(\mathcal{X}_\epsilon)$ be the variety of $\mathcal{H}$-tangents. Since all images of elements of $Hom(\mathbf{P}_1, \mathcal{X})$ are contained in the fibers of π, $\mathcal{C}$ is contained in the subbundle $\mathbf{P}T^{\pi}$ of $\mathbf{P}T(\mathcal{X})$ where T^{π} denotes the relative tangent bundle of π. $\mathcal{C}$ is irreducible and there exists some $\epsilon' < \epsilon$ so that $\mathcal{C} \cap \mathbf{P}T(X_t)$ is irreducible for $0 < |t| < \epsilon'$. In particular, $\mathcal{C} \cap \mathbf{P}T(X_t)$ is exactly $\mathcal{C}_t$, the variety of $\mathcal{H}_t$-tangents, for $0 < |t| < \epsilon'$.

Since $\mathcal{C}$ is locally an immersed submanifold near the point corresponding to h, we see that $\mathcal{C}_0$, the variety of $\mathcal{H}_0$-tangents, is contained in the closure of the union of $\mathcal{C}_t$ for $0 < |t| < \epsilon'$.

Choose a small open set $U^* \subset Y$ and shrink ϵ' if necessary, so that $f_t|_{U^*}$ is biholomorphic for small t and the image $f_t(U^*)$ is contained in the open set covered by the union of images of $\mathcal{H}_t, |t| < \epsilon'$. For each $y \in U^*$, let $\mathcal{C}_{f_t(y)}$ be the variety of rational tangents at $f_t(y)$ associated to $\mathcal{H}_t, 0 < |t| < \epsilon'$. Then the closure of the union of $\{\mathcal{C}_{f_t(y)}, 0 < |t| < \epsilon'\}$ contains $\mathcal{C}_{f_0(y)}$, the variety of rational tangents at $f_0(y)$ for $\mathcal{H}_0$. By Proposition 1.6, $\{df_t^{-1}(\mathcal{C}_{f_t(y)}), 0 < |t| < \epsilon'\}$ gives a family of varieties of distinguished tangents in $\mathbf{P}T_y(Y)$. Since there are only countably many varieties of distinguished tangents in $\mathbf{P}T_y(Y)$, $f_t^{-1}(\mathcal{C}_{f_t(y)})$ is independent of t and $f_0^{-1}(\mathcal{C}_{f_0(y)})$ is the union of some components of $f_t^{-1}(\mathcal{C}_{f_t(y)}), t \neq 0$. The biholomorphic map $\varphi_t := f_t \circ f_0^{-1}$ from $f_0(U^*)$ to $f_t(U^*)$ sends each component of $\mathcal{C}_{f_0(y)}$ to a component of $\mathcal{C}_{f_t(y)}$. By the Cartan-Fubini type extension property, it can be extended to a biholomorphic map $g_t : X_0 \to X_t$ with the desired property. □

Remark Although we do not know whether Cartan-Fubini type extension holds for the case of $p = 0$, an analogue of Theorem 1.4 for the case of $p = 0$ is proved in [HM5], by using a completely different method which cannot work for the case $p > 0$.

2 Analytic continuation along standard rational curves

For the biholomorphic map $\varphi : U \to U'$ in the statement of Theorem 1.2, we will say that φ preserves varieties of rational tangents. For the proof of Theorem 1.2 we will have to deal with locally defined meromorphic maps which preserve varieties of rational tangents at generic points. More precisely, let $\Omega \subset X$ be a connected open set and $\varphi : \Omega \to X'$ be a meromorphic map. We say that φ **preserves varieties of rational tangents** if and only if (a) φ is of maximal rank at a generic point $x \in \Omega$ and (b) for such $x \in \Omega$ we have $\varphi_* \mathcal{C}_x \subset \mathcal{C}'_{\varphi(x)}$, i.e., φ_* sends each component to $\mathcal{C}_x$ to a component of $\mathcal{C}'_{\varphi(x)}$.

Theorem 1.2 will be proved by constructing an analytic continuation of φ. This analytic continuation is different from the classical one, in the sense that we have to carry it out only along standard rational cuves. Let $C \subset X$ be a $\mathcal{K}^o$-curve intersecting U. We want to get an analytic continuation of φ along paths lying on C. This analytic continuation needs not be univalent because C is not necessarily smooth. Moreover we want to repeat this process along other standard rational curves intersecting U. For this reason, it is convenient to introduce the notion of parametrized analytic continuation along a holomorphic map from a complex space into X.

Let x_0 be a point on X and φ be a germ of meromorphic map into X' at x_0 preserving varieties of rational tangents. Let S be a complex space and $s_0 \in S$ be a base point. Let $\lambda : S \to X$ be a holomorphic map such that $\lambda(s_0) = x_0$. By the **parametrized analytic continuation of φ along λ** we mean a germ of meromorphic map F along $\Sigma := \text{Graph}(\lambda) \subset S \times X$ such that

(a) denoting by $pr_X : S \times X \to X$ the canonical projection onto the second factor, the germ of F at $(s, \lambda(s))$ agrees with $pr_X^* \nu$ for some germ of meromorphic map ν into X' at $\lambda(s) \in X$ for each $s \in S$;

(b) the germ of F at (s_0, x_0) agrees with $pr_X^* \varphi$.

We will write $\lambda : (S; s_0) \to (X; x_0)$ to indicate that $s_0 \in S$ is the base point, $\lambda(s_0) = x_0$. We sometimes write $(\varphi; x_0)$ for the germ of φ at x_0, and $(F; \Sigma)$ for the germ of F along Σ, etc.

We have analytic continuation of the meromorphic map preserving rational tangents along standard rational curves in the following way.

Proposition 2.1 *Under the assumptions of Theorem 1.2, let x_0 be a point in U and C_0 be a standard rational curve through x_0. Choose a point $u_0 \in \rho^{-1}([C_0])$ satisfying $\mu(u_0) = x_0$. (A choice of u is equivalent to the choice of a local irreducible component of C_0 at x.) Then, there exists an open neighborhood $\mathcal{B}_0$ of $[C_0]$ in $\mathcal{K}^o$, so that for $\lambda := \mu|_{\rho^{-1}(\mathcal{B}_0)} : \rho^{-1}(\mathcal{B}_0) \to X$, there exists a parametrized analytic continuation of the germ of meromorphic map $(\varphi; x_0)$ along $\lambda : (\rho^{-1}(\mathcal{B}_0); u_0) \to (X; x_0)$.*

We will prove three lemmas first.

Lemma 2.2 *Let $\Omega \subset X$ be a connected open set and $\varphi : \Omega \to X'$ be a meromorphic map preserving varieties of rational tangents. Let $x \in \Omega$ be a point and $[C] \in \mathcal{K}^o$ be a standard rational curve passing through x. Choose $u \in \rho^{-1}([C]) \subset \mathcal{U}^o$ such that $\mu(u) = x$. Then, there exist an open neighborhood $\mathcal{W}$ of u in $\mathcal{U}^o$, an open neighborhood $\mathcal{B}$ of $[C]$ in $\mathcal{K}^o$, together with meromorphic maps $\varphi^\flat : \mathcal{W} \to \mathcal{U}'$, $\varphi^\# : \mathcal{B} \to \mathcal{K}'$, such that $\tau' \circ \varphi^\flat \equiv [d\varphi] \circ \tau$ and $\rho' \circ \varphi^\flat \equiv \varphi^\# \circ \rho$. Moreover, the germs of $\varphi^\flat$ at u and of $\varphi^\#$ at $[C]$ are uniquely determined by φ and they are of maximal rank at generic points.*

Here and henceforth an open neighborhood is always understood to be connected. As is evident $\tau' : \mathcal{U}' \to \mathbf{P}T(X')$ denotes the analogue of $\tau : \mathcal{U} \to \mathbf{P}T(X)$, etc.

Proof. Consider

$$\mathcal{U}^o|_\Omega \xrightarrow{\tau} \mathcal{C}|_\Omega \xrightarrow{\varphi_*} \mathcal{C}' \xleftarrow{\tau'} \mathcal{U}'^o.$$

By Proposition 1.3, $\tau : \mathcal{U}^o \to \mathcal{C}$ and $\tau' : \mathcal{U}'^o \to \mathcal{C}'$ are birational immersions. We define $\varphi^\flat$ to be the composition $\tau'^{-1} \circ \varphi_* \circ \tau$, which is a meromorphic map from $\mathcal{U}|_\Omega$ into $\mathcal{U}'$. Let $\mathcal{W}$ be the connected component of $\mathcal{U}|_\Omega$ containing u. By Proposition 1.3, $\varphi^\flat$ sends the fibers of ρ on $\mathcal{W}$ to fibers of ρ', inducing a meromorphic map $\varphi^\# : \mathcal{B} \to \mathcal{K}'$ for some open set $\mathcal{B} \subset \mathcal{K}^o$ containing $[C]$.
□

Lemma 2.3 *Suppose we are given a connected open set $\mathcal{B} \subset \mathcal{K}^o$ and a meromorphic map $\xi : \mathcal{B} \to \mathcal{K}'$. For any $[C] \in \mathcal{K}^o$, any $x \in C$ and $u \in \mu^{-1}(x) \cap \rho^{-1}([C])$, there exists at most one germ of meromorphic map φ at x to X' preserving varieties of rational tangents, so that the germ of the induced map $\varphi^\#$ at $[C]$ with respect to u defined in Lemma 2.2 agrees with ξ.*

Proof. Suppose not. We may assume that

(i) there exist two distinct meromorphic maps $\varphi_1, \varphi_2 : \Omega \to X'$ on some neighborhood Ω of x, both of them preserving rational tangents;

(ii) the induced maps $\varphi_1^\flat$ and $\varphi_2^\flat$ are defined on the same neighborhood $\mathcal{W}$ of u;

(iii) the induced maps $\varphi_1^\#$ and $\varphi_2^\#$ are defined and equal on $\mathcal{B}$.

Let $y \in \Omega$ be a generic point. $\rho(\mu^{-1}(y))$ is a p-dimensional family of standard rational curves through y. Recall that $\varphi_1^\#$ and $\varphi_2^\#$ have maximal rank at generic points. By $\varphi_1^\#$, it will be sent to a p-dimensional family of standard rational curves on X' passing through $\varphi_1(y)$. By $\varphi_2^\#$, it will be sent to a p-dimensional family of standard rational curves passing through $\varphi_2(y)$. But $\varphi_1^\# = \varphi_2^\#$, so we get a p-dimensional family of standard rational curves on X' passing through two distinct points $\varphi_1(y) \neq \varphi_2(y)$. A contradiction to Lemma 1.1 (2). □

Lemma 2.4 *Suppose we are given a $\mathcal{K}^o$-curve $C \subset X$, a point $x \in C$, $u \in \rho^{-1}([C]) \cap \mu^{-1}(x)$, and a meromorphic map $\varphi : \Omega \to X'$ in a neighborhood of x preserving rational tangents. Choose $\mathcal{W}, \mathcal{B}, \varphi^\flat, \varphi^\#$ as in Lemma 2.2. Let Δ^p denote the p-dimensional polydisc. Given $y \in C$ and $w \in \rho^{-1}([C]) \cap \mu^{-1}(y)$ with neighborhoods $y \in D_y$ in X and $w \in \mathcal{D}_w$ in $\mathcal{U}$ satisfying*

(i) $\mathcal{D}_w \subset \rho^{-1}(\mathcal{B})$;

(ii) $\mu(\mathcal{D}_w) = D_y$ and $\mathcal{D}_w$ is biholomorphic to $D_y \times \Delta^p$ in such a way that the fiber of $\mu|_{\mathcal{D}_w}$ over $z \in D_y$ corresponds to $\{z\} \times \Delta^p$; and

(iii) $\mathcal{D}_w \cap \mathcal{W} \neq \emptyset$ and $D_y \cap \Omega \neq \emptyset$,

there exists a meromorphic map $\varphi_1 : D_y \to X'$ preserving rational tangents, so that $\varphi_1 = \varphi$ on $D_y \cap \Omega$ and the induced maps $\varphi_1^\#$ agrees with $\varphi^\#$ as germs of meromorphic maps at $[C] \in \mathcal{B}$.

