

# In Black and White: An Integrated Approach to Class Level Testing of Object-Oriented Programs

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Because of the growing importance of object-oriented programming, a number of testing strategies have been proposed. They are based either on pure black-box or white-box techniques. We propose in this paper a methodology to integrate the black- and white-box techniques. The black-box technique is used to select test cases. The white-box technique is mainly applied to determine whether two objects resulting from the program execution of a test case are observationally equivalent. It is also used to select test cases in some situations.

We define the concept of a fundamental pair as a pair of equivalent terms that are formed by replacing all the variables on both sides of an axiom by normal forms. We prove that an implementation is consistent with respect to all equivalent terms if and only if it is consistent with respect to all fundamental pairs. In other words, the testing coverage of fundamental pairs is as good as that of all possible term rewritings, and hence we need only concentrate on the testing of fundamental pairs. Our strategy is based on mathematical theorems. According to the strategy, we propose an algorithm for selecting a finite set of fundamental pairs as test cases.

Given a pair of equivalent terms as a test case, we should then determine whether the objects that result from executing the implemented program are observationally equivalent. We prove, however, that the observational equivalence of objects cannot be determined using a finite set of observable contexts (which are operation sequences ending with an observer function) derived from any black-box technique. Hence, we supplement our approach with a “relevant observable context” technique, which is a heuristic white-box technique to select a relevant finite subset of the set of observable contexts for determining the observational equivalence. The relevant observable contexts are constructed from a Data member Relevance Graph, which is an abstraction of the given implementation for a given specification. A semi-automatic tool has been developed to support this technique.

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## 1. INTRODUCTION

The special characteristics and properties of an object-oriented approach render resulting software systems more reliable, maintainable, and reusable. However, an object-oriented approach also poses new challenges to software testing as a software system is now composed of classes of objects and has unique features not found in other programming paradigms. New testing problems arise from the following facts: (1) Programs in an object-oriented system are not necessarily executed in a predefined order; the sequence of invocation of methods in a class is not specified explicitly; and there are more variations in combining methods in the same class or across different classes [1]. (2) Special testing techniques are also required to deal with inheritance, polymorphism, overloading, message passing, association, aggregation, and state-dependent behavior [2, 3, 4, 5, 6, 7]. (3) Furthermore, it is mandatory to derive an algorithm for determining the observational equivalence of the output objects so as to judge the correctness of implementations. The concept of object observational equivalence reflects the encapsulation and information hiding features of the object-oriented paradigm. In this paper, we only consider the facts (1) and (3) in *class level testing*, which concerns only the interactions of methods and data within a given class. However, inheritance, polymorphism, overloading, message passing, association, and aggregation concern the relationships and interactions among different classes in a given **cluster**, which are considered in our other paper [8]

In recent years, a number of papers on class-level testing of object-oriented programs have been published. The techniques involved can be classified into two categories. The black-box technique refers to program testing based on software specifications [9, 10, 11]; whereas the white-box technique refers to that based on information from the source code of the developed systems [1, 12, 13, 14, 15, 16, 17]. Each technique has its advantages and disadvantages. For example, if part of the specification is missing in an implementation, there is no way of revealing the problem using a pure white-box technique. On the other hand, we shall formally prove that it is impossible to determine whether two objects are observationally equivalent using a pure black-box technique. We therefore propose to integrate black- and white-box techniques in our project. We do not consider program syntax errors and specification errors in this paper.

The organization of the paper is as follows. Section 2 states the problems of test case selection, including the reasons why we use equivalent sequences of operations rather than individual operations as test cases, and our new selection strategy. In Section 3, we present a white-box technique, namely a “relevant observable context” technique, to determine the observational equivalence of objects. Section 4 is devoted to comparing our approaches with related work by other researchers. In Section 5, we conclude our current findings, summarize their limitations, and make suggestions for future work.

## 2. SELECTION OF TEST CASES

### 2.1 Background: Equivalent Terms as Test Cases

Algebraic specifications are popular in the formal specification of object-oriented programs [18, 19, 20]. An algebraic specification for a class consists of a syntax declaration and a semantic specification. The syntax declaration lists the *operations*<sup>1</sup> involved, plus their domains and co-domains, corresponding

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<sup>1</sup> In this paper, the word “operation” is used in a specification, while its counterpart in the implementation is called a “method”.

to the input and output parameters of the operations. The semantic specification consists of equational *axioms* that describe the behavioral properties of the operations. The following is an example of an algebraic specification for the class of integer stacks.

### Example 1

```

module INTSTACK is
  classes Int Bool IntStack
  inheriting INT
  operations
    new: → IntStack
    _empty: IntStack → Bool
    _push(_): IntStack Int → IntStack
    _pop: IntStack → IntStack
    _top: IntStack → Int ∪ {NIL}
  variables
    S: IntStack
    N: Int
  axioms
    a1: new.empty = true
    a2: S.push(N).empty = false
    a3: new.pop = new
    a4: S.push(N).pop = S
    a5: S.top = NIL if S.empty
    a6: S.push(N).top = N

```

□

Intuitively, a *term* is a series of operations in an algebraic specification. For example, *new.push(1).push(2).pop* is a term in the class of integer stacks above. A term is in *normal form* if and only if it cannot be further transformed by any axiom in the specification. For example, *new.push(1).push(2)* is in normal form but *new.push(1).push(2).pop* is not.

The concept of equivalent terms has been adopted in testing [9, 11, 21, 22, 23]. Two terms are said to be *equivalent* if and only if they can both be transformed to the same normal form. The terms *new.push(1).push(2).pop* and *new.push(3).pop.push(1)* are equivalent as they can both be transformed to the normal form *new.push(1)*. A term without variables is called a *ground term*. In this paper, we only consider ground terms because in dynamic testing, actual test cases involve ground terms only.

Let  $u_1$  and  $u_2$  be two ground terms and  $s_1$  and  $s_2$  be their corresponding method sequences in a given implementation. The test case  $\{u_1, u_2\}$  reveals an error of the implementation if  $u_1$  is equivalent to  $u_2$  but  $s_1$  and  $s_2$  produce observationally different objects.

The idea of using pairs of equivalent terms, rather than individual operations, as test cases in object-oriented black-box testing is justified by the following reasons:

- (1) In object-oriented programming, a series of messages are often passed to an object, and the resulting object is then evaluated for correctness. The concept of observational equivalence is very important here. Consider, for example, a word processor that maintains the history of insertions and deletions

in its document file for the purpose of undo's and redo's before it is finally saved. A user may enter a series of messages into the word processor, possibly with a number of wrong insertions followed by a number of corrective deletions. Another user may make different mistakes followed by different corrections when creating the same document. In either case, as long as they produce the same printed version, the final document files produced by the two users should be regarded as observationally equivalent. The concept of “equivalent terms” models this phenomenon very naturally. A series of messages passed to the object is modeled by a sequence of operations in a term. The objects resulting from two different series of messages would be equivalent if their observable versions, modeled by normal forms, are identical.

- (2) The conventional approach of testing the output  $B$  of an individual operation  $\_op$  using an input  $A$  is just a special case of the testing of equivalence. The equivalent terms in this case are  $A.op$  and  $B$ . However, testing pairs of equivalent terms includes the checking of interactions among operations in the terms, while testing individual operations separately does not.

Obviously, if an error occurs in a common subterm of a pair of equivalent terms, it cannot be revealed with this pair as a test case. We can, however, find another pair of equivalent terms to reveal this error. For example, if  $u_1 = new.push(1).push(2).pop$  and  $u_2 = new.push(1)$ , then  $u_1$  and  $u_2$  are equivalent. The common subterm of  $u_1$  and  $u_2$  is  $new.push(1)$ . If  $\_push(1)$  is erroneously implemented as  $\_push(11)$ , then the error cannot be detected by the test case of equivalent terms  $u_1$  and  $u_2$ , but can be revealed by another pair of equivalent terms  $new.push(1).top$  and  $1$ .

## 2.2 Basic Concepts

The following are the formal definitions of the basic concepts used in this paper. Definitions 2.1, 2.2, 2.3, and 2.6 are about algebraic specification, Definitions 2.7 to 2.10 are about implementation, while Definitions 2.4 and 2.5 are related to both.

**Definition 2.1** The sets  $T_C$  of *ground terms* in a term algebra  $T$  are defined recursively as follows:

- (a) For any constant or *constant operation*  $f: \rightarrow C$ ,  $f$  is a ground term in  $T_C$ . The *length* of  $f$  is defined to be 1.
- (b) For any operation  $\_f(\_, \_, \_): C_0 C_1 \dots C_n \rightarrow C$  (where  $n \geq 0$ ), and for any ground terms  $u_i$  in  $T_{C_i}$ ,  $0 \leq i \leq n$ ,  $u_0.f(u_1, \dots, u_n)$  is a ground term in  $T_C$ . Each  $u_i$  is a *proper subterm* of  $u_0.f(u_1, \dots, u_n)$ . Furthermore, if  $v$  is a proper subterm of  $u_i$ , then  $v$  is also a *proper subterm* of  $u_0.f(u_1, \dots, u_n)$ . The *length* of  $u_0.f(u_1, \dots, u_n)$  is defined as

$$\text{length}(u_0.f(u_1, \dots, u_n)) = 1 + \text{length}(u_0) + \text{length}(u_1) + \dots + \text{length}(u_n)$$

In Example 1, for instance,  $new.push(1).push(2).pop.top$  is a ground term, with proper subterms “ $new.push(1).push(2).pop$ ”, “ $new.push(1).push(2)$ ”, “ $new.push(1)$ ”, “ $2$ ”, “ $new$ ”, and “ $1$ ”. Their lengths are shown in Table 1. By Definition 2.1, all ground terms are of finite lengths.

**Table 1** Lengths of terms

Terms	Lengths
“1”	1
<i>new</i>	1
“2”	1
<i>new.push(1)</i>	3
<i>new.push(1).push(2)</i>	5
<i>new.push(1).push(2).pop</i>	6
<i>new.push(1).push(2).pop.top</i>	7

**Definition 2.2** Suppose

- (a)  $a_0$ :  $u = u'$  is an equational axiom such that each variable occurring in  $u'$  also appears in  $u$ .
- (b)  $u_0$  is a ground term containing a subterm that is a substitution instance of the left-hand side  $u$  of the axiom.
- (c) if we replace that subterm in  $u_0$  by the corresponding substitution instance of the right-hand side  $u'$ , the result is a ground term  $u_1$ .

Then we say that the ground term  $u_0$  can be *transformed* into the ground term  $u_1$  using the axiom  $a_0$  as a *left-to-right rewriting rule*. This is denoted by the notation:

$$u_0 \xrightarrow{a_0} u_1.$$

In Example 1, for instance, the ground term *new.push(1).push(2).pop.top* can be transformed into *new.push(1).top* using axiom  $a_4$  as a left-to-right rewriting rule, and *new.push(1).top* can be transformed further into the ground term “1” using  $a_6$ . These transformations are expressed as:

$$\begin{array}{ccc} a_4 & & a_6 \\ \text{new.push(1).push(2).pop.top} & \rightarrow & \text{new.push(1).top} \\ & \rightarrow & 1. \end{array}$$

**Definition 2.3** A ground term is said to be in *normal form* if and only if no further axiom is applicable to it as a left-to-right rewriting rule. A specification is said to be *canonical* if and only if every sequence of rewrites on the same ground term reaches a unique normal form in a finite number of steps.

**Definition 2.4** Suppose  $A_1, \dots, A_i$ , and  $E$  are classes different from class  $C$ , and  $D_1, \dots, D_k$  are classes that may or may not be different from class  $C$ , where  $i, k \geq 0$ . An operation or method  $f: A_1 \dots A_i \rightarrow C$  is called a *creator* of class  $C$ . An operation or method  $g(\_, \dots, \_): C D_1 \dots D_k \rightarrow C$  is called a *constructor* of class  $C$  if it can appear in a normal form. If an operation or method  $h(\_, \dots, \_): C D_1 \dots D_k \rightarrow C$  cannot appear in any normal form, it is called a *transformer* of class  $C$ . An operation or method  $p(\_, \dots, \_): C D_1 \dots D_k \rightarrow E$  is called an *observer* of class  $C$ .

In Example 1, for instance, the operation *new* is a creator, *.push(N)* is a constructor, *.pop* is a transformer, and *.empty* and *.top* are observers.

**Definition 2.5** Suppose  $C$  is a class of a given specification. A valid sequence of operations or methods in  $C$ , starting from a constructor or transformer but ending in an observer, is called an *observable context* on  $C$ .

The general form of an observable context  $oc$  is as follows:

$$oc = \_f_1(\dots).f_2(\dots)\dots.f_k(\dots).obs(\dots)$$

where  $\_f_1(\dots)$ ,  $\_f_2(\dots)$ , ...,  $\_f_k(\dots)$  are constructors or transformers of class  $C$  and  $\_obs(\dots)$  is an observer of class  $C$ . For any object  $O$  in  $C$ ,  $O.oc$  denotes the result of applying  $oc$  to  $O$ . For example,  $oc = \_push(1).push(2).pop.push(4).top$  is an observable context on the class of integer stacks given by Example 1. The result of applying it to  $new$  is  $new.push(1).push(2).pop.push(4).top = 4$ .

**Definition 2.6** For a given canonical specification, two ground terms  $u_1$  and  $u_2$  are said to be *equivalent* (denoted by  $u_1 \sim u_2$ ) if and only if both of them can be transformed into the same normal form by some axioms as left-to-right rewriting rules.

The following definition is adapted from [11, 22]:

**Definition 2.7** Given a canonical specification and its implementation in a class  $C$ , two objects  $O_1$  and  $O_2$  in  $C$  are said to be *observationally equivalent* (denoted by  $O_1 \approx O_2$ ) if and only if the following conditions are satisfied:

- (a) When  $C$  is a class provided by the implementation language,  $O_1$  and  $O_2$  are identical according to the built-in equality in the language.
- (b) When  $C$  is a user-defined class, for any observable context  $oc$  on  $C$ ,  $O_1.oc$  is observationally equivalent to  $O_2.oc$  in the output class.

Let  $O_i.d_j$  represent the value of a data member  $d_j$  in an object  $O_i$ . Notice that even if  $O_1.d_j \neq O_2.d_j$  for some data member  $d_j$  of the objects  $O_1$  and  $O_2$ , it does not necessarily mean that  $O_1$  and  $O_2$  are observationally nonequivalent. This point reflects the encapsulation or hiding of implementation details in the object-oriented paradigm.

For a canonical system, the observational equivalence of objects is reflexive, symmetric, and transitive.

**Definition 2.8** Given a canonical specification and its implementation, a series of methods corresponding to the operations in a ground term is called a *method sequence* corresponding to the ground term. Two such sequences  $s_1$  and  $s_2$  are said to be *equivalent* (denoted by  $s_1 \approx s_2$ ) if and only if they produce observationally equivalent objects.

For a canonical system, the equivalence of method sequences is reflexive, symmetric, and transitive.

**Definition 2.9** Suppose  $P$  is an implementation of a canonical specification  $SP$ .  $P$  is said to be *complete* if and only if, for every operation  $f$  in  $SP$ , there exists one and only one method  $m_f$  in  $P$  that implements  $f$ .

We can regard a complete implementation as a mapping  $\Psi$  from the specification to the implemented class, such that  $\Psi(f) = m_f$ . Let  $u = f_1.f_2\dots.f_k$  be a ground term in the class. We write the method sequence  $\Psi(f_1).\Psi(f_2)\dots.\Psi(f_k)$  as  $\Psi(u)$ .

**Definition 2.10** A complete implementation  $\Psi$  is said to be *consistent with respect to the equivalent ground terms*  $u_1$  and  $u_2$  if and only if the corresponding method sequences  $\Psi(u_1) \approx \Psi(u_2)$ .

Obviously, given a canonical specification, if a complete implementation is not consistent with respect to some equivalent ground terms, then there is an error in this implementation. Hence, this forms the basis of using equivalent ground terms as test cases.

### 2.3 Our Strategy: Fundamental Pairs as Test Cases

Although we have seen the rationale behind the use of equivalent ground terms as test cases, the set of all such terms for a given specification is infinite in general. Exhaustive testing is of course impossible. How do we select a finite representative subset of all equivalent ground terms as test cases? In this and the next sections, we shall propose a mathematically based strategy for selecting representative equivalent ground terms as test cases. First, we define an important concept as follows.

**Definition 2.11** For a given canonical specification, a pair of equivalent ground terms, formed by replacing all the variables on both sides of an axiom by normal forms, is called a *fundamental pair of equivalent terms* induced from the axiom. For the simplicity of expression, we shall refer to such a pair as a *fundamental pair* in this paper.

In Example 1, the pair of equivalent ground terms  $new.push(1).push(2).push(3).pop \sim new.push(1).push(2)$  can be formed by replacing the variables  $S$  and  $N$  in axiom  $a_4$  by the normal forms  $new.push(1).push(2)$  and “3”, respectively, and hence is a fundamental pair induced from axiom  $a_4$ . However, the pair of equivalent ground terms  $new.push(1).push(2).pop.push(3).pop \sim new.push(1)$  is not fundamental.

When we generate fundamental pairs from an axiom, if the right side of the axiom contains some conditions, the selected normal forms to replace variables have to satisfy these conditions.

Having defined the basic concepts, we would like to state our strategy for test case selection. We prove that, in order to test whether a complete implementation of a canonical specification is consistent with respect to all equivalent ground terms, we need only test fundamental pairs. That is, the testing coverage of all fundamental pairs remains the same as that of all equivalent ground terms. In other words, any error revealable by general equivalent ground terms can be revealed by some fundamental pairs.

**Example 2** Given the specification as shown in Example 1, consider an erroneous implementation in which the second call of *pop* returns a wrong value because of a *flag*. In this implementation, a stack is represented by an *array* and has an internal Boolean *flag* that is set to *false* when a new stack is created. The operation *pop* is implemented as follows<sup>2</sup>:

```

S.pop:
  if S.flag then return NIL; /* An error */
  else { ... /* If S.flag is false, then pop behaves properly. The code for the correct
        implementation will not be listed in full here. */
        S.flag = true; }

```

---

<sup>2</sup> This example is due to [24].

This error can be revealed by the following pair of general equivalent ground terms:

$$\begin{aligned} w_1 &= \text{new.push}(1).\text{push}(2).\text{pop}.\text{push}(3).\text{pop}, \\ w_2 &= \text{new.push}(1); \end{aligned}$$

It must also be exposed by a fundamental pair, say,

$$\begin{aligned} u_1 &= \text{new.push}(1).\text{push}(2).\text{push}(3).\text{pop}, \\ u_2 &= \text{new.push}(1).\text{push}(2). \end{aligned}$$

Executing the corresponding implementation method sequences  $s_1$  and  $s_2$  of  $u_1$  and  $u_2$ , respectively, we obtain the following objects as a result:

$$\begin{aligned} \text{array flag} &\quad /* \text{ Two data members in the implementation */} \\ O_1 &= ([1, 2], \text{true}) \\ O_2 &= ([1, 2], \text{false}) \end{aligned}$$

Since  $O_1.\text{pop.top} = \text{NIL}$  but  $O_2.\text{pop.top} = 1$ , by Definition 2.7,  $O_1$  and  $O_2$  are not observationally equivalent. Thus,  $\neg(s_1 \approx s_2)$ , but  $u_1 \sim u_2$ . By Definition 2.10, the error is detected by  $u_1 \sim u_2$ .  $\square$

In Example 1, for instance, we need only select test cases like the fundamental pair  $\text{new.push}(1).\text{push}(2).\text{push}(3).\text{pop} \sim \text{new.push}(1).\text{push}(2)$ , and need not consider general equivalent ground terms like  $\text{new.push}(1).\text{push}(2).\text{pop}.\text{push}(3).\text{pop} \sim \text{new.push}(1)$ . Obviously, the set of fundamental pairs is a proper subset of the set of equivalent ground terms.

To prove Theorem 2, we need the following lemmas and Theorem 1.

**Lemma 1** If a ground term  $u_i$  is a proper subterm of ground term  $u$ , then  $\text{length}(u) > \text{length}(u_i) > 0$ .

**Proof:**

The proof of Lemma 1 follows immediately from Definition 2.1.  $\square$

**Lemma 2** A canonical specification cannot contain any axiom  $a$  in the following form:

$$a: X = T$$

where  $X$  is a variable,  $T$  is a term, and  $X$  and  $T$  belong to the same class.

**Proof:**

Suppose the given specification contains  $a: X = T$ . Let  $u_0$  be a term of the class.

If  $T$  contains the variable  $X$ , we denote  $T = T(X)$ ,

$$a: X = T(X).$$

Thus, we have:

$$u_0 \xrightarrow{a} T(u_0) \xrightarrow{a} T(T(u_0)) \xrightarrow{a} \dots$$

which is an infinite rewriting sequence, thus contradicting the termination property of canonical specifications.  $\square$

If  $T$  does not contain the variable  $X$ , we have:

$$u_0 \xrightarrow{a} T \xrightarrow{a} T \xrightarrow{a} \dots$$

which is also an infinite rewriting sequence, thus contradicting the termination property of canonical specifications.  $\square$

For instance, if Example 1 contained an axiom  $a_{01}$ :  $S = S.push(N).pop$ , then we would obtain the infinite rewriting sequence:

$$new \xrightarrow{a_{01}} new.push(1).pop \xrightarrow{a_{01}} new.push(1).pop.push(2).pop \xrightarrow{a_{01}} \dots$$

which would contradict the terminating property of canonical specifications. If it contained an axiom  $a_{02}$ :  $S = new$ , we would have an infinite rewriting sequence:

$$new \xrightarrow{a_{02}} new \xrightarrow{a_{02}} new \xrightarrow{a_{02}} \dots$$

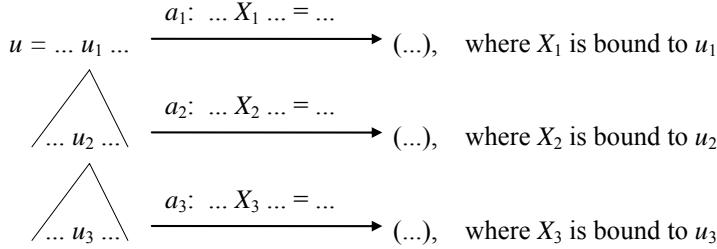
which would also contradict the terminating property of canonical specifications.

**Lemma 3** Suppose  $u$  is a ground term in a canonical specification. If  $u$  is not in normal form, then there exists an axiom  $a$  which can be applied to  $u$  as a left-to-right rewriting rule such that the following *binding condition* is satisfied:

All the variables involved in  $a$  are bound to normal forms.

**Proof:**

The basic idea of the proof is shown in Figure 1. Since  $u$  is not in normal form, according to Definition 2.3, there exists some axiom  $a_1$  that can be applied to  $u$  as a left-to-right rewriting rule. If  $a_1$  satisfies the binding condition, then the conclusion holds.



**Figure 1.** The basic idea of the proof of Lemma 3

Otherwise,  $a_1$  includes a variable  $X_1$  bound to a non-normal form  $u_1$ , where  $u_1$  is a subterm of  $u$ . We know that  $u_1$  must be a proper subterm of  $u$  because, if  $u_1 = u$ ,  $a_1$  would be of the form:

$$a_1: X_1 = T.$$

thus contradicting Lemma 2. According to Definition 2.3 again, there exists some other axiom  $a_2$  that can be applied to  $u_1$  as a left-to-right rewriting rule. Since  $u_1$  is a proper subterm of  $u$ ,  $a_2$  can also be applied to  $u$  as a left-to-right rewriting rule. If  $a_2$  satisfies the binding condition, then the conclusion holds.

Otherwise,  $a_2$  includes a variable  $X_2$  bound to a non-normal form  $u_2$ , where  $u_2$  is a subterm of  $u_1$ . Similarly to the above, according to Lemma 2,  $u_2$  must be a proper subterm of  $u_1$ . Furthermore,  $u_2$  is also a proper subterm of  $u$ . According to Definition 2.3, there exists some other axiom  $a_3$  that can be applied to  $u_2$  as a left-to-right rewriting rule. Since  $u_2$  is also a proper subterm of  $u$ ,  $a_3$  can also be applied to  $u$  as a left-to-right rewriting rule. If  $a_3$  satisfies the binding condition, then the conclusion holds.

Otherwise, continue the process similar to the above. Since the length of  $u$  is finite, and according to Lemma 1, we have

$$\text{length}(u) > \text{length}(u_1) > \text{length}(u_2) > \dots > 0,$$

the process must terminate in a finite number of steps. Thus, we obtain an axiom  $a_i$  that can be applied to  $u$  as a left-to-right rewriting rule and satisfies the binding condition.  $\square$

Consider, for instance, a term  $u = \text{new.push}(1).\text{push}(2).\text{pop.push}(3).\text{top}$  for the specification in Example 1. We can apply axiom  $a_6$  to  $u$  as a left-to-right rewriting rule. In this case, the variables  $S$  and  $N$  in  $a_6$  are bound to  $\text{new.push}(1).\text{push}(2).\text{pop}$  and 3, respectively. However,  $\text{new.push}(1).\text{push}(2).\text{pop}$  is not in normal form. We can apply axiom  $a_4$  to it, such that the variables  $S$  and  $N$  involved are bound to the normal forms “ $\text{new.push}(1)$ ” and “2”, respectively. Furthermore, since “ $\text{new.push}(1).\text{push}(2).\text{pop}$ ” is a subterm of  $u$ , we can also apply  $a_4$  directly to  $u$ . Thus, we have found an axiom  $a_4$  which can be applied directly to  $u$  as a left-to-right rewriting rule such that the binding condition is satisfied.

**Theorem 1** Suppose  $u$  is a ground term in a canonical specification. If  $u$  is not in normal form, then  $u$  can be transformed into a unique normal form  $u^*$  via a series of axioms  $a_1, a_2, \dots, a_k$ :

$$\begin{array}{cccc} a_1 & a_2 & \dots & a_{k-1} & a_k \\ u \rightarrow u_2 \rightarrow \dots \rightarrow u_k \rightarrow u_{k+1} = u^*, \end{array}$$

such that each  $a_i$  satisfies the binding condition.

**Proof:**

Since the given specification is canonical, according to Lemma 3, there exists some axiom  $a_1$  that can be applied to  $u$  as a left-to-right rewriting rule, transforming  $u$  into a ground term  $u_2$ :

$$\begin{array}{c} a_1 \\ u \rightarrow u_2, \end{array}$$

and satisfies the binding condition. If  $u_2$  is in normal form, then the theorem holds.

Otherwise, according to the same lemma, there exists another axiom  $a_2$ :

$$\begin{array}{c} a_2 \\ u_2 \rightarrow u_3, \end{array}$$

which satisfies the binding condition. If  $u_3$  is in normal form, then the theorem is satisfied.