Proof. Define $\zeta : \rho^{-1}(\mathcal{B}) \to \mathcal{K}'$ by $\zeta := \varphi^\# \circ \rho$. Identify $\mathcal{D}_w$ with $D_y \times \Delta^p$. Then choosing a point $v \in \Delta^p$ corresponds to assigning a $\mathcal{K}^o$-curve $C_{x,v}$ to each point x of D_y. Choose a generic $v \in \Delta^p$ so that ζ is holomorphic at a generic point of $D_y \times \{v\}$. This gives a $\mathcal{K}'^o$-curve $C'_{x,v}$ for each $x \in D_y$, defined by the meromorphic map $\zeta_v : D_y \to \mathcal{K}'$ by $\zeta_v(z) = \zeta(z, v)$ for $z \in D_y$. We want to show that the family of curves $C'_{x,v}$ defined by generic choices of $v \in \Delta^p$ has a unique common point and define $\phi_1(x)$ as this common point. To make it precise, we will work with their graphs.

Let $\Theta_v \subset D_y \times \mathcal{U}'$ be defined by

$$\begin{aligned}\Theta_v &:= (id, \rho')^{-1}(\mathrm{Graph}(\zeta_v)) \\ &= \{(x, u') \in D_y \times \mathcal{U}', \rho'(u') \in C'_{x,v}\}.\end{aligned}$$

Let (id, μ') be the map $D_y \times \mathcal{U}' \to D_y \times X'$ and define

$$\begin{aligned}\Pi_v &:= (id, \mu')(\Theta_v) \\ &= \{(x, x') \in D_y \times X', x' \in C'_{x,v}\}.\end{aligned}$$

Then Π_v is an analytic subvariety of $D_y \times X'$ which is proper over D_y. Consider now the intersection

$$\begin{aligned}\Pi &:= \bigcap\{\Pi_v : v \in \Delta^p, \zeta_v \text{ is holomorphic at a generic point of } D_y\} \\ &= \{(x, x') \in D_y \times X', x' \in \bigcap C'_{x,v} \text{ for generic } v \in \Delta^p\}.\end{aligned}$$

Then Π is also proper over D_y. With respect to the canonical projection $D_y \times X' \to D_y$ the fiber of $\Pi \subset D_y \times X'$ over a generic point consists of the intersection of a p-dimensional family of standard rational curves on X'.

Over a generic point $z \in D_y \cap \Omega$, this is exactly the p-dimensional family of standard rational curves passing through $\varphi(z)$, and $\Pi|_{D_y \cap \Omega}$ can be regarded as the graph of $\varphi|_{D_y \cap \Omega}$. So $\Pi|_{D_y \cap \Omega}$ is bimeromorphic over $D_y \cap \Omega$. From the properness of Π over D_y, there exists a unique component of Π which is bimeromorphic over D_y, defining a meromorphic map $\varphi_1 : D_y \to X'$. It certainly satisfies the required properties. □

Proof of Proposition 2.1. From φ at x_0 and u_0, we get $\mathcal{B}, \varphi^\flat, \varphi^\#$ as in Lemma 2.2. Since μ is submersive along $\rho^{-1}([C_0])$ by Lemma 1.1 (1), we can choose finitely many points $y_i \in C_0, w_i \in \rho^{-1}([C_0]) \cap \mu^{-1}(y_i)$ and cover $\rho^{-1}([C_0])$ by finite number of open sets $\mathcal{D}_{w_i}$'s in $\rho^{-1}(\mathcal{B})$ so that $\mathcal{D}_{w_i} \cong D_{y_i} \times \Delta^p$ for suitable D_{y_i}'s covering C_0. Choose $\mathcal{B}_0 \subset \mathcal{B}$ so that $\rho^{-1}(\mathcal{B}_0) \subset \cup \mathcal{D}_{w_i}$. By repeatedly applying Lemma 2.4, we obtain analytic continuation $\tilde{\varphi}_i$ of φ on D_{y_i}. This may not be univalent on the open set $\cup D_{y_i}$ of X. But its pull-back to $\cup \mathcal{D}_{w_i}$ must be univalent by Lemma 2.3, defining a parametrized analytic continuation of (φ, x_0) along λ. □

Let $\alpha : (\tilde{S}; \tilde{s}_0) \to (S; s_0)$ be a holomorphic map between complex spaces with base points, $\alpha(\tilde{s}_0) = s_0$. Let F be a parametrized analytic continuation of φ along $\Sigma := \text{Graph}(\lambda)$. Let $\mathcal{V} \subset S \times X$ be an open neighborhood of Σ on which F can be defined. Consider $\tilde{\lambda} : (\tilde{S}; \tilde{s}_0) \to (X; x_0)$ for $\tilde{\lambda} := \lambda \circ \alpha$. Then the graph $\tilde{\Sigma} := \text{Graph}(\tilde{\lambda}) \subset \tilde{S} \times X$ is given by $\tilde{\Sigma} = (\alpha, id)^{-1}(\Sigma)$. The meromorphic map $\tilde{F} := (\alpha, id)^* F$ is defined on $\tilde{\mathcal{V}} := (\alpha, id)^{-1}(\mathcal{V})$, and the germ of meromorphic map $\tilde{F}$ into X' along $\tilde{\Sigma}$ is a parametrized analytic continuation of the germ of meromorphic map φ at x_0 along the map $\tilde{\lambda} : (\tilde{S}, \tilde{s}_0) \to (X, x_0)$. By abuse of notations we will write $\tilde{F} = \alpha^* F$. $(\tilde{F}; \tilde{\Sigma})$ is the **parametrized analytic continuation of** $(\varphi; x_0)$ **along** $\tilde{\lambda}$ **obtained by pulling back** $(F; \Sigma)$.

The proof of Proposition 2.1 can be easily modified to give

Proposition 2.5 *Under the assumptions of Theorem 1.2, let B be a complex space and $\beta : B \to \mathcal{K}^o$ be a holomorphic map, with associated holomorphic $\mathbf{P}^1$-bundle $\hat{\rho} : \mathcal{P} = \beta^* \mathcal{U}^o \to B$ and induced tautological map $\hat{\beta} : \mathcal{P} \to \mathcal{U}^o = \rho^{-1}(\mathcal{K}^o)$. Write $b_0 \in B$ resp. $s_0 \in \hat{\rho}^{-1}(b_0)$ for chosen distinguished points on B resp. $\mathcal{P}$, such that $\mu(\hat{\beta}(s_0)) = x_0$. Suppose there exists a holomorphic section $\sigma : B \to \mathcal{P}$ such that $\sigma(b_0) = s_0$. Consider $\mu \circ \hat{\beta} : (\mathcal{P}; s_0) \to (X; x_0)$ and $\mu \circ \hat{\beta}|_{\sigma(B)} : (\sigma(B), s_0) \to (X; x_0)$. Denote by $\Sigma \subset \mathcal{P} \times X$ resp. $\Sigma_0 \subset \sigma(B) \times X$ the graphs of $\mu \circ \hat{\beta}$ resp. $\mu \circ \hat{\beta}|_{\sigma(B)}$. Assume now that there exists a parametrized analytic continuation $(F_0; \Sigma_0)$ of $(\varphi; x_0)$ along $\mu \circ \hat{\beta}|_{\sigma(B)}$. Then, there exists a parametrized analytic continuation $(F; \Sigma)$ of $(\varphi; x_0)$ along $\mu \circ \hat{\beta}$ such that the restriction of F to $\sigma(B) \times X$ agrees with F_0 as germs along Σ_0.*

Proof. As in Lemma 2.2, F_0 induces $F_0^\# : B \to \mathcal{K}'$. Choose a neighborhood $\mathcal{V}$ of Σ_0 where F_0 is defined. We can cover $\mathcal{P}$ by open subsets of the form $\hat{\beta}^{-1}(\mathcal{D}_w)$ where $\mathcal{D}_w \subset \mathcal{U}^o$ is as defined in Lemma 2.4, in such a way that for each $\hat{\beta}^{-1}(\mathcal{D}_w)$, there exists a free rational curve C and a chain of open sets $\hat{\beta}^{-1}(\mathcal{D}_{w_i}), i = 0, 1, \ldots, k$ with $w_i \in \rho^{-1}([C])$ satisfying $w_0 = w, \mathcal{D}_{w_i} \cap \mathcal{D}_{w_{i+1}} \cap \rho^{-1}([C]) \neq \emptyset$ and $\hat{\beta}^{-1}(\mathcal{D}_{w_k}) \cap \sigma(B) \neq \emptyset$. By pulling back the analytic continuation $\tilde{\varphi}$ of φ to $\mathcal{D}_w$ obtained in Lemma 2.4, we can find analytic continuation $\hat{\varphi}$ to $\hat{\beta}^{-1}(\mathcal{D}_w)$. Then $\hat{\varphi}^\# = F_0^\#$ as germs at the points of B where it is defined. Thus the analytic continuation is uniquely well-defined by Lemma 2.3 and can be patched together to define F. □

3 Adjunction of standard rational curves

Throughout this section, we assume the situation of Theorem 1.2. We say that an irreducible subvariety $A \subset X$ is **saturated** if for any C with $[C] \in \mathcal{K}^o$, either $C \subset A$ or $C \cap A = \emptyset$.

Lemma 3.1 *There exists a countable union of proper subvarieties of X, so that the only saturated subvariety of X containing a point outside this countable union is X itself.*

Proof. Otherwise the union of saturated subvarieties of dimension $< n$ cover a Zariski-open subset of X. Thus there exists an irreducible subvariety $\mathcal{A}$ of the Hilbert scheme of X whose generic point corresponds to a saturated proper subvariety of X so that the members of $\mathcal{A}$ cover the whole X. By choosing a suitable subvariety of $\mathcal{A}$, we get a hypersurface $H \subset X$ which is the closure of the union of some collection of saturated proper subvarieties of X. Choose a $\mathcal{K}^o$-curve C_1 which is not contained in H. From the Picard number condition, C_1 intersects H. Thus small deformations of C_1 intersect generic points of H by Lemma 1.1 (1). This gives standard rational curves not contained in H but intersecting saturated subvarieties lying in H, a contradiction to the definition of saturated subvarieties. □

Let $x_0 \in X$ be a generic point in the sense of Lemma 3.1. Let S be an irreducible projective variety with a distinguished Zariski-open subset $V \subset S$ and a distinguished point $s_0 \in V$. Let $\lambda : (S; s_0) \to (X; x_0)$ be a holomorphic map generically finite over its image with $\lambda(s_0) = x_0$. Assume $\lambda(S) \neq X$.

We can construct a new irreducible projective variety $\hat{S}$ with a distinguished point $\hat{s}_0$ on a distinguished Zariski-open subset $\hat{V} \subset \hat{S}$ and a holomorphic map $\hat{\lambda} : (\hat{S}; \hat{s}_0) \to (X; x_0)$ generically finite over its image with $\hat{\lambda}(\hat{s}_0) = x_0$, as follows.

Consider the natural map $\mu : \mathcal{U} \to X$, the pull-back $\lambda^*\mu : \lambda^*\mathcal{U} \to S$ and the tautological map $\beta : \lambda^*\mathcal{U} \to \mathcal{U}$. Since $\lambda(S)$ is not saturated by the choice of x_0, generic fibers of $\lambda^*\mu$ correspond to standard rational curves which do not lie on S. Choose a generic point $u \in \mathcal{U}^o \cap \mu^{-1}(x_0)$ and let $\lambda^*u \in \lambda^*\mathcal{U}$ be the lifting of u lying above s_0. Since $\lambda^*\mathcal{U}$ is projective, there exists an irreducible projective subvariety $E \subset \lambda^*\mathcal{U}$ such that $\lambda^*\mu|_E : E \to S$ is generically finite and $\lambda^*u \in E$. Let $\alpha : Q \to E$ be a normalization of E and $q_0 \in Q$ be a point such that $\alpha(q_0) = \lambda^*u$. Then $(\rho \circ \beta \circ \alpha)^*\mathcal{U}$ defines an irreducible variety $\mathcal{P}$ with a natural map $\gamma : \mathcal{P} \to Q$ which is generically a $\mathbf{P}_1$-bundle. There is a tautological section $\sigma : Q \to \mathcal{P}$ of γ where $\sigma(q)$ corresponds to the point $\beta \circ \alpha(q)$ of the fiber of $\mathcal{U}$ over $\rho \circ \beta \circ \alpha(q)$. We let $\hat{S} = \mathcal{P}$, $\hat{s}_0 = \sigma(q_0)$ and $\hat{\lambda}$ to be the natural map from $\mathcal{P}$ to X induced by μ. Then $\hat{\lambda}(\hat{S})$ is an irreducible subvariety of X containing $\lambda(S)$ but not contained in $\lambda(S)$ because $\lambda(S)$ is not saturated. Since $\dim(\hat{S}) = \dim(S) + 1$, this implies that $\hat{\lambda}$ is generically finite. Let $Q^* \subset Q$ be the open subset $(\lambda^*\mu \circ \alpha)^{-1}(V)$. We define $\hat{V}$ to be the Zariski-open subset of $\mathcal{P}|_{Q^*}$ where the fibers of γ corresponds to standard rational curves of X. By our choice of u, $\hat{s}_0 \in \hat{V}$.

We say that $(\hat{S}, \hat{s}_0, \hat{V}, \hat{\lambda})$ is obtained from (S, s_0, V, λ) by **an adjunction of standard rational curves**. This construction is not unique and depends on the choice of E. From the construction and Proposition 2.5, the following is immediate.

Proposition 3.2 *Let $s_0 \in V \subset S$ be a distinguished point of a distinguished Zariski-open subset in an irreducible projective variety. Given a morphism $\lambda : (S; s_0) \to (X, x_0)$ generically* *finite over $\lambda(S) \neq X$ and x_0 generic in the sense of Lemma 3.1, let $(\hat{S}, \hat{s}_0, \hat{V}, \hat{\lambda})$ be an adjunction of standard rational curves. If there exists a parametrized analytic continuation of $(\varphi; x_0)$ along $\lambda|_V : (V; s_0) \to (X; x_0)$, then there exists a parametrized analytic continuation of $(\varphi; x_0)$ along $\hat{\lambda}|_{\hat{V}} : (\hat{V}; \hat{s}_0) \to (X; x_0)$.*

Starting from x_0, we can repeatedly apply this construction to obtain

Proposition 3.3 *Let $x_0 \in X$ be generic in the sense of Lemma 3.1. Then for $1 \le k \le n = \dim X$, there exist a k-dimensional irreducible projective variety $S^{(k)}$ with a distinguished point $s_0^{(k)}$, a holomorphic map $\lambda^{(k)} : (S^{(k)}; s_0^{(k)}) \to (X; x_0)$ generically finite over its image, and a non-trivial Zariski-open subset $V^{(k)} \subset S^{(k)}$ with $s_0^{(k)} \in V^{(k)}$, such that, for any germ $(\varphi; x_0)$ of meromorphic map into X' preserving varieties of rational tangents, there exists a parametrized analytic continuation of $(\varphi; x_0)$ along $\lambda^{(k)}|_{V^{(k)}} : (V^{(k)}; s_0^{(k)}) \to (X; x_0)$.*

Proof. To start with, choose a $\mathcal{K}^o$-curve C passing through x_0. Let $S^{(1)} = V^{(1)} = \mathbf{P}_1$ and $\lambda^{(1)}$ be the normalization $\mathbf{P}_1 \to C$ with $s_1 \in S^{(1)}$ a point over x_0. This satisfies the required analytic continuation property by Proposition 2.1. Now apply Proposition 3.2 inductively to construct $(S^{(k+1)}, s_0^{(k+1)}, V^{(k+1)}, \lambda^{(k+1)})$ as $(\hat{S}^{(k)}, \hat{s}_0^{(k)}, \hat{V}^{(k)}, \hat{\lambda}^{(k)})$ by an adjunction of standard rational curves. □

Using the above construction, we want to extend the given map φ to a multi-valued meromorphic map defined on a Zariski dense open subset of X, in other words, a meromorphic map defined on an unramified cover of a Zariski open subset of X. Given an unramified morphism $\chi : Z \to X$ from a complex manifold Z and $z \in Z$, we identify $T_z(Z)$ with $\chi^* T_{\chi(z)}(X)$ canonically and define $\mathcal{C}_z$ to be $[d\chi_z]^{-1}\mathcal{C}_{\chi(z)} \subset \mathbf{P}T_z(Z)$.

Proposition 3.4 *Let $x_0 \in X$ be a generic point, $\dim(X) = n$. Then, there exists an n-dimensional normal projective variety $\bar{\Sigma}$ with a distinguished point $\sigma_0 \in \bar{\Sigma}$, a generically finite holomorphic map $\chi : \bar{\Sigma} \to X$, $\chi(\sigma_0) = x_0$, and a non-empty smooth Zariski-open subset $Z \subset \bar{\Sigma}$ such that, writing $\pi = \chi|_Z$*

(a) $\pi : Z \to X - D$ is unramified for some divisor D;

(b) for any open neighborhood U of x_0 in X, and any meromorphic map $\varphi : U \to X'$ preserving varieties of rational tangents, there exists a meromorphic map $\psi : Z \to X'$ preserving varieties of rational tangents in the sense $\psi_*(\mathcal{C}_z) = \mathcal{C}'_{\psi(z)}$ at points $z \in Z$ at which ψ is locally biholomorphic, such that for some open neighborhood W of σ_0 on $\bar{\Sigma}$ for which $\chi(W) \subset U$, we have $\psi \equiv \chi^*\varphi$ on $W \cap Z$.

Note that x_0 may lie on D.

Proof. For $k = n$ in Proposition 3.3, $\lambda^{(n)} : (S^{(n)}; s_0^{(n)}) \to (X; x_0)$ is a surjective generically finite morphism. Write $\lambda = \lambda^{(n)}$, etc. and assume without loss of generality that S is normal. We have a non-empty Zariski-open subset $V \subset S$, $s_0 \in V$, such that, for any germ $(\varphi; x_0)$ of meromorphic map into X' preserving varieties of rational tangents, there exists a parametrized analytic continuation of $(\varphi; x_0)$ along $\lambda|_V : (V; s_0) \to (X; x_0)$. We need to extract a meromorphic map out of this parametrized analytic continuation.

Write $\Sigma \subset V \times X$ for $\mathrm{Graph}(\lambda|_V)$ and $(F; \Sigma)$ for the parametrized analytic continuation of $(\varphi; x_0)$ along $\lambda|_V$. Let $pr_X : V \times X \to X$ be the natural projection and $\chi = pr_X|_\Sigma$. Let $\bar{\Sigma}$ be a suitable projective variety compactifying Σ so that χ can be extended to a holomorphic map $\chi : \bar{\sigma} \to X$. Let $Z \subset \Sigma$ be a smooth Zariski-open set so that χ is unramified on Z. Write $\psi = F|_Z$. At any point $(s, \lambda(s)) \in Z \subset \Sigma$ the germ of $F|_{\{s\}\times X}$ at $(s, \lambda(s))$ preserves varieties of rational tangents at generic points, when $\{s\} \times X$ is identified with X canonically. By the condition (a) of the definition of parametrized analytic continuation, the germ of F at $(s, \lambda(s))$ is of the form $pr_X^* \nu$ for some germ ν at $\lambda(s)$, where ν preserves varieties of rational tangents. It follows that $\psi : Z \to X'$ is a meromorphic map preserving varieties of rational tangents. $\square$

4 Global extension of a meromorphic map

In this section, we will finish the proof of Theorem 1.2. Starting with the unramified covering $\pi : Z \to X - D$ of Proposition 3.4, first we are going to construct a meromorphic map Φ from X to X' extending a given germ of meromorphic map $(\varphi; x_0)$ preserving varieties rational tangents, and then show that Φ is biholomorphic. There are two problems for the construction of Φ: Z is not univalent and the meromorphic map $\psi : Z \to X'$ may have essential singularities along D.

Proposition 4.1 *In the notation of Proposition 3.4, let $(\varphi; x_0)$ be any germ of meromorphic map into X' preserving varieties of rational tangents, and $\psi : Z \to X'$ be the meromorphic map arising from $(\varphi; x_0)$ by parametrized analytic continuation. Let $x \in X - D$ and $z_1, z_2 \in V$ be two points lying above x, i.e., $\pi(z_1) = \pi(z_2) = x$. Then, the germs of meromorphic maps $(\psi; z_1)$ and $(\psi; z_2)$ agree in the sense that $(\psi; z_1) = (\pi^*\xi; z_1)$, $(\psi; z_2) = (\pi^*\xi; z_2)$ for some germ of meromorphic map $(\xi; x)$ at x into X'.*

Proof. Introduce an equivalence relation on Z by writing $z_1 \sim z_2$ whenever (i) $\pi(z_1) = \pi(z_2)$ and (ii) for each germ $(\varphi; x_0)$ the germs of the extended map $(\psi; z_1)$, resp. $(\psi; z_2)$ at z_1 resp. z_2 agree with each other. Write $\bar{Z} = Z/\!\sim$. Then the canonical map $Z \to \bar{Z}$ and the associated covering $\bar{\pi} : \bar{Z} \to X - D$ are unramified. Replacing Z by $\bar{Z}$ and $\pi : Z \to X - D$ by $\bar{\pi} : \bar{Z} \to X - D$ we may assume without loss of generality that given $z_1 \neq z_2$ with $\pi(z_1) = \pi(z_2)$, there exists some germ $(\varphi; x_0)$ of meromorphic map into X' preserving varieties of rational tangents so that the extended map ψ has distinct germs at z_1 and z_2. For this new meaning of Z, Proposition 4.1 amounts to saying that π is bijective.

We need the following lemma which holds for any Fano manifold with Picard number 1.