Otherwise, continue the process similar to the above. Since the specification is canonical, according to Definition 2.3, the process must terminate at a unique normal form, thus yielding a finite series of axioms  $a_1, a_2, \dots, a_k$  that satisfies the theorem.  $\square$

**Theorem 2** Given a canonical specification, a complete implementation is consistent with respect to all equivalent ground terms if and only if it is consistent with respect to all fundamental pairs.<sup>3</sup>

**Proof:**

Obviously, if a complete implementation is consistent with respect to all equivalent ground terms, then it is consistent with respect to all fundamental pairs.

Suppose a complete implementation is consistent with respect to all fundamental pairs. Let  $u_1 \sim u_2$  be any two equivalent ground terms. Since the implementation is complete, it can be regarded as a mapping  $\Psi$ . Let  $\Psi(u_1) = s_1$  and  $\Psi(u_2) = s_2$ . We wish to prove that  $s_1 \approx s_2$ .

By Definition 2.6,  $u_1$  and  $u_2$  can be transformed into the same normal form  $u^*$ . Since the given specification is canonical, according to Theorem 1, we can find a series of  $k$  axioms that transform  $u_1$  to  $u^*$ :

$$\begin{array}{cccc} a_1 & a_2 & \dots & a_{k-1} & a_k \\ u_1 \rightarrow u_{12} \rightarrow u_{13} \rightarrow \dots \rightarrow u_{1k} \rightarrow u^*. \end{array}$$

---

<sup>3</sup> In spite of this theorem, we should note that an infinite set of observable contexts may be required to check the observational equivalence of objects resulting from the fundamental pairs. This problem cannot be solved by any black-box technique. We shall present in Section 3 a heuristic white-box technique that selects a relevant finite subset of the set of observable contexts.

where all the axioms  $a_1, \dots, a_k$  satisfy the binding condition. Let  $\Psi(u_{1j}) = s_{1j}$ ,  $j = 2, \dots, k$ , and  $\Psi(u^*) = s^*$ .

$u_1$  must be of the form  $f_0(v_0).f_1(v_1)\dots f_n(v_n)$ . Each  $v_i$ ,  $i = 0, 1, \dots, n$ , is a list of parameters (possibly an empty list) of the form

$$v_i = (h_{10}(v_{10}).h_{11}(v_{11})\dots h_{1p}(v_{1p}), h_{20}(v_{20}).h_{21}(v_{21})\dots h_{2q}(v_{2q}), \dots, h_{r0}(v_{r0}).h_{r1}(v_{r1})\dots h_{rs}(v_{rs}))_i,$$

where each parameter contains only ground terms, each  $v_{10}, v_{11}, \dots, v_{rs}$  may further be expressed in a form similar to  $v_i$ , and so on.

Since  $u_1$  can be rewritten by applying  $a_1$  as a rewriting rule, the left hand side of  $a_1$  should match a subterm of  $u_1$ . Hence,  $a_1$  must be of one of the following two forms:

$$X.f_j(X_j).f_{j+1}(X_{j+1})\dots f_k(X_k) = X.g_1(Y_1).g_2(Y_2)\dots g_m(Y_m), \quad 1 \leq j \leq k \leq n,$$

or

$$X.f_j(X_j).f_{j+1}(X_{j+1})\dots f_k(X_k) = Y.g_1(Y_1).g_2(Y_2)\dots g_m(Y_m), \quad 1 \leq j \leq k \leq n,$$

where  $X$  and  $Y$  ( $\neq X$ ) are creators or object variables<sup>4</sup>,  $X_j, X_{j+1}, \dots, X_k, Y_1, Y_2, \dots, Y_m$  are lists of parameters (possibly empty lists) containing variables or ground terms, and  $g_1(Y_1).g_2(Y_2)\dots g_m(Y_m)$  may be absent<sup>5</sup>.

Thus,  $u_{12}$  will be of the corresponding form

$$f_0(v_0).f_1(v_1)\dots f_{j-1}(v_{j-1}).g_1(w_1).g_2(w_2)\dots g_m(w_m).f_{k+1}(v_{k+1})\dots f_n(v_n)$$

or

$$w.g_1(w_1).g_2(w_2)\dots g_m(w_m).f_{k+1}(v_{k+1})\dots f_n(v_n),$$

where  $w, w_1, w_2, \dots, w_m$  are substitution instances of  $Y, Y_1, Y_2, \dots, Y_m$ , respectively. Without loss of generality, we will only discuss the more complex case

$$u_{12} = f_0(v_0).f_1(v_1)\dots f_{j-1}(v_{j-1}).g_1(w_1).g_2(w_2)\dots g_m(w_m).f_{k+1}(v_{k+1})\dots f_n(v_n)$$

for the remaining part of this proof. Thus,

---

<sup>4</sup> Since  $a_1$  satisfies the binding condition and  $X$  in  $a_1$  is bound to  $f_0(v_0).f_1(v_1)\dots f_{j-1}(v_{j-1})$ , we need to show that any ground term  $u_1$  contains at least a normal form at the beginning. In fact, any ground term consists of a creator at its beginning followed by constructors, transformers, or observers. According to Lemma 2, a canonical specification cannot contain any axiom of the form “ $X = T$ ”. Thus, any creator cannot be rewritten in a canonical specification, and hence is a normal form. Besides the creator, in the ground term  $u_1$ , any subterm beginning with this creator and followed only by some constructors is also a normal form. See footnote 6. We can therefore conclude that any ground term  $u_1$  contains at least some normal form at its beginning.

<sup>5</sup> For example, the axiom  $X.credit(M).balance = X.balance.add(M)$  is of the first form such that  $g_1(Y_1).g_2(Y_2)\dots g_m(Y_m)$  is present. The axiom  $X.push(N).pop = X$  is also of the first form such that  $g_1(Y_1).g_2(Y_2)\dots g_m(Y_m)$  is absent. The axiom  $X.push(N).top = N$  is of the second form such that  $Y = N$  and  $g_1(Y_1).g_2(Y_2)\dots g_m(Y_m)$  is absent. The axiom  $X.push(N).empty = false$  is also of the second form such that  $Y$  is the creator  $false$  and  $g_1(Y_1).g_2(Y_2)\dots g_m(Y_m)$  is absent.

$$\begin{aligned}s_1 &= \Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_n(v_n)), \\ s_{12} &= \Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_{j-1}(v_{j-1})) \cdot \Psi(g_1(w_1)) \cdot \Psi(g_2(w_2)) \dots \Psi(g_m(w_m)) \\ &\quad \cdot \Psi(f_{k+1}(v_{k+1})) \dots \Psi(f_n(v_n)).\end{aligned}$$

Since axiom  $a_1$  satisfies the binding condition in the transformation from  $u_1$  to  $u_{12}$ ,

$$f_0(v_0) \cdot f_1(v_1) \dots f_{j-1}(v_{j-1}) \cdot f_j(v_j) \cdot f_{j+1}(v_{j+1}) \dots f_k(v_k) \sim f_0(v_0) \cdot f_1(v_1) \dots f_{j-1}(v_{j-1}) \cdot g_1(w_1) \cdot g_2(w_2) \dots g_m(w_m)$$

must be a fundamental pair induced from  $a_1$ . According to the assumption that the implementation is consistent with respect to all fundamental pairs, we have

$$\begin{aligned}\Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_k(v_k)) \\ \approx \Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_{j-1}(v_{j-1})) \cdot \Psi(g_1(w_1)) \cdot \Psi(g_2(w_2)) \dots \Psi(g_m(w_m))\end{aligned}\tag{a}$$

If  $f_n(v_n)$  is an observer, then

$$\Psi(f_{k+1}(v_{k+1})) \dots \Psi(f_n(v_n))\tag{b}$$

is an observable context. According to Definitions 2.7 and 2.8, by applying (b) to both sides of (a), we have

$$\begin{aligned}\Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_n(v_n)) \\ \approx \Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_{j-1}(v_{j-1})) \cdot \Psi(g_1(w_1)) \cdot \Psi(g_2(w_2)) \dots \Psi(g_m(w_m)) \\ \cdot \Psi(f_{k+1}(v_{k+1})) \dots \Psi(f_n(v_n))\end{aligned}$$

In other words,  $s_1 \approx s_{12}$ .

If  $f_n(v_n)$  is not an observer, for any observable context  $oc$  of the given class,

$$\Psi(f_{k+1}(v_{k+1})) \dots \Psi(f_n(v_n)) \cdot oc\tag{c}$$

is still an observable context on the given class. According to Definitions 2.7 and 2.8, by applying (c) to both sides of (a), we have

$$\begin{aligned}\Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_n(v_n)) \cdot oc \\ \approx \Psi(f_0(v_0)) \cdot \Psi(f_1(v_1)) \dots \Psi(f_{j-1}(v_{j-1})) \cdot \Psi(g_1(w_1)) \cdot \Psi(g_2(w_2)) \dots \Psi(g_m(w_m)) \\ \cdot \Psi(f_{k+1}(v_{k+1})) \dots \Psi(f_n(v_n)) \cdot oc\end{aligned}$$

This means that the objects produced by  $s_1$  and  $s_{12}$  are observationally equivalent. According to Definition 2.8, we have  $s_1 \approx s_{12}$ .

By the same argument, we have

$$s_{12} \approx s_{13} \approx \dots \approx s_{1k} \approx s^*.$$

Therefore,  $s_1 \approx s^*$ . Similarly, we can prove that  $s_2 \approx s^*$ . Hence,  $s_1 \approx s_2$ .  $\square$

## 2.4 Algorithm GFT for Generating a Finite Number of Test Cases

In general, the axioms of an algebraic specification may contain branch conditions. An axiom may induce an infinite number of different fundamental pairs by assigning different normal forms to its variables. Exhaustive testing is impossible. How do we select a finite number of representative test cases from the infinite set of fundamental pairs? We present the following Algorithm GFT to deal with this problem. We give a related definition first.

**Definition 2.12** Let  $m_f$  be an implemented method. We say we apply the *path-based domain partition technique* (PDP technique) to  $m_f$  if we:

- (a) Partition the *input domain* of  $m_f$  into subdomains such that all the test points in each subdomain cause a particular path in the implementation of  $m_f$  to be executed. Here, the partition concept follows White and Cohen [25], but the *path generation algorithm* is the same as the one proposed by Jeng and Weyuker [26].
- (b) Use the *simplified domain-testing strategy* presented by Jeng and Weyuker [26] to select some *test points* from each subdomain, if the assumptions required by the strategy are satisfied.
- (c) Otherwise, randomly select a test point from each subdomain.

Since the PDP technique is path-oriented, it obviously inherits the problems associated with path testing, such as an infinite number of paths and the identification of infeasible paths. Recently, Jeng and Weyuker [26] proposed an innovative technique for detecting domain errors. Instead of the traditional approach of testing whether a border is correct, they test whether or not there is a displaced area. Their new perspective has greatly improved the practicality of White and Cohen's domain-testing strategy by removing most of the unrealistic constraints in its original model. Furthermore, although their new technique has a lower cost, the effectiveness is comparable. They also propose a path generation algorithm in which all the selected paths are executable, and hence infeasible paths are no longer an issue in the implementation. In view of all the above merits, we have adopted Jeng and Weyuker's method in our algorithm.

**Algorithm GFT (Generating a Finite number of Test cases)** The algorithm asks the analyst to supply a canonical specification, and requests the designer to identify the mapping from the set of specified operations to the set of implemented methods. According to Theorem 2, we need not produce general equivalent ground terms as test cases. We need only construct fundamental pairs, which are produced from each axiom in the given canonical specification. Suppose the given specification contains  $n$  axioms  $a_1, a_2, \dots, a_n$ . For each axiom  $a_i$  ( $i = 1, 2, \dots, n$ ), conduct the following steps:

- (a) If a variable  $V$  of type  $T$  involved in  $a_i$  is not observable, use the syntax part of the given specification as a grammar [27] to construct all patterns of normal forms from the creators and constructors<sup>6</sup> of type  $T$ , such that their lengths do not exceed some positive integer  $k$ . Then replace each occurrence of  $V$  in  $a_i$  by these patterns to unfold  $a_i$  into several new equations, which are further unfolded until all the variables involved in the new equations  $a_{ij}$  are of observable types.

---

<sup>6</sup> We can infer from Definition 2.4 that a normal form contains only a creator and some constructors, but no transformer.

The above positive integer  $k$  may be determined by a white-box technique, such as by referring to the maximum sizes of arrays, or the boundary values of variables declared in the implemented code. If the maximum sizes or the boundary values are too large for the generation of test sets of reasonable sizes, ask the user to specify an acceptable value of  $k$ .

- (b) Suppose the right hand side of a new equation  $a_{ij}$  obtained in step (a) contains a defined operation  $f$ . Use the conditions of the set of axioms defining  $f$  to partition the input domain of  $f$  into subdomains.
- (c) Randomly select an element from each subdomain obtained in step (b), and use these elements to replace all occurrences of the corresponding input variables in equation  $a_{ij}$  to obtain a group of fundamental pairs induced from axiom  $a_i$ .
- (d) If the above group of fundamental pairs reveals an error, exit from the algorithm. Otherwise go to step (e).
- (e) Suppose the defined operation  $f$  in step (b) is implemented by method  $m_f$ . Apply the PDP technique to  $m_f$  for selecting input data points to replace all occurrences of the corresponding variables in equation  $a_{ij}$ , and hence obtain another group of fundamental pairs for axiom  $a_i$ .  $\square$

Example 3 below is used to illustrate Algorithm GFT.

**Example 3** The specification is nearly the same as Example 1, except the following additional entries:

**operations**

...  
 $_\text{ascending}: \text{IntStack} \rightarrow \text{Bool}$

**variables**

...  
 $N1\ N2: \text{Int}$

**axioms**

...  
 $a_7: \text{new.ascending} = \text{true}$   
 $a_8: \text{new.push}(N).\text{ascending} = \text{true}$   
 $a_9: S.\text{push}(N1).\text{push}(N2).\text{ascending} = N1 \leq N2 \text{ and } S.\text{push}(N1).\text{ascending}$

Suppose the following axioms in class *INT* define the operation “ $\leq$ ”:

- $b_1: (N1 \leq N2) = \text{true} \text{ if } N1 == N2$
- $b_2: (N1 \leq N2) = \text{true} \text{ if } N1 < N2$
- $b_3: (N1 \leq N2) = \text{false} \text{ if } N2 < N1$

Following Algorithm GFT, we should conduct steps (a), (b), and (c) for each of the axioms  $a_1$  to  $a_9$ . For simplicity, however, we shall only illustrate the procedure for axiom  $a_9$ .

- (a)  $a_9$  includes a variable  $S$  that is not observable. Determine a positive integer  $k$  for  $S$ . For the sake of illustration, suppose  $k = 3$ . The patterns of normal forms of  $S$  of lengths  $\leq 3$  are as follows:

$S = new,$   
 $S = new.push(N0).$

By replacing  $S$  in  $a_s$  with the above patterns, we unfold  $a_s$  into the following new equations:

$a_{91}$ :  $new.push(N1).push(N2).ascending = (N1 \leq N2)$  and  $new.push(N1).ascending$   
 $a_{92}$ :  $new.push(N0).push(N1).push(N2).ascending$   
 $= (N1 \leq N2)$  and  $new.push(N0).push(N1).ascending$

- (b) The right-hand side of  $a_{91}$  contains an operation  $\leq$  defined by axioms  $b_1$ ,  $b_2$ , and  $b_3$ . Use the conditions of  $b_1$ ,  $b_2$ , and  $b_3$  to partition the input domain of the operation  $\leq$  into the following subdomains:

- (1)  $N1 = N2$   
 (2)  $N1 < N2$   
 (3)  $N2 < N1$

Furthermore, we partition the input domain into the following subdomains for axiom  $a_{92}$ :

- (1)  $N1 = N2$  and  $N0 = N1$
  - (2)  $N1 = N2$  and  $N0 < N1$
  - (3)  $N1 = N2$  and  $N1 < N0$
  - (4)  $N1 < N2$  and  $N0 = N1$
  - (5)  $N1 < N2$  and  $N0 < N1$
  - (6)  $N1 < N2$  and  $N1 < N0$
  - (7)  $N2 < N1$  and  $N0 = N1$
  - (8)  $N2 < N1$  and  $N0 < N1$
  - (9)  $N2 < N1$  and  $N1 < N0$

- (c) Replace the variables  $N1$ ,  $N2$ , and  $N0$  in the above axioms  $a_{91}$  and  $a_{92}$  by some integers randomly selected from the corresponding subdomains above, thus resulting in the following fundamental pairs induced from  $a_9$  as a part of test cases:

$new.push(1).push(1).ascending \sim (1 \leq 1)$  and  $new.push(1).ascending$   
 $new.push(1).push(2).ascending \sim (1 \leq 2)$  and  $new.push(1).ascending$   
 $new.push(2).push(1).ascending \sim (2 \leq 1)$  and  $new.push(2).ascending$   
 $new.push(2).push(2).push(2).ascending$   
 $\sim (2 \leq 2)$  and  $new.push(2).push(2).ascending$   
 $new.push(1).push(2).push(2).ascending$   
 $\sim (2 \leq 2)$  and  $new.push(1).push(2).ascending$   
 $new.push(3).push(2).push(2).ascending$   
 $\sim (2 \leq 2)$  and  $new.push(3).push(2).ascending$   
 $new.push(3).push(3).push(4).ascending$   
 $\sim (3 \leq 4)$  and  $new.push(3).push(3).ascending$

```

new.push(2).push(3).push(4).ascending
  ~ (3 ≤ 4) and new.push(2).push(3).ascending
new.push(5).push(3).push(4).ascending
  ~ (3 ≤ 4) and new.push(5).push(3).ascending
new.push(4).push(4).push(2).ascending
  ~ (4 ≤ 2) and new.push(4).push(4).ascending
new.push(3).push(4).push(2).ascending
  ~ (4 ≤ 2) and new.push(3).push(4).ascending
new.push(5).push(4).push(2).ascending
  ~ (4 ≤ 2) and new.push(5).push(4).ascending

```

□

It should be noted that if  $k$  in step (a) is not well chosen, some implementation errors may not be revealed, as illustrated in Example 4. Thus, the selection of  $k$  is important, but difficult as indicated in Section 2.5.2. This may warrant further investigation.

**Example 4** Let us refer to the specification in Example 1 again. Suppose the implementation is as follows, where  $array[100]$  is the top of the stack and  $array[1]$  is the bottom.

```

#include <iostream.h>
#define SIZE 100
#define NIL 0

class intStack {
    int array[SIZE];      /* Only one data member */
public: ...
};

...

void intStack :: newStack( )
{
    for ( int j = 1; j <= 100; j++ )
        array[j] = NIL;
}

void intStack :: push(int i)
{
    for ( int j = 1; j <= 99; j++ )
        array[j] = array[j+1];
    array[100] = i;
}

void intStack :: pop( )
{
    for ( int j = 100; j >= 2; j-- )
        array[j] = array[j-1];
    array[1] = NIL;
}

```

```

int intStack::top( )
{
    return array[100];
}
...

```

Let  $u_1 = new.push(1).push(2)...push(100).push(101).pop$   
 $u_2 = new.push(1).push(2)...push(100)$ .

Obviously,  $u_1 \sim u_2$  is a fundamental pair. The following objects  $O_1$  and  $O_2$  are produced by  $u_1$  and  $u_2$ , respectively:

$$O_1 = [NIL, 2, \dots, 100]$$

$$O_2 = [1, 2, \dots, 100].$$

Since  $O_1$  and  $O_2$  are not observationally equivalent, this implementation contains an error. In step (a) above, suppose for argument's sake we have chosen  $k = 10$  for the variable  $S$  in axiom  $a_4$ . Then the error cannot be revealed.  $\square$

## 2.5 Discussions on Algorithm GFT

In this section, we discuss a number of important issues on Algorithm GFT including assumptions, limitations, applicability, and complexity.

### 2.5.1 Assumptions

We have assumed in Theorem 2 and Algorithm GFT that a canonical specification and a complete implementation are given. How restrictive are these requirements?

Intuitively, the normal form of a ground term denotes the “abstract object value” [28] of this ground term. Two ground terms having the same normal form would have the same “abstract object value”. Thus, according to Definition 2.3, every ground term under a canonical specification would have a unique “abstract object value”, hence avoiding any ambiguity. In other words, if we relax the canonical requirement for a specification, the ground terms may be ambiguous.

If an implementation is not complete, there exists some operation  $f_0$  such that (1) no method implements it or (2) two or more distinct methods implement it in the same class. Case (1) is obviously an error, since the implemented system will fail when  $f_0$  is called. Case (2), on the other hand, is ambiguous, since the implemented system can have two distinct outcomes. In both cases, the problem can easily be detected by comparing a checklist of all the operations in the specification against the corresponding methods in the implementation.

To summarize, in order to avoid omissions and ambiguities, it is acceptable to require a specification to be canonical and an implementation to be complete.

### 2.5.2 Limitations

It is difficult to determine the positive integer  $k$  in step (a) of Algorithm GFT. It will be helpful to apply a white-box technique, such as referring to the maximum sizes of arrays, or the boundary values of

variables declared in the implemented code. However, when the maximum sizes or the boundary values are large, the sizes of the test cases may be of the order  $O(m^k)$  (where  $m$  is the number of constructors in the class under test) and hence unreasonable. Furthermore, even when the maximum sizes or the boundary values are not large, some of the faults on the capabilities for handling excess of the maximum sizes or the boundary values may not be identified. This is a natural limitation of step (a). Similarly, the PDP technique in step (e) of Algorithm GFT cannot be fully automated.

As an optional heuristic, we may supplement the algorithm by the “weak class graph” and “weak coverage criteria” approaches proposed by [15] for selecting the normal forms in step (a) of Algorithm GFT. (See Section 4.3 for more details.)

Alternatively, we may have to ask the user to choose  $k$  for the algorithm (similarly to [21]) when the white-box technique fails. Thus, Algorithm GFT can be implemented as a semi-automatic CASE tool that interacts with users when the above problems are encountered.

### 2.5.3 Effectiveness and Applicability Issues

In step (a) of Algorithm GFT, we replace every variable of nonobservable types by a finite number of patterns of normal forms with limited lengths. In fact, this is a common practice in testing, and has been formalized in [21] by means of a regularity hypothesis. The random selection of a value from each subdomain in step (c) of Algorithm GFT and PDP technique is also a common practice in testing. It has also been formalized in [21] by means of a uniformity hypothesis.

For example, consider a program “*if*  $X > 0$  *then*  $Y = f(X)$  *else*  $Y = -X$ ”. Suppose “ $Y = f(X)$ ” is a computational error that should be corrected to “ $Y = g(X)$ ”. We partition the input domain of  $X$  into two subdomains  $sb_1 = \{X \mid X > 0\}$  and  $sb_2 = \{X \mid X \leq 0\}$ . Let  $solutionSet_1 = \{X \mid X > 0 \text{ and } f(X) = g(X)\}$ . Suppose  $t$  is some randomly selected test data from  $sb_1$ . If  $t \in solutionSet_1$ , the error cannot be revealed. However, if  $t \in (sb_1 \setminus solutionSet_1)$ , the error can be exposed. In many practical cases, we can expect the cardinal number of the set  $(sb_1 \setminus solutionSet_1)$  to be much greater than that of the  $solutionSet_1$ . For instance, in many programs,  $f(X)$  and  $g(X)$  are arithmetic expressions. In this case, the  $solutionSet_1$  is finite but the set  $(sb_1 \setminus solutionSet_1)$  is infinite. This means that the probability of  $t \in (sb_1 \setminus solutionSet_1)$  is much greater than that of  $t \in solutionSet_1$ . In other words, in such cases, the probability of revealing the error by the randomly selected  $t$  from the subdomain is much greater than that of not revealing it.

The “simplified domain-testing strategy” and its corresponding “path generation algorithm”, adopted from [26] and used in Definition 2.12 and step (e) of Algorithm GFT of this paper, have been shown by the original authors to be effective and applicable.

### 2.5.4 Complexity Issue

Algorithm GFT is similar to that used in the tools described in Sections 5 and 6 of Bouge et al. [9], except the following differences: (1) We suggest in Algorithm GFT to use a white-box technique to determine the positive number  $k$  in step (a), whereas Bouge et al. regard  $k$  as a part of the regularity hypothesis. (2) In Algorithm GFT, we replace all the variables in the unfolded equations by normal forms, while Bouge et al. replace them by ground terms.

In practice, the complexity of Algorithm GFT depends heavily on the actual number of normal forms generated for a given positive integer  $k$ . According to Definition 2.4, a normal form contains only

a creator and a number of constructors, but no transformer, whereas a ground term may contain all three types of operations. In most situations, a class contains more transformers than creators and constructors. Hence, our proposal in the algorithm to replace variables by normal forms, rather than ground terms in general, enhances the efficiency of testing.

### 3. DETERMINING THE OBSERVATIONAL EQUIVALENCE OF TWO OBJECTS

Suppose the fundamental pair  $u_1 \sim u_2$  is selected as a test case for a given specification. To apply this test case to an implementation, we should map each operation in  $u_1$  and  $u_2$  to a method in the program. As a result, this mapping generates two method sequences  $s_1$  and  $s_2$  in the program corresponding to  $u_1$  and  $u_2$ , respectively. For a complete implementation, this mapping exists and can be indicated manually by the implementation designer, or be derived automatically from a given interface specification. Let  $O_1$  and  $O_2$  be two objects resulting from the execution of  $s_1$  and  $s_2$ , respectively. After executing  $s_1$  and  $s_2$  in the program and obtaining results  $O_1$  and  $O_2$ , in order to judge whether the test case  $\{u_1, u_2\}$  reveals an implementation error, we have to decide whether  $O_1$  and  $O_2$  are observationally equivalent (denoted by  $O_1 \approx O_2$ ). In Section 3.1, we explain why this problem is undecidable using black-box techniques, and indicate that we have to use a heuristic white-box technique to select a relevant finite subset of the set of observable contexts.

#### 3.1 Reason for Using a Heuristic White-Box Technique

According to Definition 2.7, we can use the observable contexts on class  $C$  to determine whether  $O_1 \approx O_2$ . Unfortunately, the set of all observable contexts in class  $C$  is infinite in general. How do we select a finite subset? For the stack example ( $O_1$  and  $O_2$  are stacks), it intuitively seems that

$$\begin{aligned}
 (O_1 \approx O_2) \Leftrightarrow & (O_1.height \approx O_2.height) \text{ and} \\
 & (O_1.top \approx O_2.top) \text{ and} \\
 & (O_1.pop.top \approx O_2.pop.top) \text{ and} \\
 & \dots \text{ and} \\
 & (O_1.pop^{(O_1.height)}.top \approx O_2.pop^{(O_2.height)}.top)
 \end{aligned} \tag{Formula I}$$

Let  $SS = \{height\} \cup \{pop^i.top \mid i = 0, 1, \dots, height\}$ . Although the subset  $SS$  of observable contexts is finite, Formula I is unfortunately still incorrect. A counterexample<sup>7</sup> is given in Bernot et al. [21]. The authors then added, “we get the depressing result that the only credible alternative is to consider the set of all observable contexts, which is infinite (and consequently impracticable).” They simply regard Formula I as a hypothesis, known as an “oracle hypothesis”<sup>8</sup>, for the class of integer stacks.