Lemma 4.2 *Let $\pi : Y \to X$ be a generically finite morphism from a normal irreducible variety Y onto a Fano manifold X with Picard number 1. Suppose for a generic standard rational curve $C \subset X$ belonging to a chosen standard component $\mathcal{H}^o$, each component of the inverse image $\pi^{-1}(C)$ is birational to C by π. Then $\pi : Y \to X$ itself is birational.*

Proof of Lemma. Suppose π is not birational. From the simply-connectedness of X (e.g. [Kb]), there exists a ramification divisor $R \subset Y$ of π so that $\pi(R)$ is a divisor in X. By genericity of C, we may assume that $\pi^{-1}(C)$ lies on the smooth part of the normal variety Y. Let C_1 be any irreducible component of $\pi^{-1}(C)$ which is birational to C by π. Let $h : \mathbf{P}_1 \to C_1$ be the normalization. Then $\pi \circ h$ is the normalization of C. Thus a deformation $h_t : \mathbf{P}_1 \to Y$ of C_1 induces a deformation $\pi \circ h_t$ of C. On the other hand, by the genericity of C, any small deformation of C can be lifted to a small deformation of C_1. It follows that the space of deformations of C and the space of deformations of C_1 have equal dimensions. So we have $K_Y \cdot C_1 = K_X \cdot C$ (cf. [Kl] II.1.2). This implies C_1 is disjoint from the ramification divisor $R \subset Y$. Since this holds for any component C_1 of $\pi^{-1}(C)$, C is disjoint from the divisor $\pi(R)$, a contradiction to the assumption that X is of Picard number 1. □

Now we prove that π is bijective. Suppose not. By Lemma 4.2, for a standard rational curve C intersecting U, there exists an irreducible quasi-projective curve C^* on Z such that $\pi(C^*) \subset C$ and $\pi|_{C^*}$ is not birational. For a generic point $x \in C \cap U$, we have $z_1 \neq z_2$ on C^* such that $\pi(z_1) = \pi(z_2) = x$. We can find a germ φ of meromorphic map to X preserving rational tangents so that the germs $(\psi; z_1)$ and $(\psi; z_2)$ obtained by analytically continuing φ are distinct. Choose an arc Υ on C^* starting from z_1 ending at z_2. The analytic continuation of $(\psi; z_1)$ along Υ gives $(\psi; z_2)$. However there is an analytic continuation of φ along C by Proposition 2.1. So the analytic continuation along the loop $\pi(\Upsilon)$ on C must give the same germ at x. The analytic continuation along C^* should agree with the one pulled back from C via $\pi|_{C^*}$. Thus follows $(\psi; z_1) = (\psi; z_2)$, a contradiction. □

Proposition 4.3 *Any germ of meromorphic map $(\varphi; x_0)$ to X' preserving varieties of rational tangents extends to a meromorphic map from X to X'.*

Proof. From Proposition 4.1, we see that there exists a Zariski-open set $X^o \subset X$ such that any germ of meromorphic map $(\varphi; x_0)$ to X' preserving varieties of rational tangents extends to a meromorphic map Φ from X^o to X'. Suppose there exists a divisor $D \subset X - X^o$. Since X is of Picard number 1, we have a $\mathcal{K}^o$-curve C through a generic point b of D by Lemma 1.1 (1). Pick an irreducible branch of the germ of C at b. Then by Proposition 2.1, we can extend Φ to the union of X^o and a neighborhood U_b of b. Applying this to each codimension 1 component of $X - X^o$, Φ can be extended outside a codimension > 1 set, and we are done by Hartogs extension for meromorphic maps. □

Let $\Phi : X \rightarrow X'$ be the meromorphic map in Proposition 4.3. Since Φ preserves varieties of rational tangents, the strict transform of $\mathcal{C} \subset \mathbf{P}T(X)$ by Φ_* must be a component of $\mathcal{C}'$, and must agree with $\mathcal{C}'$ from the irreducibility of $\mathcal{C}'$. It follows that $\varphi_*(\mathcal{C}_x) = \mathcal{C}'_{\varphi(x)}$ for generic $x \in U$. This means that $\varphi^{-1} : U' \rightarrow U$ preserves varieties of rational tangents. Now applying Proposition 4.3 to φ^{-1} which is a germ of meromorphic map at $\varphi(x_0) \in X'$ to X preserving varieties of rational tangents, we get a meromorphic map $\Phi^{-1} : X' \rightarrow X$. Thus Theorem 1.2 follows from the following, whose proof is given in 3.2.5 of [HM4]. Since the proof there is stated in the case when both X and X' are irreducible Hermitian symmetric spaces, let us rewrite it.

Proposition 4.4 *Let X, X' be as in Theorem 1.2 and $\Phi : X \rightarrow X'$ be a birational map preserving varieties of rational tangents. Then Φ is biholomorphic.*

Proof. We denote by $B \subset X$ the subvariety on which Φ fails to be a local biholomorphism and call B the bad locus of Φ. We claim that Proposition 4.4 will follow if we show that B is of codimension ≥ 2. Since X and X' are Fano we may choose k large enough so that both K_X^{-k} and $K_{X'}^{-k}$ are very ample. Let s be a pluri-anticanonical section on X' in $\Gamma(X', K_{X'}^{-k})$. Then $\Phi^* s$ is a well-defined pluri-anticanonical section on $X - B$. It extends across B under the assumption that B is of codimension ≥ 2. It follows that Φ induces a linear monomorphism $\theta : \Gamma(X', K_{X'}^{-k}) \rightarrow$

$\Gamma(X, K_X^{-k})$ and hence a linear isomorphism $\theta^* : \Gamma(X, K_X^{-k})^* \to \Gamma(X', K_{X'}^{-k})^*$ by taking adjoints. Identifying X resp. X' as a complex submanifold of $\mathbf{P}\Gamma(X, K_X^{-k})^*$ resp. $\mathbf{P}\Gamma(X', K_{X'}^{-k})^*$, Φ is nothing other than the restriction of the projectivization $[\theta^*] : \mathbf{P}(\Gamma(X, K_X^{-k})^*) \to \mathbf{P}(\Gamma(X', K_{X'}^{-k})^*)$ to X, thus a biholomorphism.

It remains to show that the bad locus B of $\Phi : X \to Y$ is of codimension ≥ 2. Otherwise let $R \subset B$ be an irreducible component of codimension 1 in X. We can choose a generic $\mathcal{K}^o$-curve C disjoint from the indeterminacy locus of Φ. Φ is a holomorphic map in a neighborhood of C and $\Phi(C)$ is a standard rational curve C' on X' from Proposition 1.3. Let $h : \mathbf{P}_1 \to C$ and $h' : \mathbf{P}_1 \to C'$ be the normalizations. From the birationality of Φ, we may assume that $\Phi|_C$ lifts to the identity map on $\mathbf{P}_1$. Furthermore, the bundle homomorphism $d\Phi|_C : T(X)|_C \to T(X')|_{C'}$ lifts to a bundle homomorphism $\Psi : h^*T(X) \to h'^*T(X')$.

We have $h^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. At $t \in \mathbf{P}_1$ we write $P_t = (\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_t \subset (h^*T(X))_t$, which is independent of the choice of Grothendieck decomposition. If $[(h^*T_t(\mathbf{P}_1)] \in \mathbf{P}T_{h(t)}(X)$ is a smooth point of $\mathcal{C}_{h(t)}$, then $T_t(h^*\mathcal{C}) = P_t \bmod T_t(\mathbf{P}_1)$. Define $P_t' \subset (h'^*T(X))_t$ analogously. For a generic t, $[h_*T_t(\mathbf{P}_1)]$ is a smooth point of $\mathcal{C}_{h(t)}$, and $[h'_*T_t(\mathbf{P}_1)]$ is a smooth point of $\mathcal{C}'_{h'(t)}$. It follows that for a generic $t \in \mathbf{P}_1$, $\Psi(P_t) = P_t'$ since $d\Phi_{h(t)}(\mathcal{C}_{h(t)}) = \mathcal{C}'_{h'(t)}$.

Since X is of Picard number 1, C intersects the ramification locus R at some point $x_1 = h(t_1)$. Choose a non-zero tangent vector $\eta \in T_{x_1}(X)$ such that $d\Phi(\eta) = 0$. Either $h^*\eta \notin P_{t_1}$ or $h^*\eta \in P_{t_1}$. In both cases we are going to derive a contradiction.

Since $h^*T(X)$ is semipositive there exists $s \in \Gamma(\mathbf{P}_1, h^*T(X))$ such that $s(x_1) = h^*\eta$. Suppose $h^*\eta \notin P_{t_1}$. Then, $s(t) \notin P_t$ for a generic $t \in \mathbf{P}_1$. It follows that $\Psi s(t) \notin P_t'$ for a generic t. On the other hand, $\Psi s(1) = 0$ since $d\Phi(\eta) = 0$, implying that $\Psi s(t) \in (\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_t$ for every $t \in \mathbf{P}_1$, a contradiction.

Suppose now $h^*\eta \in P_{t_1}$. Then, there exists $s \in \Gamma(\mathbf{P}_1, h^*T(X))$ such that $s(0) = 0$ and $s(1) = \eta, s(t) \notin \mathcal{O}(2)_t$ for generic $t \in \mathbf{P}_1$ (i.e., h_*s is not tangent to C). Then, for $\Psi s \in \Gamma(\mathbf{P}_1, h'^*T(X')), \Psi s(0) = 0$ since $s(0) = 0; \Psi s(1) = 0$ since $d\Phi(\eta) = 0$, while $\Psi s \notin \Gamma(\mathbf{P}_1, \mathcal{O}(2))$. Since $h^*T(X)/\mathcal{O}(2) \cong [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ does not admit any non-trivial holomorphic section vanishing at two points, we have again derived a contradiction. □

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Jun-Muk Hwang
Korea Institute for Advanced Study
207-43 Cheongryangri-dong
Seoul 130-012, Korea
e-mail: jmhwang@ns.kias.re.kr

Ngaiming Mok
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
e-mail: nmok@hkucc.hku.hk

Deformation rigidity of the rational homogeneous space associated to a long simple root

Jun-Muk Hwang [1] **and** **Ngaiming Mok** [2]

As a continuation of our previous works [HM1] and [Hw1], we study the following conjecture on the rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation. For the background of this conjecture, see the introduction of [HM1].

Conjecture *Let G be a complex simple Lie group and P be a maximal parabolic subgroup. Let $\pi : \mathcal{X} \to \Delta = \{t \in \mathbf{C}, |t| < 1\}$ be a smooth projective morphism from a complex manifold to the unit disc. If $X_t := \pi^{-1}(t)$ is biholomorphic to G/P for all $t \neq 0$, then X_0 is also biholomorphic to G/P.*

A natural approach is to construct a geometric structure on X_0 using the tangent vectors to minimal rational curves. In [HM1] (resp. [Hw1]), we constructed a G-structure (resp. a contact structure) this way and proved the Conjecture. By the work of Yamaguchi ([Ya]), for the cases different from the symmetric or the contact cases, it suffices to recover the nilpotent Lie algebra structure of a differential system to prove the Conjecture. The purpose of this paper is to show this when P is associated to a long simple root, including the cases of all maximal parabolic subgroups when all roots of G are of the same length:

Main Theorem *Let G be a complex simple Lie group and P be a maximal parabolic subgroup associated to a long simple root. Let $\pi : \mathcal{X} \to \Delta = \{t \in \mathbf{C}, |t| < 1\}$ be a smooth projective morphism from a complex manifold to the unit disc. It $X_t := \pi^{-1}(t)$ is biholomorphic to G/P for all $t \neq 0$, then X_0 is also biholomorphic to G/P.*

As in [HM1] and [Hw1] our approach consists of studying distributions derived from *varieties of minimal rational tangents* (see Section 2 for the definition), notably on questions of integrability. There is however an essential difference in that we have to deal with a nilpotent Lie algebra structure of the differential system, which is much more complicated than a G-structure or a contact structure. The hypothesis on P enters in a crucial way in the proof. In fact, P is associated to a long simple root if and only if the minimal G-invariant distribution on G/P is spanned by varieties of minimal rational tangents.

With some oversimplification to streamline the comparison with earlier works the proof of the Main Theorem breaks down into three steps. The first step, which parallels the first steps of [HM1] and [Hw1], is to show that the normalized space $\mathcal{K}_x$ of minimal rational curves at a generic point x of X_0 agrees with that of the model G/P. The proof of this step is a refinement of arguments in [HM1] or [Hw1] requiring deeper knowledge of the geometry of Hermitian symmetric spaces (as varieties of minimal rational tangents). The second step is to show that the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbf{P}T_x(X_0)$, which is the image of $\mathcal{K}_x$ under the tangent map, agrees with that of the model as a projective subvariety. The third step is then to show that

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the differential system generated by the varieties of minimal rational tangents has the same nilpotent Lie algebra structure as the model G/P. The second and the third steps are closely intertwined and handled together in Section 3. Here new difficulties arise which were not present in [HM1] or [Hw1]. As a matter of fact, while the third step is completely trivial for the Hermitian symmetric case and is rather straightforward for the contact case, it is highly non-trivial in other cases covered by the Main Theorem. An analogue of the third step is also the main obstacle in extending the Main Theorem to the case when P is associated to a short root.