---

<sup>7</sup> Example 5 in Section 3.2 is another counter-example. It concludes that the objects  $O_1 = ([1, 2], 1, 0, 0)$  and  $O_2 = ([1, NIL], 1, 1, 0)$  are not observationally equivalent, but the finite subset  $SS$  of observable contexts shown above will report that they are observationally equivalent.

<sup>8</sup> The oracle hypothesis is an attempt by [21] to formalize the basic assumptions about the oracle problem in software testing. According to the authors, “the oracle problem [is related with] how to decide if a program execution returns a correct result. The solutions to this problem depend both on the kind of formal specification and program; a property required by the specification may not be observable using the program under test. Most of the formal specification methods provide a way to express observability. In this case, the program is assumed to satisfy the observability requirements (for instance, to decide correctly the equality of two integers); it is [known as] an oracle hypothesis.”

In fact, we can formally prove that the observational equivalence of two objects cannot be decided by a black-box technique.

**Theorem 3** For any given class, let  $AllOCs$  be the set of observable contexts,  $CT$  be the set of constructors and transformers, and  $OBS$  be the set of observers. If  $CT$  and  $OBS$  are non-empty, then  $AllOCs$  is infinite. Furthermore, if the class has at least one pair of equivalent ground terms  $u_1 \sim u_2$  and a constructor or transformer  $f(\dots) \in CT$  such that the numbers of appearances of  $f(\dots)$  in  $u_1$  and  $u_2$  are different, then the observational equivalence of objects cannot be determined using a finite set of observable contexts selected independently of implementations. In other words, given any finite subset  $SubOCs$  of  $AllOCs$ , there exist some implementation and two objects  $O_1$  and  $O_2$  such that

$$(\forall oc \in SubOCs) (O_1.oc \approx O_2.oc) \wedge (\exists oc_0 \in AllOCs) (\neg(O_1.oc_0 \approx O_2.oc_0)) \quad (\text{Formula II})$$

That is, there exist objects that are not observationally equivalent, but appear to be so when only a finite subset of the observational contexts are applied.

**Proof:**

Since  $CT$  and  $OBS$  are non-empty, there exist a constructor or transformer  $g(\dots)$  in  $CT$  and an observer  $obs(\dots)$  in  $OBS$ . Given any positive integer  $i$ , we can construct an  $oc_i = g(\dots) \dots g(\dots).obs(\dots)$  in  $AllOCs$  that contains  $i$   $g(\dots)$ 's. Hence,  $AllOCs$  is infinite.

Suppose  $u_1$  contains  $m f(\dots)$ 's and  $u_2$  contains  $n f(\dots)$ 's, such that  $0 \leq n < m$ . Consider any given finite subset  $SubOCs$  of  $AllOCs$ , selected independently of implementations. We can find an  $oc_1$  from this  $SubOCs$  such that the number  $k$  of  $f(\dots)$ 's in  $oc_1$  is maximal in  $SubOCs$  (where  $k \geq 0$ ). We can then construct an implementation  $\Psi$  that contains an error in the  $(m+k+1)$ th call of  $f(\dots)$  but correct otherwise. Let  $O_1$  and  $O_2$  be the objects resulting from executing the method sequences corresponding to  $u_1$  and  $u_2$ , respectively. For simplicity, we write  $O_1 = \Psi(u_1)$  and  $O_2 = \Psi(u_2)$ . Since  $O_1.oc_1 = \Psi(u_1).oc_1$  contains  $m+k$   $f(\dots)$ 's and  $O_2.oc_1 = \Psi(u_2).oc_1$  contains  $n+k$   $f(\dots)$ 's such that  $n < m$ , for any  $oc \in SubOCs$ , there must be no more than  $m+k$   $f(\dots)$ 's in  $O_1.oc$  or  $O_2.oc$ . Thus, according to the construction of implementation  $\Psi$ ,

$$O_1.oc \approx O_2.oc$$

for any  $oc$  in  $SubOCs$ . By Definition 2.5,  $oc_1$  must end with an observer. Let  $u$  be the result of removing the observer from  $oc_1$ . Consider  $u^* = u.f(\dots)$ . Obviously, since  $u^*$  contains  $k+1$   $f(\dots)$ 's,

$$\begin{aligned} O_1.u^* &= \Psi(u_1).u^* \text{ contains } m+k+1 \text{ } f(\dots) \text{'s, and} \\ O_2.u^* &= \Psi(u_2).u^* \text{ contains } n+k+1 \leq m+k \text{ } f(\dots) \text{'s,} \end{aligned}$$

according to the construction of implementation  $\Psi$ , we have  $\neg(O_1.u^* \approx O_2.u^*)$ . By Definition 2.7, there exists an observable context  $oc_2$  such that  $\neg(O_1.u^*.oc_2 \approx O_2.u^*.oc_2)$ . Let  $oc_0 = u^*.oc_2$ . It follows that

$$\neg(O_1.oc_0 \approx O_2.oc_0).$$

Since  $u^*$  contains  $k+1$   $f(\dots)$ 's,  $oc_0$  must contain at least  $k+1$   $f(\dots)$ 's, and hence  $oc_0 \notin SubOCs$  but  $oc_0 \in AllOCs$ . Thus we arrive at Formula II.  $\square$

In the above proof, we realize that the number  $m+k+1$  is closely related to this error. In order to reveal this error, we must catch the number  $m+k+1$ . Obviously, this can be done only by using a heuristic white-box technique, rather than a black-box technique. This is the reason why we propose to supplement our axiom-based black-box approach by the following heuristic white-box technique. (On the other hand, see also Section 3.3.2 that discusses why a white-box technique cannot be a substitute for the black-box technique.)

### 3.2 Relevant Observable Context Technique

The basic idea behind our heuristic technique is as follows. Suppose we want to determine whether  $O_1 \approx O_2$ . Suppose further that  $O_1$  and  $O_2$  have different values for the same data member  $d_i$  of the implemented class. Such different values may or may not have an effect on the observable attributes of  $O_1$  and  $O_2$ . If no observable attribute is affected,  $d_i$  need not be considered. If some observable attribute is affected,  $d_i$  must have affected the attribute through some series of methods in the implemented class. Such a series of methods is called a *relevant observable context*. We need only use the relevant observable contexts to determine whether  $O_1 \approx O_2$ . We can ignore any other observable contexts for this decision. We shall give a formal definition for the concept of relevant observable context, and how it can be produced by means of a Data member Relevance Graph constructed from the implemented class  $C$ .

If a relevant observable context itself contains an implementation error, by applying it to determine the observational equivalence of objects, we may increase the chance of having the error revealed. If, after applying the relevant observable context to objects  $O_1$  and  $O_2$ , we find inconsistencies in some observable attribute, we can conclude an implementation error in  $s_1$  or  $s_2$ , or in the relevant observable context itself, or both. The worst case scenario happens when the error(s) in  $s_1$  and  $s_2$  offset the error(s) in the relevant observable context, so that neither can be revealed. We note, however, that the possible offsetting of errors cannot be avoided in testing. Even if we test a single method sequence, errors in two of the methods may happen to cancel each other. This is the well-known phenomenon of fault masking (see, for example, Morell [1990]).

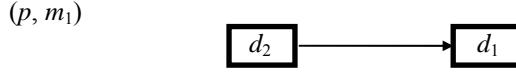
In the relevant observable context technique, we assume a program model without pointers. This kind of program model is gaining popularity in the latest object-oriented programming languages such as JAVA.

**Definition 3.1** If a data member  $d_1$  is defined or revised by a data member  $d_2$  in a method  $m$  under a condition  $p(\dots, d_3, \dots)$ , we say that  $d_2$  *directly affects*  $d_1$  in  $m$  under  $p(\dots)$ , and that  $d_3$  *directly affects*  $d_1$  in  $m$  under  $p(\dots)$ .

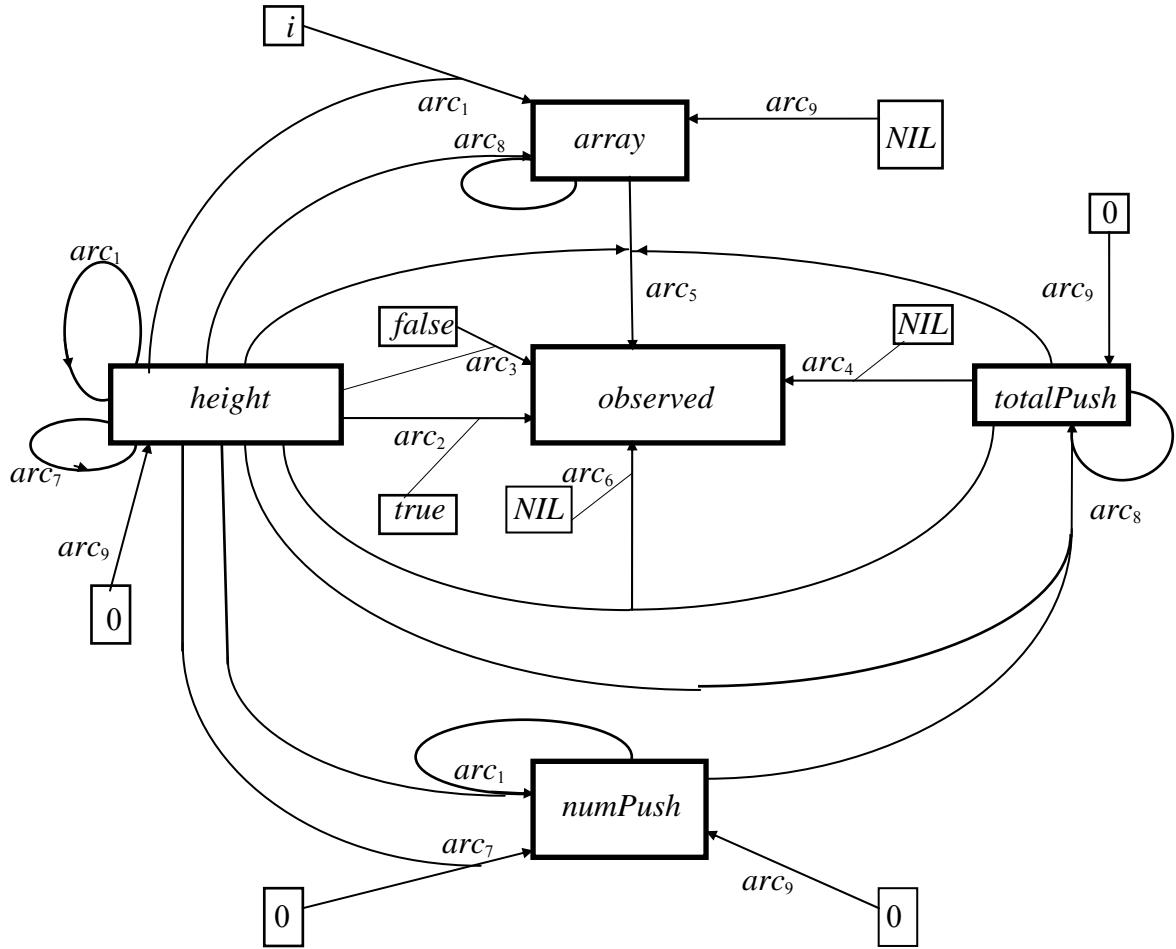
In Example 5 below, for instance, we say that the data member *numPush* directly affects the data member *totalPush* in the method *incTop* under the condition *height > 0*.

**Definition 3.2** Given an implemented class  $C$ , its *Data member Relevance Graph* (DRG) is constructed as follows. Each data member of  $C$  is represented as a bold rectangle *node* in the DRG. The DRG also contains some thin rectangle *nodes*, which denote some constants coming from the given program. If the data member  $d_2$  directly affects the data member  $d_1$  in the method  $m_1$  under a condition  $p(\dots)$ , then there is an *arc*, labeled by  $(p, m_1)$ , from  $d_2$  to  $d_1$ . (See Figure 2.) We call  $[d_2, (p, m_1), d_1]$  a *segment* of the DRG,  $d_2$  a *start node* of arc  $(p, m_1)$ ,  $d_1$  an *end node* of arc  $(p, m_1)$ ,  $(p, m_1)$  an *output arc* of  $d_2$ , and  $(p, m_1)$  an *input arc* of  $d_1$ . If  $d_2$  is identical to  $d_1$ , the segment is said to be a *cycle*. Otherwise it is said to be *acyclic*. Each DRG contains a special *node* called *observed*, which is the ending node of each

arc with an observer as the second component of its label. An arc with *observed* as an ending node is called an *observer arc*. An example of a DRG is given in Figure 3.



**Figure 2.** Nodes and arc in a DRG



- $arc_1: (height \neq size, push(i));$
- $arc_2: (height = 0, empty);$
- $arc_3: (height \neq 0, empty);$
- $arc_4: (totalPush = 3, top);$
- $arc_5: (totalPush \neq 3 \wedge height > 0, top);$
- $arc_6: (totalPush \neq 3 \wedge height = 0, top);$
- $arc_7: (height > 0, pop);$
- $arc_8: (height > 0, incTop);$
- $arc_9: (true, newStack);$

**Figure 3.** DRG of integer stacks

**Definition 3.3** Suppose  $d_1$  is a data member of an implemented class  $C$ ,  $O_1$  and  $O_2$  are two given objects of  $C$ . If

- (1)  $O_1.d_1 \neq O_2.d_1$ ,
- (2) there is a path  $P$  from the node  $d_1$  to the node *observed* in the DRG of class  $C$ , and

(3) the methods in the labels of the arcs in path  $P$  are  $op_1, op_2, \dots, op_t, obs$  successively, then we call  $op_1.op_2...op_t.obs$  a *relevant observable context* induced from path  $P$  with respect to  $O_1$  and  $O_2$ , and say that  $d_1$  *affects* the observable attribute of  $O_1$  and  $O_2$ . Notice that, the concept of *paths* in this paper is the same as that in directed graphs in general, except they end at a special node *observed*.