For the second step, in the Hermitian symmetric case it is enough to show that the varieties of minimal rational tangents at the central fiber span the full tangent bundle; in the contact case it is enough to show that they must span a distribution of codimension 1 (as on a generic fiber). In both cases assuming the contrary we would have obtained an integrable distribution spanned by varieties of minimal rational tangents, leading to a contradiction since X_0 is of Picard number 1. Essential to this line of proof is the particular projective geometry of the variety of minimal rational tangents of the model space G/P. For instance, in the model contact case, varieties of minimal rational tangents span the contact distribution D, and are Legendrian subvarieties of the projectivization of $\mathbf{P}D$. From this it followed that any drop in the rank (when compared to D) of the distribution W spanned by varieties of minimal rational tangents in the central fiber X_0 would force W to be integrable by results from [HM1]. In other words, we relied on the fact that the contact distribution in the model contact manifold is just short of being integrable.

In the situation of the Main Theorem and assuming that G/P is neither Hermitian symmetric nor of the contact type, the distribution D spanned by varieties of minimal rational tangents on the model space can be very far away from being integrable. The problem is to prove that the failure of integrability, in a sense to be made precise, is stable under deformation. Even on the model space the differential system may have many levels, and jumps of simple numerical invariants such as ranks of distributions are far from being enough to lead to contradictions. On X_0 we have to consider the differential system obtained by augmenting W by taking successive Lie brackets. The nilpotent Lie algebra structure associated to the differential system is precisely the algebraic structure in which the failure of integrability is encoded. The novel point of the proof of our Main Theorem is Proposition 6, which shows that a *natural integrability condition obtained in [HM1] coming from the deformation theory of minimal rational curves turns out to be equivalent to the finiteness condition in the Serre presentation of the simple Lie algebra.* This is essentially a result on the model G/P and is expected to be useful in the study of geometry of G/P itself, independent from the deformation problem.

In a sense, the main motivation for studying the Conjecture for us is that it is a good testing ground for the study of Fano manifolds of Picard number 1 through minimal rational curves. The problem of recovering the structure of a given Fano manifold of Picard number 1 from the information on the minimal rational curves is broader and of greater importance to us than the Conjecture itself. From this perspective the study of the large classes of G/P in the Main Theorem reveals that the deformation theory of rational curves provides a powerful tool to unravel the algebraic structures of differential systems arising from distributions spanned by varieties of minimal rational tangents. In the case at hand it provides a means of identifying varieties of minimal rational tangents and recovering the complex structure of these rational homogeneous spaces. It is in this context that we believe that our result enhances the general perspective in our geometric study of Fano manifolds as put forth in [HM1,2,3].

1 Rational homogeneous spaces associated to long simple roots

In this section, we will review some basic facts about the rational homogeneous space associated to a long simple root (see e.g. [Ya] or Section 2 of [HM2]).

Let $\mathbf{g}$ be a complex simple Lie algebra. Choose a Cartan subalgebra $\mathbf{h}$ and the root system $\Phi \subset \mathbf{h}^*$ of $\mathbf{g}$ with respect to $\mathbf{h}$. Fix a system of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ and a distinguished choice of a simple root α_i. Given an integer $k, -m \leq k \leq m$, we define Φ_k as the set of all roots $\sum_{q=1}^{l} m_q \alpha_q$ with $m_i = k$. Here m is the largest integer such that $\Phi_m \neq 0$. For $\alpha \in \Phi$, let $\mathbf{g}_\alpha$ be the corresponding root space. Define

$$\begin{aligned} \mathbf{g}_0 &= \mathbf{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathbf{g}_\alpha \\ \mathbf{g}_k &= \bigoplus_{\alpha \in \Phi_k} \mathbf{g}_\alpha, \quad k \neq 0. \end{aligned}$$

The decomposition $\mathbf{g} = \bigoplus_{k=-m}^{m} \mathbf{g}_k$ gives a graded Lie algebra structure on $\mathbf{g}$. Define

$$\begin{aligned} \mathbf{p} &= \mathbf{g}_0 \oplus \mathbf{g}_{-1} \oplus \cdots \oplus \mathbf{g}_{-m} \\ \mathbf{l} &= \mathbf{g}_0 \\ \mathbf{u} &= \mathbf{g}_{-1} \oplus \cdots \oplus \mathbf{g}_{-m} \end{aligned}$$

We say that $\mathbf{p}$ is **the maximal parabolic subalgebra associated to the simple root** α_i. $\mathbf{u}$ is the unipotent radical of $\mathbf{p}$ and $\mathbf{p} = \mathbf{u} + \mathbf{l}$ is a Levi decomposition. Let us remark that our choice of $\mathbf{p}$ has signs of roots different from the choice in some references, e.g., [Ya]. We prefer this choice because positive roots correspond to positive line bundles.

Each $\mathbf{g}_j, 1 \leq j \leq m$, is an irreducible $\mathbf{l}$-module. Let $W \subset \mathbf{g}_1$ be the cone of highest weight vectors of the irreducible $\mathbf{l}$-module $\mathbf{g}_1$. Its projectivization $\mathbf{P}W \subset \mathbf{P}\mathbf{g}_1$ will be called the **highest weight variety**. $\mathbf{l}$ has 1-dimensional center. The semi-simple part of $\mathbf{l}$ has rank $l-1$ and its Dynkin diagram is obtained by removing α_i from the Dynkin diagram of $\mathbf{g}$. From this, one can easily determine the highest weight variety in $\mathbf{P}\mathbf{g}_1$. We list the pairs $(\alpha_i; \mathbf{P}W)$ below. For the numbering of simple roots, we will use the convention of [Ya].

- $\mathbf{g} = A_l$
  $(\alpha_i; \mathbf{P}_{i-1} \times \mathbf{P}_{l-i})$
- $\mathbf{g} = B_l$
  $(\alpha_i; \mathbf{P}_{i-1} \times \mathbf{Q}_{2(l-i)-1})$ for $1 \leq i \leq l-1$, $(\alpha_l; Gr(2, l-2))$

- $\mathbf{g} = C_l$
  $(\alpha_i; \mathbf{P}_{i-1} \times \mathbf{P}_{2(l-i)-1})$ for $1 \leq i \leq l-1, (\alpha_l; v_2(\mathbf{P}_{l-1}))$
- $\mathbf{g} = D_l$
  $(\alpha_i; \mathbf{P}_{i-1} \times \mathbf{Q}_{2(l-i)-2})$ for $1 \leq i \leq l-2, (\alpha_{l-1}; Gr(2, l-2)), (\alpha_l; Gr(2, l-2))$
- $\mathbf{g} = E_6$
  $(\alpha_1; Gr^{II}(5,5)), (\alpha_2; Gr(3,3)), (\alpha_3; \mathbf{P}_1 \times Gr(2,3)), (\alpha_4; \mathbf{P}_1 \times \mathbf{P}_2 \times \mathbf{P}_2)$
- $\mathbf{g} = E_7$
  $(\alpha_1; Gr^{II}(6,6)), (\alpha_2; Gr(3,4)), (\alpha_3; \mathbf{P}_1 \times Gr(2,4)), (\alpha_4; \mathbf{P}_1 \times \mathbf{P}_2 \times \mathbf{P}_3), (\alpha_5; \mathbf{P}_2 \times Gr(2,3)),$
  $(\alpha_6; \mathbf{P}_1 \times Gr^{II}(5,5)), (\alpha_7; \mathbf{E}_6)$
- $\mathbf{g} = E_8$
  $(\alpha_1; Gr^{II}(7,7)), (\alpha_2; Gr(3,5)), (\alpha_3; \mathbf{P}_1 \times Gr(2,5)), (\alpha_4; \mathbf{P}_1 \times \mathbf{P}_2 \times \mathbf{P}_4), (\alpha_5; \mathbf{P}_3 \times Gr(2,3)),$
  $(\alpha_6; \mathbf{P}_2 \times Gr^{II}(5,5)), (\alpha_7; \mathbf{P}_1 \times \mathbf{E}_6), (\alpha_8; \mathbf{E}_7)$
- $\mathbf{g} = F_4$
  $(\alpha_1; Gr^{III}(3,3)), (\alpha_2; \mathbf{P}_1 \times v_2(\mathbf{P}_2)), (\alpha_3; \mathbf{P}_1 \times \mathbf{P}_2), (\alpha_4; Gr^{II}(3,3))$
- $\mathbf{g} = G_2$
  $(\alpha_1; \mathbf{P}_1), (\alpha_2; v_3(\mathbf{P}_1))$

In the list, $\mathbf{Q}_k$ denotes the k-dimensional smooth hyperquadric, $Gr(k,l)$ denotes the Grassmannian of k-dimensional subspaces in $(k+l)$-dimensional vector space, $Gr^{II}(k,k)$ denotes the orthogonal Grassmannian of k-dimensional isotropic subspaces in a $2k$-dimensional orthogonal vector space, $Gr^{III}(k,k)$ denotes the Lagrangian Grassmannian of a $2k$-dimensional symplectic vector space, and $\mathbf{E}_6$ (resp. $\mathbf{E}_7$) denotes the Hermitian symmetric space with the group E_6 (resp. E_7). $v_2(\mathbf{P}_k)$ (resp. $v_3(\mathbf{P}_k)$) denotes the 2nd (resp. 3rd) Veronese embedding of the projective space. Except these Veronese embeddings of projective spaces, all other irreducible Hermitian symmetric spaces are embedded in a minimal way and the product stands for the Segre embedding coming from tensor product of the embeddings of each factor.

Now let G (resp. P) be a complex Lie group with Lie algebra $\mathbf{g}$ (resp. $\mathbf{p}$). The quotient variety G/P is called the **rational homogeneous space associated to the simple root** α_i. The quotient map $G \to G/P$ defines a P-principal bundle on G/P. The left action of P on the reductive group $L = P/U$ where U is the unipotent radical of P, induces an L-principal bundle $\mathbf{L}$ on G/P. The Picard group of G/P is generated by an ample line bundle $\mathcal{L}$. This line bundle $\mathcal{L}$ is homogeneous and is associated to $\mathbf{L}$ by a 1-dimensional representation of L. This representation can be described as follows. Let α_i be the simple root defining P. Let $H_{\alpha_i} \in \mathbf{h}$ be its coroot. The center of the reductive group $L = P/U$ has Lie algebra $\mathbf{C}H_{\alpha_i}$. Hence a $\mathbf{Z}$-functional on $\mathbf{Z}H_{\alpha_i}$ induces a character of L, giving rise to a homogeneous line bundle on G/P. The line bundle $\mathcal{L}$

is the one associated to the functional having value 1 on H_{α_i}. It is well-known that $\mathcal{L}$ is very ample.

For example, when $G/P = \mathbf{P}_1$, $\mathbf{g} = \mathbf{sl}_2$ has a unique simple root and corresponding coroot. A functional having value $k \in \mathbf{Z}$ on the coroot gives rise to the line bundle $\mathcal{O}(k)$ on $\mathbf{P}_1$.

On our rational homogeneous space G/P, we have rational curves which are lines under the embedding defined by $\mathcal{L}$. Let α_i be the simple root defining P and $H_{\alpha_i} \in \mathbf{h}$ be its coroot. Let $\mathbf{s}_{\alpha_i} \subset \mathbf{g}$ be the subalgebra isomorphic to $\mathbf{sl}_2$ such that $\mathbf{s}_{\alpha_i} \cap \mathbf{h} = \mathbf{C}H_{\alpha_i}$ and H_{α_i} is the coroot for $\mathbf{s}_{\alpha_i}$. The orbit of $o \in G/P$ under the subgroup $S_{\alpha_i} \subset G$ with Lie algebra $\mathbf{s}_{\alpha_i}$ is a rational curve and will be denoted by C_{α_i}. Note that the character of L defining $\mathcal{L}$ has value 1 on H_{α_i}. Thus C_{α_i} is a line under the embedding of G/P defined by $\mathcal{L}$. Under the natural identification of $\mathbf{g}_1$ as a subspace of the tangent space $T_o(G/P)$, H_{α_i} is a tangent vector of the line C_{α_i} at the point $o \in G/P$.

So far all our discussions work for any simple root α_i. But for the next Proposition we need to assume that α_i is a long simple root.

Proposition 1 *If α_i is a long simple root of $\mathbf{g}$, then the Chow space of lines through the base point $o \in G/P$ is isomorphic to the highest weight variety $\mathbf{P}W \subset \mathbf{P}\mathbf{g}_1$.*

Proof. Each point $w \in \mathbf{P}W$ can serve as the highest weight vector H_{α_i} under a suitable choice of the Cartan subalgebra $\mathbf{h}$ and the Weyl chamber. Thus we have a line C_w whose tangent vector at o is given by w. Thus $\mathbf{P}W$ is a subvariety of the Chow space of lines through o.

We claim that $\mathbf{P}W$ is an irreducible component of the Chow space. It suffices to show that the dimension of the deformation of a line fixing a point on G/P cannot exceeds the dimension of $\mathbf{P}W$. The former is bounded by $h^0(C_w, N \otimes \mathcal{O}(-1))$ where N is the normal bundle of the line in G/P. Since the normal bundle is semi-positive, $h^0(C_w, N \otimes \mathcal{O}(-1)) = C_w \cdot K^{-1}_{G/P} - 2$. To calculate the anti-canonical degree of C_w, we use Grothendieck's splitting theorem for principal bundles on $\mathbf{P}_1$ with reductive structure groups and associated vector bundles([Gr]).

Theorem (Grothendieck) *Let $\mathcal{O}(1)^*$ be the $\mathbf{C}^*$-principal bundle on $\mathbf{P}_1$ corresponding to the line bundle $\mathcal{O}(1)$. Let L be a reductive complex Lie group. Up to conjugation, any L-principal bundle on $\mathbf{P}_1$ is associated to $\mathcal{O}(1)^*$ by a group homomorphism from $\mathbf{C}^*$ to a maximal torus of L. If H is the coroot of $\mathbf{sl}_2$, such a group homomorphism is determined by the image of H in $\mathbf{h}$, a fixed Cartan subalgebra of L. Given a representation of L with weights $\mu_1, \ldots, \mu_l \in \mathbf{h}^*$, the associated vector bundle on $\mathbf{P}_1$ splits as $\mathcal{O}(\mu_1(H)) \oplus \cdots \oplus \mathcal{O}(\mu_l(H))$, where $\mu_j(H)$ denotes the value of μ_j on the image of H in $\mathbf{h}$.*

Note that $T_o(G/P)$ can be naturally identified with $\mathbf{g}/\mathbf{p}$. So the Chern number of $T(G/P)$ is equal to the sum of Chern numbers of the vector bundles associated to the L-principal bundle $\mathbf{L}$ via the representations of L on $\mathbf{g}_1, \ldots, \mathbf{g}_m$. Hence by Grothendieck's theorem, the Chern number of $T(G/P)$ restricted to C_{α_i} is $\sum_{\beta \in \Phi_1 \cup \cdots \cup \Phi_m} \beta(H_{\alpha_i})$. Since α_i is a long root,

$$\beta(H_{\alpha_i}) = \begin{cases} 2 & \text{if } \beta = \alpha_i \\ 1 & \text{if } \beta \neq \alpha_i \text{ and } \beta - \alpha_i \in \Phi \\ -1 & \text{if } \beta \neq \alpha_i \text{ and } \beta + \alpha_i \in \Phi. \end{cases}$$

From $\alpha_i \in \Phi_1$, the Chern number is

$$\begin{aligned}\sum_{\beta\in\Phi_1\cup\cdots\cup\Phi_m} \beta(H_{\alpha_i}) &= 2 + \sharp\{\beta \in \Phi_1 \cup \cdots \cup \Phi_m, \beta \neq \alpha_i, \beta - \alpha_i \in \Phi\} \\ &\quad -\sharp\{\beta \in \Phi_1 \cup \cdots \cup \Phi_m, \beta \neq \alpha_i, \beta + \alpha \in \Phi\} \\ &= 2 + \sharp\{\beta \in \Phi_1, \beta - \alpha_i \in \Phi_0\} \\ &= 2 + \sharp\{\gamma \in \Phi_0, \alpha + \gamma \in \Phi_0\} \\ &= 2 + \dim([\mathbf{g}_0, H_{\alpha_i}]).\end{aligned}$$

It follows that $h^0(C_w, N \otimes \mathcal{O}(-1)) = \dim([\mathbf{l}, H_{\alpha_i}])$. But $\dim([\mathbf{l}, H_{\alpha_i}])$ is exactly the dimension of $\mathbf{P}W$. This proves that $\mathbf{P}W$ is an irreducible component of the Chow space of lines through o.

It remains to show that the Chow space of lines through o is irreducible. A line is determined by its tangent vector at o. Thus if there exists a line different from C_w, its tangent vector will be contained in $\mathbf{T}_o(G/P) - \mathbf{P}W$. From Proposition 5.2 in [HM2], the closure of the P-orbit of such a vector intersects $\mathbf{P}W$. Since the limit of a family of lines is again a line, this implies that the component $\mathbf{P}W$ is not smooth. However $\mathbf{P}W$ is homogeneous and $h^1(C_w, N \otimes \mathcal{O}(-1)) = 0$ since N is semi-positive, so the Chow component $\mathbf{P}W$ is smooth, a contradiction. □

Remark 1 As complex manifolds, the rational homogeneous space associated to α_l for $\mathbf{g} = B_l$ is biholomorphic to that associated to α_l for $\mathbf{g} = D_{l+1}$. Also the rational homogeneous space associated to α_1 for $\mathbf{g} = G_2$ is isomorphic to $\mathbf{Q}_5$ which is associated to α_1 for $\mathbf{g} = B_3$. Thus when we study complex structure of G/P, these two cases can be regarded as rational homogeneous spaces associated to long simple roots.

Remark 2 When α_i is a short simple root, Proposition 1 does not hold. The Chow space of lines through o contains, but is strictly bigger than, $\mathbf{P}W$. It is not contained in $\mathbf{P}\mathbf{g}_1$ and excepting the cases mentioned in Remark 1, it is not homogeneous.

2 Rigidity of the normalized Chow spaces

Let us recall some basic facts from deformation theory of rational curves (cf. Section 2 of [HM1] or [Kl]). Let X be a Fano manifold of Picard number 1 and $x \in X$ be a generic point. Let $\mathcal{K}_x$ be an irreducible component of the normalized Chow space of rational curves of minimal degree through x. Then $\mathcal{K}_x$ is a smooth projective variety. If the anti-canonical degree of members of $\mathcal{K}_x$ is $p+2$, then $\mathcal{K}_x$ has dimension p, and for a generic member C of $\mathcal{K}_x$,

$$T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^{n-1-p}.$$

Define the tangent map $\tau_x : \mathcal{K}_x \to \mathbf{P}T_x(X)$ by assigning the tangent vector at x to each member of $\mathcal{K}_x$ which is smooth at x. This is a generically finite rational map and its strict image is denoted by $\mathcal{C}_x$, called the **variety of minimal rational tangents** at x. Suppose X is embedded in some projective space $\mathbf{P}_N$ and a minimal rational curve through a generic point x is a line in $\mathbf{P}_N$. Since lines through x in $\mathbf{P}_N$ are determined by their tangent vectors at x, τ_x is an embedding. This is the case for our G/P. In particular, when P is associated to a long simple root, Proposition

1 implies that $\mathcal{K}_o \cong \mathcal{C}_o \cong \mathbf{P}W$ and τ_o is an embedding described in the list of highest weight varieties in Section 1.

We now go to the situation of the Main Theorem. Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective morphism from a complex manifold to the unit disc. Suppose the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to G/P associated to a long simple root for each $t \neq 0$. Let us use the same symbol $\mathcal{L}$ to denote the line bundle on $\mathcal{X}$ whose restriction to X_t is equivalent to the line bundle $\mathcal{L}$ on G/P. Choose a generic point $x \in X_0$ and a section $\sigma : \Delta \to \mathcal{X}$ of π satisfying $\pi(0) = x$. Let $\rho : \mathcal{K}_\sigma \to \Delta$ be the family of normalized Chow spaces $\mathcal{K}_{\sigma(t)}$ of minimal rational curves through $\sigma(t)$ in X_t. Then ρ is a smooth projective morphism by the same proof as Proposition 4 in [HM1] or Proposition 8 in [Hw1]. The goal of this section is to prove the following.

Proposition 2 *The family $\rho : \mathcal{K}_\sigma \to \Delta$ is a trivial family, namely, its fiber at $t = 0$ is also isomorphic to $\mathbf{P}W$.*

Proof. From the list of highest weight varieties in Section 1, we see that $\mathbf{P}W$ belongs to (at least) one of the following.

(i) $\mathbf{P}W$ is an irreducible Hermitian symmetric space.

(ii) $\mathbf{P}W$ is the product of two projective spaces.

(iii) $\mathbf{P}W \cong S_1 \times S_2$ where $S_1 \cong \mathbf{P}_k$ and S_2 is a hyperquadric.

(iv) $\mathbf{P}W \cong S_1 \times S_2$ where $S_1 \cong \mathbf{P}_k$ and S_2 is a Hermitian symmetric space of rank 2 with $\dim(S_2) > k := \dim(S_1)$.

For the case (i), Proposition 3 follows from the result of [HM1]. For the case (ii), Proposition 3 was proved in Section 3 of [HM1]. Thus we will only consider the cases (iii) or (iv). In these case, either S_2 is irreducible or the product of two projective spaces. Let ζ_1 be the hyperplane line bundle on $S_1 \cong \mathbf{P}_k$ and ζ_2 be the ample line bundle on S_2 which is the generator of $Pic(S_2)$ if S_2 is irreducible and is the tensor product of hyperplane bundles of each factor when S_2 is the product of two projective spaces. Let $\zeta = \zeta_1 \otimes \zeta_2$. We say that a curve on $\mathcal{C}_o$ is a line (resp. a conic), if it has degree 1 (resp. 2) with respect to ζ. Let ξ (resp. ξ_1, resp. ξ_2) be the line bundle on $\mathcal{K}_\sigma$ so that its restriction to $\rho^{-1}(t)$ is ζ (resp. ζ_1, resp. ζ_2) for $t \neq 0$.

Lemma 1 *Let $l_t \subset \mathcal{K}_{\sigma(t)}$ be a family of curves so that l_t is a line on $S_1 \times S_2 \cong \mathcal{K}_{\sigma(t)}$ for all $t \neq 0$. Then l_0 is irreducible and reduced as a cycle in $\mathcal{K}_{\sigma(0)}$.*

Proof. For $t \neq 0$, a line in $\mathcal{K}_{\sigma(t)}$ corresponds to a family of lines in G/P passing through a fixed point $o \in G/P$, which span a surface of degree 1 with respect to $\mathcal{L}$. Given a family of rational curves $l_t \subset \mathcal{K}_{\sigma(t)}$ of degree 1 with respect to ξ, we have a corresponding family of surfaces $R_t \subset \mathcal{X}_t$ of degree 1 with respect to $\mathcal{L}$. Since $\mathcal{L}$ is ample on $\mathcal{X}_0$, the limit R_0 must be a reduced irreducible surface. It follows that the limit l_0 is a reduced irreducible rational curve on $\mathcal{K}_{\sigma(0)}$. □

Note that for any polarized projective manifold X and an integer N, there exists a non-empty Zariski open subset $X^* \subset X$ with the property that for any irreducible rational curve C of degree $\leq N$ with respect to the given polarization, $T(X)|_C$ is semipositive if C contains a point of X^* (e.g. the argument of [Kl] II.3.11).