**Definition 3.4** Let  $O_1$  be an object of the implemented class  $C$ . Suppose the data members of class  $C$  are  $d_1, d_2, \dots, d_n$ . In the DRG of class  $C$ , if all the conditions in the labels of the arcs in a given path  $P$  are satisfied by  $O_1.d_i$  as initial data, then the path  $P$  is said to be *executable for  $O_1$* . Otherwise  $P$  is said to be *unexecutable for  $O_1$* .

**Algorithm DOE (Determining Observational Equivalent)** Suppose  $O_1$  and  $O_2$  are two objects of the same implemented class, resulting from the execution of the method sequences  $s_1$  and  $s_2$ , respectively. The steps for deciding whether  $O_1 \approx O_2$  by means of relevant observable contexts are as follows:

- (a) If  $s_1$  and  $s_2$  end with an observer, then  $O_1$  and  $O_2$  are values of some import class. We can therefore directly decide whether  $O_1 \approx O_2$  in the import class.
- (b) Otherwise, suppose  $O_1$  and  $O_2$  belong to the implemented class  $C$ . Construct the Data member Relevance Graph of class  $C$  from the coding of class  $C$ .
- (c) Suppose the data members of the implemented class  $C$  are  $d_1, d_2, \dots, d_n$ . In general, the classes of  $d_i$  are imported, observable, and there are known methods to determine the equivalence of values of the classes. Suppose further that  $O_t.d_i$  denotes the value of  $d_i$  of  $O_t$  for  $i = 1, 2, \dots, n$  and  $t = 1, 2$ .

Check whether the tuples  $(O_1.d_1, O_1.d_2, \dots, O_1.d_n)$  and  $(O_2.d_1, O_2.d_2, \dots, O_2.d_n)$  are equal. If yes, we have  $O_1 \approx O_2$ , and exit from Algorithm DOE. Otherwise proceed to step (d).

- (d) Suppose  $O_1$  and  $O_2$  have different values with respect to the data members  $d_{x1}, d_{x2}, \dots, d_{xk}$ , where  $1 \leq x_1 < x_2 < \dots < x_k \leq n$ . In other words, suppose  $O_1.d_{xj} \neq O_2.d_{xj}$  for  $j = 1, 2, \dots, k$ .

For each  $d_{xj}$ , check whether there is a path from the node  $d_{xj}$  to the node *observed* in the DRG. If not, skip this  $d_{xj}$ . If yes, proceed as follows:

- (1) If  $d_{xj}$  is a simple data type, traverse every acyclic executable path  $P$  once (using the original  $O_1.d_i$  as **initial data** and **backtracking** if necessary) and obtain the relevant observable context  $oc$  induced from  $P$ . If there are uninstantiated input variables in  $oc$ , apply the PDP technique to select values for the input variables.

If a cycle  $l_i$  is encountered when traversing an executable path for  $O_1$ , the user should manually decide on a ceiling  $t_i$  for the number of iterations of  $l_i$ , or supply a global ceiling  $T$  allowed by the system.

Check whether at least one of these relevant observable contexts, say  $oc_0$ , fails, that is,  $\neg(O_1.oc_0 \approx O_2.oc_0)$ . If so, we have  $\neg(O_1 \approx O_2)$ , and exit from Algorithm DOE. Otherwise we say this  $d_{xj}$  has successfully passed the check, and proceed to step (3).

- (2) If  $d_{xj}$  is a compound data type (such as an array or structure in C++), construct relevant observable contexts by the following process: For each value  $V$  of the component index or element variable of  $d_{xj}$  that satisfies  $O_1.d_{xj}.V \neq O_2.d_{xj}.V$ , select every method sequence  $msEl$  to change the current value of the component index or element variable to  $V$ , then traverse each executable path (not exceeding the iteration ceilings in the case of cycles, if any) for  $O_1$  from the node  $d_{xj}$  to the node *observed* to obtain method sequence  $msOb$ , and create a relevant observable context  $oc = msEl.msOb$ . (If  $V$  is already the current value of the component index or the current element variable, then  $msEl$  is empty.) The process of traversing each executable path (not exceeding the iteration ceilings in the case of cycles, if any) for  $O_1$  in this step is the same as that in step (1). If  $oc$  contains uninstantiated input variables, the PDP technique is applied to determine values for the input variables. If at least one of these relevant observable contexts, say  $oc_0$ , fails, then we have  $\neg(O_1 \approx O_2)$ , and exit from Algorithm DOE. Otherwise we say this  $d_{xj}$  has successfully passed the check, and proceed to step (3).
- (3) If all the  $d_{xj}$  have successfully passed the checks, then we have  $O_1 \approx O_2$ , and exit from Algorithm DOE. Otherwise continue to check the next  $d_{xj}$  such that  $O_1.d_{xj} \neq O_2.d_{xj}$ .  $\square$

**Example 5** The specification is the same as Example 1, except for following additional operations and axioms:

**operations**

...  
 $_incTop: IntStack \rightarrow IntStack$

**axioms**

...  
 $a_7: S.incTop.top = \begin{cases} S.empty & \text{then } S.top \\ & \text{else } S.top + 1 \end{cases}$

In the following implementation, the internal data member *numPush* is used to count the number of continuous calls to *\_push*, and *totalPush* is applied to record the total number of calls to *\_push*. For the sake of illustration, we have embedded some errors in the implementation.

```
#include <iostream.h>

#define SIZE 100
#define NIL 0

enum bool { false, true };
```

```

class intStack {
    /* intStack consists of 4 data members: */
    int array[SIZE];
    int height;
    int numPush;
    int totalPush;
public:
    void newStack( );
    bool empty( );
    void push(int i);
    void pop( );
    void incTop( );
    int top( );
};

void intStack :: newStack( )
{
    height = 0;
    numPush = 0;
    totalPush = 0;
    for ( int j = 1; j <= 100; j++ )
        array[j] = NIL;
}

bool intStack :: empty( )
{
    if (height == 0) return true;
    else return false;
}

void intStack :: push(int i)
{
    if (height == SIZE)
        cout << "Stack is full";
    else {
        height = height + 1;
        array[height] = i;
        numPush = numPush + 1;
    }
}

void intStack :: pop( )
{
    if (height > 0) {
        height = height - 1;
        numPush = 0;
    }
}

```

```

void intStack :: incTop( )
{
    if (height > 0) {
        array[height] = array[height] + 1;
        totalPush = totalPush + numPush; /* Error 1: This statement should be in the
                                         method pop but has been placed here by
                                         mistake. */
    }
}

int intStack :: top( )
{
    if (totalPush == 3) return NIL; /* Error 2: The condition should be totalPush == 0. */
    else {
        if (height > 0) return array[height];
        else return NIL;
    }
}

```

The fundamental pair  $u_1 = \text{new.push}(1).\text{push}(2).\text{pop}$  and  $u_2 = \text{new.push}(1)$  can be induced from axiom  $a_4$ . Let us denote their corresponding implemented method sequences as  $s_1$  and  $s_2$ , respectively. Suppose the execution results of  $s_1$  and  $s_2$  are objects  $O_1$  and  $O_2$ , respectively. We would like to illustrate how to use Algorithm DOE to determine whether  $O_1 \approx O_2$ .

- (a) Since the sequences  $s_1$  and  $s_2$  do not end with an observer, proceed to step (b).
- (b) Construct the Data member Relevance Graph of class  $C$  from the coding of class  $C$ . The DRG is shown in Figure 3.
- (c) The execution results are:

$$\begin{aligned}
& (\text{array}, \text{height}, \text{numPush}, \text{totalPush}) \\
O_1 &= ([1, 2], 1, 0, 0), \\
O_2 &= ([1, NIL], 1, 1, 0).
\end{aligned}$$

Check whether the tuples  $([1, 2], 1, 0, 0)$  and  $([1, NIL], 1, 1, 0)$  are equal. The answer is no.

- (d)  $O_1$  and  $O_2$  have different values on the data members  $array$  and  $numPush$ :

- (1) For  $array$ , we follow step (d)(2) of Algorithm DOE. Here, the component index is  $height$ . The current values of both  $O_1.height$  and  $O_2.height$  are 1, whereas the value of  $height$ , which satisfies  $O_1.array[height] \neq O_2.array[height]$ , is 2. From the cycle  $height$ - $arc_1$ - $height$  in Figure 3, we see that the only method sequence  $msEl_1$  which changes the value of  $height$  from 1 to 2 is  $push(i)$ . On the other hand, by traversing the executable paths for  $O_1$  from the node  $array$  to the node  $observed$  in Figure 3, we obtain the method sequences  $msOb_1 = \text{top}$  and  $msOb_2 = \text{incTop.top}$ , corresponding to the paths  $array$ - $arc_5$ - $observed$  and  $array$ - $arc_8$ - $array$ - $arc_5$ - $observed$ , respectively. Thus, by concatenating  $msEl_1$  with  $msOb_1$  and

$msOb_2$ , respectively, we obtain the relevant observable contexts  $push(i).top$  and  $push(i).incTop.top$ . Then apply the PDP technique<sup>9</sup> to determine  $i = 8$ , and check whether

$$\begin{aligned} O_1.push(8).top &= O_2.push(8).top \text{ and} \\ O_1.push(8).incTop.top &= O_2.push(8).incTop.top \end{aligned}$$

They are both successful.

- (2) For  $numPush$ , since it is a simple data type, we follow step (d)(1) of Algorithm DOE to traverse every executable path for  $O_1$  by backtracking. Suppose the global ceiling  $T$  supplied by the user for the number of iterations of cycles is 2.
  - (i) In Figure 3, the node  $numPush$  has two output arcs  $arc_8$  and  $arc_1$  (where  $arc_1$  is a cycle). All the conditions of  $arc_8$  and  $arc_1$  are satisfied by  $O_1.height = 1$ . Let us consider  $arc_8$  first. Its label contains the method  $incTop$ . Execute the method and obtain  $O_1.incTop = ([2, 2], 1, 0, 0)$ . The end node of  $arc_8$  is  $totalPush$ , which has four output arcs  $arc_4$ ,  $arc_5$ ,  $arc_6$ , and  $arc_8$  (where  $arc_8$  is a cycle). The current state,  $O_1.incTop = ([2, 2], 1, 0, 0)$ , satisfies the conditions of  $arc_5$  and  $arc_8$  but not those of  $arc_4$  and  $arc_6$ . Hence,  $arc_5$  and  $arc_8$  are executable for the current state but  $arc_4$  and  $arc_6$  are unexecutable. Consider  $arc_5$  first. The label of  $arc_5$  contains the method  $top$ . Execute the method and obtain  $(O_1.incTop).top = 2$ . The end node of  $arc_5$  is the node  $observed$ . Thus, we obtain an executable path  $p_1 = numPush-arc_8-totalPush-arc_5-observed$  for the given object  $O_1$ . The relevant observable context corresponding to path  $p_1$  is  $oc_1 = incTop.top$ . Execute  $O_2.oc_1$  and obtain  $O_2.oc_1 = 2$ . Hence,  $oc_1$  succeeds because  $O_1.oc_1 = O_2.oc_1$ .
  - (ii) To obtain the other executable path for  $O_1$ , backtrack to the node  $totalPush$  and consider the other executable output arc  $arc_8$  (which is a cycle). The label of  $arc_8$  contains the method  $incTop$ . Execute the method and obtain  $(O_1.incTop).incTop = ([2, 2], 1, 0, 0).incTop = ([3, 2], 1, 0, 0)$ . The end node of  $arc_8$  is  $totalPush$ , which has four output arcs  $arc_4$ ,  $arc_5$ ,  $arc_6$ , and  $arc_8$ . The current state,  $O_1.incTop.incTop = ([3, 2], 1, 0, 0)$ , satisfies the conditions of  $arc_5$  and  $arc_8$  but not those of  $arc_4$  and  $arc_6$ . Hence,  $arc_5$  and  $arc_8$  are executable for the current state but  $arc_4$  and  $arc_6$  are unexecutable. Consider  $arc_5$  first. The label of  $arc_5$  contains the method  $top$ . Execute the method and obtain  $(O_1.incTop.incTop).top = 3$ . The end node of  $arc_5$  is the node  $observed$ . Thus, we obtain an executable path  $p_2 = numPush-arc_8-totalPush-(arc_8-totalPush)^1-arc_5-observed$  for the given object  $O_1$ . The relevant observable context corresponding to path  $p_2$  is  $oc_2 = incTop.incTop.top$ . Execute  $O_2.oc_2$  and obtain  $O_2.oc_2 = 3$ . Hence,  $oc_2$  also succeeds because  $O_1.oc_2 = O_2.oc_2$ .
  - (iii) Similarly to step (ii), in order to obtain the other executable path for  $O_1$ , backtrack to the node  $totalPush$ , and consider the other executable output arc  $arc_8$  (which is a cycle). The label of  $arc_8$  contains the method  $incTop$ . Execute the method and obtain  $(O_1.incTop.incTop).incTop = ([4, 2], 1, 0, 0)$ . The end node of  $arc_8$  is  $totalPush$ , which has four output arcs  $arc_4$ ,  $arc_5$ ,  $arc_6$ , and  $arc_8$ . The current state,  $O_1.incTop = ([4, 2], 1, 0, 0)$ , satisfies the conditions of  $arc_5$  and  $arc_8$  but not those of  $arc_4$  and  $arc_6$ . Hence,  $arc_5$  and  $arc_8$  are executable for the current state but  $arc_4$  and  $arc_6$  are unexecutable. Consider  $arc_5$  first.