Lemma 2 *Let $y \in \mathcal{K}_{\sigma(0)}$ be a generic point. Let $c_t \subset \mathcal{K}_{\sigma(t)}$ be a family of curves so that c_t is a conic on $S_1 \times S_2 \cong \mathcal{K}_{\sigma(t)}$ for all $t \neq 0$ and c_0 contains y. Then c_0 is either irreducible or has two components of degree 1 with respect to ξ.*

Proof. A conic on $S_1 \times S_2$ can be degenerated to a union of two lines. Thus for $t \neq 0$, a conic on $\mathcal{K}_{\sigma(t)}$ corresponds to a surface of degree 2 in X_t with respect to $\mathcal{L}$. By the same argument as in Lemma 1, c_0 can have at most two components. Suppose it has two components c_{00} and c_{01}. One of them, say c_{00}, contains y and we may assume $T(\mathcal{K}_{\sigma(0)})|_{c_{00}}$ is semipositive from the genericity of y. From $H^1(c_{00}, T(\mathcal{K}_{\sigma(0)})) = 0$ and Kodaira's stability ([Kd]), we have a family of rational curves $C'_t \subset \mathcal{K}_{\sigma(t)}$ so that $C'_0 = c_{00}$. In particular, c_{00} has positive degree with respect to ξ. Suppose that c_{00} has degree > 1 with respect to ξ. Then the surface in X_t corresponding to C'_t is of degree > 1 with respect to $\mathcal{L}$. It follows that the surface in X_0 corresponding to c_{00} has degree > 1 with respect to $\mathcal{L}$. This is not possible because the total degree of the surfaces corresponding to $c_{00} \cup c_{01}$ is 2. Hence c_{00} has degree 1 with respect to ξ and so does c_{01}. □

We have two foliations $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{K}_\sigma$ so that the leaves of $\mathcal{E}|_{\mathcal{K}_{\sigma(t)}}, t \neq 0$ (resp. $\mathcal{F}|_{\mathcal{K}_{\sigma(t)}}$) are the S_1-factors (resp. S_2-factors) of $\mathcal{K}_{\sigma(t)} \cong S_1 \times S_2$. They define meromorphic foliations on $\mathcal{K}_{\sigma(0)}$.

Lemma 3 *Let $y \in \mathcal{K}_{\sigma(0)}$ be a generic point and $\mu : \Delta \to \mathcal{K}_\sigma$ be a section of ρ with $\mu(0) = y$. Let P_t be the $\mathcal{E}$-leaf and Q_t be the $\mathcal{F}$-leaf through $\mu(t)$ on $\mathcal{K}_{\sigma(t)}, t \neq 0$. Then the limits P_0 and Q_0 are irreducible and reduced as cycles in $\mathcal{K}_{\sigma(0)}$.*

Proof. Since P_t and Q_t has intersection number 1 for all $t \in \Delta$, the reducedness of P_0 and Q_0 are immediate if they are irreducible.

Suppose P_0 is reducible. We can choose two families of distinct points $\alpha_t, \beta_t \in P_t$ so that α_0 and β_0 lie on different components of P_0. Since $P_t \cong \mathbf{P}_k$ for $t \neq 0$, there exists a line $l_t \subset P_t$ joining α_t and β_t. By Lemma 1, the limit l_0 must be irreducible while $\alpha_0, \beta_0 \in l_0$, a contradiction. Thus P_0 is irreducible.

To prove the irreducibility of Q_0, we consider the case (iii) and the case (iv) separately.

For the case (iii), we will use the following property of the hyperquadric S_2: given two generic points $A, B \in S_2$, the union of all conics passing through A and B covers S_2. This is because the tangent bundle of the hyperquadric splits as a direct sum of $\mathcal{O}(2)$'s over a conic. Suppose Q_0 is reducible. Choose two generic points A_0, B_0 in one of the component of Q_0 so that both A_0 and B_0 are very general. Choose two families of points $A_t, B_t \in Q_t$ converging to A_0 and B_0. Consider the union of all conics through A_t and B_t. By the above mentioned property of S_2, the limits of these conics must cover Q_0. Since Q_0 is reducible, this means that for any family of conics c_t passing through A_t and B_t its limit is reducible and one of the component is a line passing through A_0 and B_0. The union of such lines must cover one component of Q_0. By Mori's bend-and-break ([Kl] II.5), this family of lines through A_0 and B_0 must degenerate to a union of two rational curves. But this gives a contradiction to the degree of corresponding surface in X_0 as in the proofs of Lemma 1 and Lemma 2.

For the case (iv), we will use the following property of Hermitian symmetric space S_2 of rank 2: conics through a given point on S_2 cover S_2. This is a consequence of the polydisc theorem (Ch. 5 (1.1) in [Mk]). If Q_0 is reducible, choose $A_t, B_t \in Q_t$ so that A_0 and B_0 are generic points of distinct components of Q_0. We may assume that A_0 is a very general point. We can find a family of conics $c_t \subset Q_t$ containing A_t and B_t. The limit c_0 cannot be irreducible, and must be the union of two irreducible curves of degree 1 with respect to ξ by Lemma 2. Fixing A_0 and varying B_0, we get irreducible rational curves of degree 1 through A_0 which cover a component of Q_0. Since A_0 is very general, we may assume that these degree 1 curves through A_0 are limits of families of degree 1 curves through A_t in $\mathcal{K}_{\sigma(t)}$ by Kodaira's stability ([Kd]) as in the proof of Lemma 2. Thus on $\mathcal{K}_{\sigma(t)}$, we get a $(\dim(S_2)-1)$-dimensional family of lines through a fixed point, but this is impossible because S_2 is not a projective space and $k < \dim(S_2)$. □

Lemma 4 *For a generic point $y \in \mathcal{K}_{\sigma(0)}$, the $\mathcal{E}$-leaf P through y and the $\mathcal{F}$-leaf Q through y intersects transversally at y.*

Proof. Suppose not. From the genericity of y, there exists a positive dimensional component R of $P \cap Q$ through y. Let P_t (resp. Q_t) be a family of leaves of $\mathcal{E}$ (resp. $\mathcal{F}$) with $P_0 = P$ (resp. $Q_0 = Q$). Choose two distinct points on R generically. Then there exist a family of lines l_t on P_t so that l_0 contains these two points on R. We can choose a section of ξ_1 whose zero section H is a hypersurface consisting of $\mathcal{F}$-leaves so that $Q \subset H$. Since l_0 has degree 1 with respect to ξ_1 and contains at least two points of H, we see that $l_0 \subset H$. This implies that $l_0 \subset Q$. From the genericity of y, we can assume that l_0 passes through a generic point of Q. We know that ξ_2 is big on Q because it is ample on Q_t. On the other hand, $l_0 \cdot \xi_2 = 0$, a contradiction. □

We are ready to finish the proof of Proposition 2. From above, $\mathcal{E}$ and $\mathcal{F}$ define two transversal foliations at generic points of $\mathcal{K}_{\sigma(0)}$. So we get a direct sum decomposition of the relative tangent bundle of ρ outside a codimension 2 set in $\mathcal{K}_\sigma$. Then it extends to a direct sum decomposition everywhere on $\mathcal{K}_\sigma$, because the set of all possible direct sum decompositions of a given vector space is an affine variety. It follows that the foliations $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{K}_\sigma$ have no singularity. Since $\mathcal{K}_{\sigma(0)}$ is simply connected, $\mathcal{K}_{\sigma(0)}$ is biholomorphic to the product of smooth deformations of S_1 and S_2. This finishes the proof when S_2 is irreducible by the result of [HM1]. When S_2 is the product of two projective spaces, we apply the same argument as above to the family of leaves Q_t, as was done in Section 3 of [HM1], to conclude. □

3 Symbol algebra of the differential system

Let us recall some definitions in the theory of differential systems ([Ya]). Given a distribution D on a complex manifold X, define the weak derived system D^k inductively by

$$\begin{aligned} D^1 &= D \\ D^k &= D^{k-1} + [D, D^{k-1}]. \end{aligned}$$

For a generic point $x \in X$ in a neighborhood of which D^k's are subbundles of $T(X)$, we define the **symbol algebra** of D at x as the graded nilpotent Lie algebra $D^1_x + D^2_x/D^1_x + \cdots + D^l_x/D^{r-1}_x$ where r is chosen so that $D^{r+1} = D^r$.

When X is a Fano manifold of Picard number 1, choose a component $\mathcal{K}$ of the Chow spaces of rational curves of minimal degree covering X. For each generic $x \in X$, let $\mathcal{K}_x$ be the subscheme consisting of curves passing through x and $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ be the variety of minimal rational tangents. Let $\mathcal{V}_x \subset T_x(X)$ be the linear span of $\mathcal{C}_x$ and $\mathcal{V}$ be the meromorphic distribution defined by $\mathcal{V}_x$'s. As an example, consider our G/P associated to a long simple root. We have the L-principal bundle $\mathbf{L}$ on G/P induced by the P-principal bundle $G \to G/P$. The L-module $\mathbf{g}_1$ induces a vector bundle $\mathcal{D}$ on G/P. By definition, since $\mathcal{C}_o$ is nondegenerate in $\mathbf{g}_1$, the distribution $\mathcal{V}$ for G/P agrees with $\mathcal{D}$. Moreover, it is easy to see that the symbol algebra of $\mathcal{D}$ is isomorphic to $\mathbf{g}_1 + \cdots + \mathbf{g}_m$.

Remark 3 As mentioned in Remark 2, if G/P is associated to a short simple root, the distribution $\mathcal{V}$ need not agree with the distribution defined by $\mathbf{g}_1$. For example, $\mathcal{V}$ is the trivial distribution $T(G/P)$ when G is of type C (symplectic group) and P is associated to a short simple root.

For any Fano manifold X of Picard number 1 and for any choice of $\mathcal{K}$, the distribution $\mathcal{V}$ has the following two properties.

Proposition 3 *Let* $[,] : \Lambda^2\mathcal{V}_x \to T_x(X)/\mathcal{V}_x$ *be the Frobenius bracket tensor at a generic point* $x \in X$. *Then for a generic smooth point* $v \in \mathcal{C}_x$ *and* v' *in the tangent space of* $\mathcal{C}_x$ *at* v, $[v, v'] = 0$ *when* v *and* v' *are regarded as vectors in* $\mathcal{V}_x$.

Proof. This is just a restatement of Proposition 10 of [HM1]. Section 4 of [HM1] was presented under the assumption that $\mathcal{C}_x$ is irreducible, but the proof of Proposition 10 did not use this assumption. □

Proposition 4 *At a generic point* $x \in X$, *the symbol algebra of* $\mathcal{V}$ *has dimension* $n = \dim(X)$.

Proof. By definition, the symbol algebra has dimension $\leq n$. If it is strictly less than n, $\mathcal{V}$ is contained in an integrable distribution. This is a contradiction to the assumption that X is of Picard number 1 by Proposition 2 in [Hw2]. □

Now let us go to the situation of the Main Theorem. Let $\mathcal{K}$ be a component of the Chow space of X_0 parametrizing rational curves covering X_0 which are limits of lines on $X_t, t \neq 0$. Let $\tau_x : \mathcal{K}_x \to \mathcal{C}_x \subset \mathbf{P}T_x(X)$ be the tangent map at a generic $x \in X_0$. Let $\mathcal{V}_x \subset T_x(X_0)$ be the linear span of $\mathcal{C}_x$ and $\mathcal{V}$ be the meromorphic distribution defined by $\mathcal{V}_x$'s.