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<sup>9</sup> Here, as a special case, the partition only contains a unique subdomain.

The label of  $arc_5$  contains the method  $top$ . Execute the method and obtain  $(O_1.incTop.incTop.incTop).top = 4$ . The end node of  $arc_5$  is the node  $observed$ . Thus, we obtain an executable path  $p_3 = numPush-arc_8-totalPush-(arc_8-totalPush)^2-arc_5-observed$  for the given object  $O_1$ . The relevant observable context corresponding to path  $p_3$  is  $oc_3 = incTop.incTop.incTop.top$ . Execute  $O_2.oc_3$  and obtain  $O_2.oc_3 = ([1, NIL], 1, 1, 0).incTop.incTop.incTop.top = NIL$ . However, as evaluated above,  $O_1.oc_3 = O_1.incTop.incTop.incTop.top = 4$ . Hence,  $oc_3$  fails because  $O_1.oc_3 \neq O_2.oc_3$ . Report  $\neg(O_1 \approx O_2)$ . Then exit from Algorithm DOE.  $\square$

### 3.3 Discussions on Algorithm DOE

We discuss in this section a number of important issues on Algorithm DOE including effectiveness, limitations, and complexity.

#### 3.3.1 Effectiveness (1): Skipping Irrelevant Observable Contexts

By adopting Algorithms DOE, we can skip the testing of many irrelevant cases. Referring to step (d) in Example 5, none of the method sequences of the form  $push(i_1).push(i_2)^j.top$  or  $push(i)^j.pop^k.incTop.top$  (where  $j, k = 1, 2, \dots$ ) are relevant observable contexts with respect to  $O_1$  and  $O_2$ . Hence, we need not consider them. In fact, none of them reveals the error.

#### 3.3.2 Effectiveness (2): Overcoming the “Missing Path” Problem

A common drawback of white-box techniques is the failure to detect “missing paths”, which are parts of the specification omitted from the implementation. However, even though Algorithm DOE is a white-box technique by itself, it can help to expose some of the missing paths when integrated with a black-box technique, such as our axiom-based approach to generate fundamental pairs as test cases. This is the main idea behind our proposal to integrate black- and white-box techniques in program testing. In Example 5, for instance, suppose the branch “*if (height>0) return array[height]*” is missing from the code of the method  $top()$ . Then the path “*totalPush-arc<sub>5</sub>-observed*” in Figure 3 will be missing. The originally selected fundamental pair  $u_1 = new.push(1).push(2).pop$  and  $u_2 = new.push(1)$  (see the paragraph before step (a) of Example 5) cannot reveal this error, since  $O_1.oc_3 = NIL = O_2.oc_3$  (see step (iii) of Example 5). However, following Algorithm GFT in our axiom-based approach, this error will be exposed by another fundamental pair  $new.push(8).top \sim 8$  induced from axiom  $a_6$  in Example 1, since  $new.push(8).top = NIL \neq 8$ .

#### 3.3.3 Limitation (1): Infinite Cycles

If a DRG contains cycles, the set of relevant observable contexts is infinite. We can, however, only choose a finite subset as test cases. Thus some program faults may remain undetected. This is an inherent limitation of program testing. To select such a finite subset, step (d) of Algorithm DOE uses a positive integer  $t_i$  or  $T$  to control the number of iterations of the cycles. The determination of  $t_i$  or  $T$  remains a difficult problem. In the current phase, these integers are supplied manually by user. Alternatively, we may consider the feasibility of adding further heuristics to the algorithm. For instance, in step (d)(2) of Example 5, we may find that the required number  $t_i$  of iterations of the cycle  $l_i = -totalPush-(arc_8-totalPush)^{t_i}$  is closely related with the number 3 in the branch condition of the method  $top$ , which can be identified in the labels of the output arcs of the node  $totalPush$ .

### 3.3.4 Limitation (2): Fault Masking

A new concern may be raised on our relevant observable context technique. If an observable context  $oc$  itself contains an error, can we determine whether  $O_1 \approx O_2$ ? Let  $u_1 \sim u_2$  be two equivalent ground terms and  $s_1$  and  $s_2$  be their corresponding method sequences in an implementation. There are four possible cases:

- (a) There exists some error in  $s_1$  or  $s_2$  such that  $\neg(O_1 \approx O_2)$ :
  - (1) The error in  $oc$  does not affect the decision whether  $O_1 \approx O_2$ . In this case, our procedure finds that  $\neg(O_1 \approx O_2)$  and reports an error.
  - (2) The error in  $oc$  causes an erroneous decision on the observational equivalence of  $O_1$  and  $O_2$ . In this case, our procedure finds that  $O_1 \approx O_2$  and does not report any error.
- (b) There is no error in  $s_1$  and  $s_2$ , and hence we should have  $O_1 \approx O_2$ :
  - (1) The error in  $oc$  does not affect the decision whether  $O_1 \approx O_2$ . In this case, our procedure finds that  $O_1 \approx O_2$  and does not report any error.
  - (2) The error in  $oc$  causes an erroneous decision whether  $O_1 \approx O_2$ . In this case, our procedure finds that  $\neg(O_1 \approx O_2)$  and reports an error. In spite of the erroneous decision, the error report is actually correct because there is an implementation error in  $oc$ .

It is well-known that program testing does not necessarily guarantee correctness [30, 31]. It is generally considered *acceptable* that a test may not reveal all the errors in an implementation. If a test reports an implementation error, we say that the test is *useful*. It would be *unacceptable*, however, if a test reports an error that does not exist in an implementation.

In the above, the cases (a)(1) and (b)(2) are useful, while the cases (a)(2) and (b)(1) are acceptable. Hence, our approach does not produce unacceptable cases.

### 3.3.5 Size of DRG

The size of a DRG can be represented by a tuple  $(N, S)$ , where  $N$  is the number of nodes in the DRG, and  $S$  is the number of segments. If the corresponding implementation contains  $D$  data members and  $M$  methods, and  $P$  is the maximum number of conditions in each method, then  $N = D+1$ , and  $S \leq D^2 \times M \times P$ . In the worst case, “*directly affects*” is a universal relation, which corresponds to  $S = D^2 \times M \times P$ . In fact, this worst case very seldomly occurs, if ever. We expect the DRG to be rather simple in most practical situations, since the DRG models the class level, which is a relatively low level in an object-oriented system. The number of nodes in the DRG of a class, equivalent to the number of data members in a given concrete class, is usually small, and “*directly affects*” is generally far from a universal relation. For conventional programming, many authors have supplied statistical data to show that simple program structures are used more often than complex structures [25, 32, 33, 34, 35]. Since the class level is relatively low in an object-oriented system, the situation is very similar. For example, we have analyzed statistically the source code of one of our projects entitled **FOOD** (Functional Object-Oriented Design) [36]. We have reviewed 16 classes and found that the average numbers of the data members and methods in each class were 6 and 8, respectively. We have also examined 21 classes in another case study on bank accounts and found that the average numbers of data members and methods in each class were 4 and 7, respectively.

### 3.3.6 Executability of a Given Path for a Given Object

Note that the concept of executability of a given path **for a given object** defined in Definition 3.4 is very different from the concept of feasibility of a path in other flow graph techniques [25]. An *infeasible path* is normally defined as a path whose conditions cannot be satisfied by **any** input value, and is well-known to be undecidable. However, since executable and unexecutable paths defined in Definition 3.4 **are related to some object**  $O_1$ , they can be determined from the known values  $O_1.d_i$ , that is, the values of the data members of the given object  $O_1$ . Thus, unlike the concept of feasibility, the executability of a given path for a given object as defined in this paper is decidable.

### 3.3.7 Complexity of Traversing Executable Paths

Referring to Algorithm DOE and Example 5, let  $L$  be the maximum length of all acyclic paths from any node to the node *observed*. Let  $n$  be the maximum number of Boolean conditions in the output arcs of any node that are *true* for the current values of  $O_1.d_i$  and  $O_2.d_i$ . Let  $T$  be the ceiling supplied by the user for the number of iterations of cycles. Since the maximum number of selective branches at any node in a path is  $n(T + 1)$ , and the longest path contains  $L$  nodes, the maximum number of executable paths is  $(n(T + 1))^L$ . We note that, for a given DRG of the class under test,  $n$  is a variable according to the different objects  $O_1$  and  $O_2$ , but  $L$  and  $T$  are constants. Hence, the complexity of traversing executable paths is  $O(n^L)$ , in the worst case.

Furthermore, by the same reasoning as that of Section 3.3.5, we do not expect the constant  $L$  to be large in most practical situations.

## 3.4 An Implementation of Algorithm DOE

As indicated in Section 2.5.4, Algorithm GFT is analogous to that used in the tools described in [9]. Their implementations are also similar, so that we have not included it in the present phase of our prototyping study. Instead, we have focused our attention on the implementation of Algorithm DOE, and implemented a prototype on a Pentium/120. In summary, it is a reformed C++ interpreter, constructed by embedding Algorithm DOE into a C++ interpreter. The prototype consists of five modules: *parser.c*, *drg.c*, *pigeonC.h*, *subLib.c*, and *pigeonC.c*. The modules *pigeonC.h* and *subLib.c* contain the definitions of the main data structures and the interfaces to internal library functions, respectively. The module *parser.c* includes a lexical analyzer and a recursive descent parser. It also performs the initialization for *drg.c* and other modules. The module *drg.c* constructs the DRG, traverses executable paths by backtracking, and generates and executes the corresponding relevant observable contexts for two given objects. Finally, *pigeonC.c* serves as the main module of the prototype. It reads the C++ program code for a given class under test, allocates memory for the program, prescans it, and calls and coordinates other modules to perform the task.

Note that Algorithm DOE as specified in this paper shows only an abstract summary. It is, in fact, refined into several sub-algorithms that call many other functions, as described in Chen et al. [37]. Readers may find the following additional notes useful:

### 3.4.1 Pigeon C++

The prototype has been implemented using Borland C++. It requires the program for a given class under test is written in a subset of C++ language. We call this subset *Pigeon C++*, which is an extension

of *Little C* [38]. In the present phase, *Pigeon C++* contains the following features of *C++* language: parameterized or non-parameterized functions with local variables; recursion; *if* statements; *do-while*, *while*, and *for* statements; *return* statements; integer, character, and array variables; instance variables of classes; global variables; string constants; some standard library functions; member functions of classes;  $+$ ,  $-$ ,  $*$ ,  $/$ ,  $\%$ ,  $<$ ,  $>$ ,  $<=$ ,  $>=$ ,  $==$ ,  $!=$ , unary  $-$ , unary  $+$ ; and comments. As a limitation, *Pigeon C++* does not contain pointers in the present phase. The implementation of the relevant observable context technique with pointers is much more complex and needs further investigation.

### 3.4.2 Construction and Traversal of a DRG

Suppose an implemented class contains  $k$  methods  $m_1, m_2, \dots, m_k$ . Let  $d_1$  and  $d_2$  be two data members in the implemented class. The following is a schematic summary of the tasks required for each method  $m_i$  in the construction of the DRG:

- (1) Scan the code of  $m_i$ .
- (2) Suppose  $c$  denotes a constant and  $p$  denotes a predicate. When a statement of the form “ $d_1 = c$ ” or “ $d_1 = f(\dots, d_2, \dots)$ ” is found, put the *arc\_label* (*true*,  $m_i$ ) into the table of arc labels, and put the segment [ $c$  or index of  $d_2$ , index of *arc\_label* (*true*,  $m_i$ ), index of  $d_1$ ] into the list of output arcs. When a statement such as “*if* ( $p$ ) { $\dots$ ;  $d_1 = c$  or  $f(\dots, d_2, \dots)$ ;  $\dots$ }” is found, put the *arc\_label* ( $p$ ,  $m_i$ ) into the table of arc labels, and put the segment [ $c$  or index of  $d_2$ , index of *arc\_label* ( $p$ ,  $m_i$ ), index of  $d_1$ ] into the list of output arcs. If  $p = p(\dots, d_3, \dots)$ ,  $d_3$  is a data member different from  $d_2$ , we should also put the segment [index of  $d_3$ , index of *arc\_label* ( $p$ ,  $m_i$ ), index of  $d_1$ ] into the list of output arcs. If the statement also contain “*else* { $\dots$ ;  $d_4 = c_0$  or  $g(\dots, d_5, \dots)$ ;  $\dots$ }”, we must also put the *arc\_label* ( $\neg p$ ,  $m_i$ ) into the table of arc labels, and put the segment [ $c_0$  or index of  $d_5$ , index of *arc\_label* ( $\neg p$ ,  $m_i$ ), index of  $d_4$ ] into the list of output arcs.

For every such statement, the time for performing this task is bounded. Assume that it is no more than  $T_1$ . If the method  $m_i$  contains  $s_{i1}$  such statements, the time for handling these statements is no more than  $T_1 \cdot s_{i1}$ .

- (3) Skip the other statements in the method  $m_i$ . The time for skipping such a statement is also bounded. Assume that it is no more than  $T_2$ . Since the method  $m_i$  contains  $(s_i - s_{i1})$  such statements, the time for handling these statements is no more than  $T_2(s_i - s_{i1})$ .