Proposition 5 *At a generic point* $x \in X_0$, *the symbol algebra of* $\mathcal{V}$ *is isomorphic to* $\mathbf{g}_1 + \cdots + \mathbf{g}_m$ *as graded nilpotent Lie algebras.*

To prove Proposition 5, we need a characterization of the graded nilpotent Lie algebra $\mathbf{g}_1 + \cdots + \mathbf{g}_m$. We need the following Lemma which follows immediately from the proof of Serre's Theorem in [Hu, 18.3], using the fact that the subalgebra generated by $\{x_i, 1 \leq i \leq l\}$ in the Lie algebra L_o constructed there is free (see also [Se, pp.48-49]). The latter fact is proved in [Bo, Ch.8, 4.2] or [Ka, Theorem 1.2(b)].

Lemma 5 *Let* $\{\alpha_1, \ldots, \alpha_l\}$ *be a set of simple roots for* $\mathbf{g}$ *and* $< \alpha_i, \alpha_j >$ *be the entries of the Cartan matrix. Let* $\{x_i, y_i, h_i | 1 \leq i \leq l\}$ *be the generators of the Serre presentation of* $\mathbf{g}$ *as given in [Hu, 18.1]. Then the subalgebra of* $\mathbf{g}$ *generated by* $\{x_1, \ldots, x_l\}$ *is the quotient of the free Lie algebra generated by* $\{x_1, \ldots, x_l\}$ *by the relations*

$$(ad\ x_i)^{-<\alpha_j,\alpha_i>+1}(x_j) \quad = \quad 0$$

for $i \neq j$.

Using Lemma 5, we get the following characterization of the graded Lie algebra $\mathbf{g}_0 + \cdots + \mathbf{g}_m$.

Proposition 6 *Let* $\mathbf{n} = \sum_{i=0}^{\infty} \mathbf{n}_i$ *be a graded Lie algebra generated by* $\mathbf{n}_0$ *and* $\mathbf{n}_1$ *so that* $\mathbf{n}_0 = \mathbf{g}_0$ *and* $\mathbf{n}_1$ *is isomorphic to* $\mathbf{g}_1$ *as a* $\mathbf{g}_0$*-module. Let* $W \subset \mathbf{n}_1$ *be the highest weight cone for the representation of* $\mathbf{g}_0$ *on* $\mathbf{n}_1$. *Assume that for any vector* $v \in W$, *the Lie bracket of* $\mathbf{n}$ *satisfies* $[v, [\mathbf{g}_0, v]] = 0$. *Then* $\mathbf{n}$ *is a quotient of the graded Lie algebra* $\mathbf{g}_0 + \cdots + \mathbf{g}_m$.

Proof. Let $\mathbf{m} \subset \mathbf{g}_0$ be the subalgebra generated by $\{x_i, i \neq k\}$ where α_k is the long simple root defining $\mathbf{p}$. Consider the subalgebra $\mathbf{n}' = \mathbf{m} + \mathbf{n}_1 + \cdots + \mathbf{n}_l$ of $\mathbf{n}$. As an abstract Lie algebra, $\mathbf{n}'$ is generated by $\{x_i, 1 \leq i \leq l\}$. It satisfies all the relations

$$(ad\ x_i)^{-<\alpha_j, \alpha_i>+1}(x_j) \ = \ 0$$

for $i \neq j$. In fact, if $j \neq k$ and $i \neq k$, this relation is just one of the Serre relations for $\mathbf{g}_0$. If $j = k$, this relation concerns the action of $\mathbf{m}$ on $\mathbf{n}_1$, which we assumed to be equivalent to the action of $\mathbf{m}$ on $\mathbf{g}_1$ for which the relation is just one of the Serre relations. When $i = k$ and $< \alpha_j, \alpha_k >= 0$, this follows again from the action of $\mathbf{m}$ on $\mathbf{n}_1$. Since α_k is a long root, the only remaining case is when $i = k$ and $< \alpha_j, \alpha_k >= -1$, for which the relation is just

$$[x_k, [x_k, x_j]] \ = \ 0.$$

But this is satisfied from the assumption that $[v, [\mathbf{g}_0, v]] = 0$ for any $v \in W$. It follows from Lemma 5 that $\mathbf{n}'$ is a quotient of the subalgebra of $\mathbf{g}$ generated by $\{x_1, \cdots, x_l\}$, which implies Proposition 6. □

Now we have the following characterization of the graded Lie algebra $\mathbf{g}_1 + \cdots + \mathbf{g}_m$.

Proposition 7 *Let* $W \subset \mathbf{g}_1$ *be the cone of highest weight vectors as a* $\mathbf{g}_0$*-module and* $\mathbf{F}(\mathbf{g}_1)$ *be the graded free Lie algebra generated by* $\mathbf{g}_1$. *We consider the ideal* I *of* $\mathbf{F}(\mathbf{g}_1)$ *generated by the relations* $[v, [\mathbf{g}_0, v]] = 0$ *for all* $v \in W$. *Let us denote the quotient graded algebra* $\mathbf{F}(\mathbf{g}_1)/I$ *by* $\mathbf{n}_1 + \mathbf{n}_2 + \cdots$. *Then* $\mathbf{n}_1 + \mathbf{n}_2 + \cdots$ *is isomorphic to the nilpotent graded Lie algebra* $\mathbf{g}_1 + \cdots + \mathbf{g}_m$.

Proof. $\mathbf{g}_0$-action on $\mathbf{g}_1$ induces a $\mathbf{g}_0$-action on the tensor algebra of $\mathbf{g}_1$ as a derivation, making $\mathbf{g}_0 + \mathbf{F}(\mathbf{g}_1)$ into a graded Lie algebra whose 0-degree part is exactly $\mathbf{g}_0$. Since the ideal I is invariant under the action of $\mathbf{g}_0$, $g_0 + \mathbf{n}_1 + \mathbf{n}_2 + \cdots$ becomes a graded Lie algebra. Setting $\mathbf{n}_0 = \mathbf{g}_0$, we can apply Proposition 6 to identify $\mathbf{n}_1 + \mathbf{n}_2 + \cdots$ with $\mathbf{g}_1 + \cdots + \mathbf{g}_m$. □

Now we are ready to finish the proof of Proposition 5.

Proof of Proposition 5. Choose a section $\sigma : \Delta \rightarrow \mathcal{X}$ of $\pi : \mathcal{X} \rightarrow \Delta$ so that $x = \sigma(0)$ is a generic point of $X_0 = \pi^{-1}(0)$. The family $\mathcal{K}_\sigma$ of normalized Chow spaces of minimal rational curves through σ is a trivial family of $\mathbf{P}W$ by Proposition 2. For $t \neq 0$, the tangent map $\tau_{\sigma(t)} : \mathcal{K}_{\sigma(t)} \rightarrow \mathbf{P}T_{\sigma(t)}(X_t)$ is an embedding into $\mathbf{P}\mathcal{D}_{\sigma(t)} = \mathbf{P}\mathbf{g}_1$ given by a complete linear system of the line bundle ξ on $\mathcal{K}_\sigma$ defined in Section 2. Thus $\tau_{\sigma(0)}$ is a rational map defined by a subsystem of this complete linear system. Namely, $\tau_{\sigma(0)}$ is induced by a projection $\mathbf{g}_1 \rightarrow \mathcal{V}_x$. Let $W' \subset \mathcal{V}_x$ be the image of the highest weight cone $W \subset \mathbf{g}_1$ under the projection. Then $\mathbf{P}W' = \mathcal{C}_x$, the variety of minimal rational tangents at x.

Consider the free lie algebra $\mathbf{F}(\mathcal{V}_x)$ generated by $\mathcal{V}_x$ and let J be its ideal generated by the relations given by $[v, v']$ where v is a smooth point of W' and v' is a vector in the tangent space of the cone W' at v. Then the quotient graded Lie algebra $\mathbf{F}(\mathcal{V}_x)/J$ is a quotient of $\mathbf{g}_1 + \cdots + \mathbf{g}_m$ by Proposition 7 because J contains the image of I under the natural graded Lie algebra homomorphism $\mathbf{F}(\mathbf{n}_1) \to \mathbf{F}(\mathcal{V}_x)$. From Proposition 3, the symbol algebra of $\mathcal{V}$ at x is a quotient algebra of $\mathbf{F}(\mathcal{V}_x)/J$, thus a quotient algebra of $\mathbf{g}_1 + \cdots + \mathbf{g}_m$. If the symbol algebra is not isomorphic to $\mathbf{g}_1 + \cdots + \mathbf{g}_m$, it has dimension strictly smaller than $n = \dim(G/P)$, a contradiction to Proposition 4. □

Our Main Theorem follows from Proposition 5 via the works of Tanaka and Yamaguchi ([Ta] and p.479 of [Ya]. See also 3.10 of [Mo] for a more general treatment). Let us briefly summarize their works. Let G/P be a rational homogeneous space associated to a simple root. Assume that G/P is not a symmetric space or a homogeneous contact manifold. Given a differential system D on a complex manifold whose symbol algebra at a generic point is isomorphic to $\mathbf{g}_1 + \cdots + \mathbf{g}_m$, there exists a natural holomorphic P-principal bundle $\mathcal{P}$ over an open neighborhood $\mathcal{U}$ of a generic point with a canonical choice of $\mathbf{g}$-valued 1-form ω, called the Cartan connection, so that if the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$ holds, then there exists a biholomorphic map of $\mathcal{U}$ to an open subset of G/P which sends the distribution D to the distribution $\mathcal{D}$ on G/P induced by $\mathbf{g}_1$. The construction of ω given in [Ta] or 3.10 of [Mo] can be carried out when we are given a family of complex manifolds with a family of differential systems whose symbol algebras are isomorphic to $\mathbf{g}_1 + \cdots + \mathbf{g}_m$.

Proof of Main Theorem. From [HM1] and [Hw1], we may assume that G/P is not a symmetric space or a homogeneous contact manifold. By Proposition 5, we are given a family of meromorphic distributions $\mathcal{V}_t$ on $\mathcal{X}$ whose symbol algebra at a generic point of X_t is $\mathbf{g}_1 + \cdots + \mathbf{g}_m$ for all $t \in \Delta$. We can apply the construction of [Ta] or 3.10 of [Mo] to a family of neighborhoods $\mathcal{U}_t$ of $x \in X_0$ to get a P-principal bundle $\mathcal{P}$ over $\mathcal{U} := \cup_{t \in \Delta} \mathcal{U}_t$ with the Cartan connection ω on $\mathcal{P}$. Since the Maurer-Cartan equation holds for $t \neq 0$, it holds also for $t = 0$. Thus there exists a biholomorphic map from $\mathcal{U}_0$ to an open subset of G/P sending $\mathcal{V}$ to $\mathcal{D}$. From the upper-semi-continuity of $h^0(X_t, T(X_t))$, the Lie algebra $aut(X_0)$ of infinitesimal automorphisms of X_0 has dimension $\geq \dim(\mathbf{g})$. By Corollary 5.4 of [Ya], the Lie algebra of infinitesimal automorphisms of $\mathcal{U}_0$ preserving $\mathcal{V}_0$ is isomorphic to $\mathbf{g}$. Thus $aut(X_0) \cong \mathbf{g}$ and the isomorphism is induced by the biholomorphism from $\mathcal{U}_0$ to an open set in G/P. In particular, G acts on X_0 with the isotropy subgroup at a generic point isomorphic to P, implying $X_0 \cong G/P$. □

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Jun-Muk Hwang
Korea Institute for Advanced Study
207-43 Cheongryangri-dong
Seoul 130-012, Korea
e-mail: jmhwang@ns.kias.re.kr

Ngaiming Mok
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
e-mail: nmok@hkucc.hku.hk