Thus, the time  $t$  for constructing the DRG of the class satisfies the following formula:

$$t \leq \sum_{i=1, \dots, k} (T_1 \cdot s_{i1} + T_2(s_i - s_{i1})) \leq \sum_{i=1, \dots, k} (T \cdot s_{i1} + T(s_i - s_{i1})) = \sum_{i=1, \dots, k} (T \cdot s_i) = T \cdot s$$

where  $T = \max \{T_1, T_2\}$  and  $s = s_1 + s_2 + \dots + s_k$ . Hence, the time  $t$  for constructing the DRG is  $O(s)$ .

When traversing the executable paths, if backtracking is necessary, the algorithm traverses the observer arcs first, followed by the acyclic output arcs, and finally the output arcs for cycles.

Further implementation details, such as the internal representations and the actual procedures for the construction and traversal of the DRG, can be found in Chen et al. [37].

### 3.4.3 Interactions with Users

The prototype is a semi-automatic tool. In the present phase, it requires the users to supply the following information manually:

- (1) Two equivalent method sequences corresponding to a selected fundamental pair of equivalent ground terms for the given class under test.
- (2) A list of methods in the class that are observers.
- (3) In step (d) of Algorithm DOE, when there are uninstantiated input variables in the *oc* just obtained, we should apply the PDP technique to select values for the input variables. This selection may be semi-automatic, but is only manual at present.
- (4) In the traversal of an executable path in step (d), if a cycle  $l_i$  is encountered, the user should supply a ceiling  $t_i$  for the number of iterations of  $l_i$ , or determine a global ceiling  $T$  allowed by the system for the number of iterations of any cycle.

If Algorithm GFT is implemented, (1) can be semi-automatic. Even then, we shall reserve the manual interface as a supplement.

### 3.4.4 Empirical Results

We have experimented with Examples 4 and 5 on the prototype. The experimental result of Example 5 (on *Pentium/120*), as shown in the following table, is the same as predicted.

Global ceiling supplied by the user for the number of iterations of any cycle	Number of observable contexts generated by the prototype	Number of error-revealing observable contexts	Total run time for all observable contexts	Run time for the first observable context that reports the error
0	2	0	0.093407 s	—
1	6	0	0.164835 s	—
2	12	3	0.283516 s	0.108791 s
3	20	5	0.437363 s	0.107692 s
4	30	5	0.634066 s	0.129670 s

Here, the run time includes the time for generating the two objects under test, traversing the executable paths for the objects in the DRG, constructing observable contexts from the executable paths, executing the observable contexts, and reporting the detected error, if any. However, it does not include the time for constructing the DRG. In Example 5, the time for constructing the DRG is 0.043297 seconds.

Suppose that the user indicates a global ceiling of 3 iterations. From the above table, we know there are 20 observable contexts to be generated in total. The run time for executing all of them would be 0.437363 seconds. In fact, Algorithm DOE does not test all the 20 observable contexts. When the first observable context reporting an error is encountered, the algorithm will exit, ignoring other observable contexts. Hence, the actual run time is found to be 0.107692 seconds.

Some trouble was encountered in the experiment on Example 4, since “`#define SIZE 100`” was too large for generating test sets of reasonable sizes. The allowable maximum size of array in the experiment on this example is 50. After changing the *SIZE* to 50, the experiment succeeds in reporting the error, but the run time on *Pentium/120* is 13.571429 seconds.

We also wrote a correct *C++* program for the specification in Example 1, embedded common errors into the program, such as writing  $height > 0$  as  $height < 0$  or  $height > 2$ , and then experimented with them on the prototype. The experimental results showed that all such common errors could be exposed by the prototype. As an illustration, the empirical results for the erroneous implementation with  $height > 2$  are listed as follows:

Global ceiling supplied by the user for the number of iterations of any cycle	Number of observable contexts generated by the prototype	Number of error-revealing observable contexts	Total run time for all observable contexts	Run time for the first observable context that reports the error
0	5	3	0.092308 s	0.041758 s
1	14	6	0.226374 s	0.041758 s
2	24	7	0.404396 s	0.041758 s
3	34	7	0.602198 s	0.041758 s
4	44	7	0.836264 s	0.041758 s

The time for constructing the DRG of this example is 0.024835 seconds.

## 4. RELATED WORK

There are two ways to use algebraic specifications in software testing. One was originally presented by Jalote [39], and extended by Frankl and Doong [11, 22, 23]. The other was initially proposed by Gannon et al. [40], and extended by Gaudel and others [9, 10, 21].

The former considers the axioms as rewriting rules, suggests to choose test cases from all legal combinations of operations (or terms), and derives their equivalent terms by means of the rewriting rules. The latter selects test cases from “the set of ground instances of the axioms obtained by replacing each variable by all ground terms of the right sort” under well-defined hypotheses [10]. Our Theorem 2 reveals an essential relationship between these two approaches.

Our approach is motivated by the ASTOOT black-box approach of Frankl and Doong and the testing theory of Gaudel and others. For completeness, we shall also compare our approach with the white-box dataflow-based approach of Parrish and others.

## 4.1 The Work of Frankl and Doong

In general, there are a number of advantages in Frankl and Doong’s functional approach [11, 22, 23] to test object-oriented programs. Using algebraic specifications, it helps to solve the oracle problem. By taking sequences of operations as test cases, instead of individual operations, this approach does not depend on a predefined calling method but on a suite of messages passing among objects. This concept is especially suitable for object-oriented programming. It can support an integrated set of semi-automatic tools covering test case generation, test driver generation, test execution, and test checking.

Our approach hopefully inherits the above advantages. However, there are a few substantial distinctions between Doong and Frankl’s approach and ours:

- (1) Frankl and Doong define two terms  $u_1$  and  $u_2$  to be equivalent “if we can use the axioms as rewrite rules to transform  $u_1$  to  $u_2$ ” [11]. There is a problem in this definition. Consider, for example, two terms  $\text{new.push}(1).\text{push}(2).\text{pop}$  and  $\text{new.push}(3).\text{pop.push}(1)$  for the specification of the class of integer stacks in Example 1. They produce observationally equivalent results when implemented correctly. However, they cannot be transformed from one to the other by left-to-right rewriting rules, and hence are not equivalent according to this definition. As a result, Frankl and Doong’s approach cannot derive this kind of test cases. Our Definition 2.6 avoids this problem. However, as a supplement, Doong and Frankl’s approach allows the user to add manually generated test cases that may remedy such kinds of situations.
- (2) Doong and Frankl’s approach requires the user to select a finite number of original (operation) sequences from the set of terms, which is infinite in general. They offer the following tentative guidelines to guide the selection and generation of test cases: “Use (at least some) long original sequences, with a variety of relative frequencies of different constructors and transformers” and “If the specification has conditional axioms (with comparison operators), choose a variety of test cases for each original sequence, with various parameters chosen over a large range. Equivalently, choose a variety of different paths through the ADT tree arising from each original sequence.” [11] These guidelines are supported by two empirical case studies.

The selection domain of our “fundamental pair” strategy is much smaller than that of the set of equivalent ground terms, but the coverage of testing fundamental pairs remains the same. Using our strategy, two of Doong and Frankl’s tools, the compiler and simplifier, can be replaced by a generator that induces fundamental pairs as test cases directly from the two sides of each axiom. Our strategy is based on mathematical theorems, thus more precise.

- (3) Frankl and Doong give an “approximate check” [11] for object observational equivalence, known as the EQN method, consisting of two techniques. One produces a recursive version of  $\text{eqn}$  from specification. The other is at the “implementation level”. The former requires the analyst to supply a special axiom  $\text{eqn}$  to the specification of each class to define equivalence of two objects in the class. Different  $\text{eqn}$  axioms are attached to different classes. The approach also requires the designer and programmer to implement a special recursive method  $\text{eqn}$  for the respective  $\text{eqn}$  axiom in each class. If one of the attached axioms for  $\text{eqn}$ , or its implementation, is problematic, then the test report may show an error even if the original class is implemented correctly. Having said that, if we consider the  $\text{eqn}$  function to be a part of the class under test, the above situation is acceptable. The technique at the “implementation level” suggests to use white-box information about how the data type is represented and manipulated in the implementation. “For example, knowing that a

FIFO queue is represented as a linked list, one can traverse the two lists comparing the elements”. If the corresponding elements of the two lists are equal, we can indeed conclude that the two queues are observationally equivalent. If some corresponding elements of the two lists are not equal, however, we cannot immediately conclude that the two queues are observationally not equivalent, since the object-oriented paradigm allows encapsulation and the hiding of internal information. As discussed by Frankl and Doong, there are some advantages and disadvantages of the two techniques. As an option, our relevant observable context technique checks observational equivalence of objects using a different idea, which concentrates on checking relevant observable contexts only, skipping irrelevant observable contexts.

## 4.2 The Work of Gaudel, Bernot, Bouge, Choquet, Dauchy, Fribourg, and Marre

Bernot, Gaudel, and Marre [21] have proposed a general theory for software testing based on algebraic specifications. This theory includes several hypotheses such as a regularity hypothesis and a uniformity hypothesis for selecting test cases, and some oracle hypotheses to constrain the oracle problem. These hypotheses are important from a theoretical point of view, because they formalize common test practices and express the gap between testing and correctness proving. In our approach, we combine our strategy with the regularity hypothesis and the decomposition technique of uniform subdomains to select a finite set of fundamental pairs as test cases.

Furthermore, the oracle hypothesis and the related counterexample in [21] have inspired us to propose the relevant observable context technique.

An important distinction between our approach and the work of Gaudel and others is that the latter replaces all the variables in the axioms by ground terms according to the regularity hypothesis [10], whereas our approach replaces them by normal forms according to Theorem 2. The benefit of replacing variables by normal forms, rather than by general ground terms, has been described in Section 2.5.4.

## 4.3 The Work of Parrish, Borie, and Cordes

Parrish, Borie, and Cordes [15] proposed a white-box dataflow-based approach to testing classes. Their approach uses a *class graph*, which is a collection of  $\langle N, E, D, U, I \rangle$ , where  $N$  is the set of nodes,  $E$  is the set of edges. A node represents an operation. An edge from a node  $A$  to a node  $B$  means that it is permissible to invoke the operation  $B$  after the operation  $A$ .  $D$  denotes the set of *definitions of data*.  $U$  denotes the set of *uses of data*.  $I$  refers to the set of infeasible subpaths.  $N, E, D$ , and  $U$  are obtained from the class interface in the implementation. The purpose of introducing the concept of  $I$  is to allow us to eliminate sequences that are inappropriate or impossible to test. For this purpose, the authors set up a *weak class graph* and the corresponding *weak coverage criteria*, and added two further restrictions. They then proved that a minimum-length sequence of operations which satisfies *weak node coverage criteria*, *weak branch coverage criteria*, *weak definition coverage criteria*, or *weak use coverage criteria* could be found in polynomial time. Hence, the approach can be automated efficiently. However, as admitted by the authors, these weak criteria and the two additional restrictions substantially weaken the degree of testing demanded, and reduce the significance of their results.

We can compare Parrish’s class graph approach and our DRG approach as follows:

- (a) Both of these two approaches are white-box techniques.
- (b) The class graph approach only deals with syntax problems. However, the DRG approach is used to determine whether two given objects are observational equivalent, which is a semantics problem.
- (c) In a class graph, a node represents an operation and an edge  $(op_1, op_2)$  denotes that the concatenation of two operations,  $op_1 \cdot op_2$ , is legal in syntax. On the other hand, in a DRG, a node represents a data member and an arc  $(d_1, d_2)$  denotes that the data member  $d_1$  directly affects data member  $d_2$ .
- (d) In DRGs, the counterpart to the weak branch coverage criteria in the class graph approach ensures each cycle is traversed only once. In general, this is very insufficient for the purpose of deciding whether two given objects are observationally equivalent. See Example 5 and the table in Section 3.4.4.
- (e) As indicated by [15], the class graph approach can also be based on the syntax section of algebraic specifications. Hence, this approach can be considered as an optional technique to select normal forms in step (a) of Algorithm GFT without a choice on the positive integer  $k$ .

## 5. CONCLUSION

In this paper, we define a fundamental pair as a pair of equivalent ground terms formed by replacing all the variables on both sides of an axiom by normal forms. We prove that a complete implementation of a canonical specification is consistent with respect to all equivalent ground terms if and only if it is consistent with respect to all fundamental pairs. In other words, the testing coverage of fundamental pairs as good as that of all equivalent ground terms, and hence we need only concentrate on the testing of fundamental pairs. Our strategy is based on mathematical theorems. Combining this strategy with the regularity hypothesis and the decomposition technique of uniform subdomains, we construct an algorithm for selecting a finite set of fundamental pairs as test cases.

On the other hand, we prove that the observational equivalence of objects cannot be determined using a finite set of observable contexts derived from any black-box technique. Instead, we propose a relevant observable context approach, which is a heuristic white-box technique, and have implemented a prototype for it.

Many authors have indicated that program testing cannot thoroughly expose all the errors in the program under consideration [30, 31]. In this sense, testing in general is an incomplete and undecidable problem. Our approach cannot avoid this inherent limitation of testing. We decompose the testing problem into several sub-tasks, separate the decidable sub-tasks from the undecidable or difficult ones, and put them into a unified methodological framework via two algorithms. The undecidable or difficult sub-tasks are analyzed separately.

As future work, we shall investigate into the selection of nonequivalent terms as test cases, and the testing of interactions among groups of cooperating classes at the *cluster level*. We shall also consider the following problems: Is it feasible to abstract heuristic information from program code to alleviate the problems of deciding on the length of normal forms in step (a) of Algorithm GFT, and determining the number of iterations of cycles in step (d) of Algorithm DOE? How do we extend *Pigeon C++* and its implementation with the relevant observable context technique to include aliasing and pointers? How do

we use compiler techniques instead of interpreter techniques to improve the efficiency of the prototype? How can we make the prototype more practical and user-friendly?

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