POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS
ASSOCIATED WITH REAL FORMS

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Dedicated to Alan Weinstein on the occasion of his 60th birthday

ABSTRACT. For a complex semisimple Lie group $G$ and a real form $G_0$ we define a Poisson structure on the variety of Borel subgroups of $G$ with the property that all $G_0$-orbits in $X$ as well as all Bruhat cells (for a suitable choice of a Borel subgroup of $G$) are Poisson submanifolds. In particular, we show that every non-empty intersection of a $G_0$-orbit and a Bruhat cell is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

1. Introduction

Let $G$ be a connected and simply-connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $X$ be the variety of Borel subalgebras of $\mathfrak{g}$. In this paper we use a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ to define a Poisson structure on $X$. This Poisson structure depends on a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Instead of dealing with each real form individually, we fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Then, as is shown in [6], a real form $\mathfrak{g}_v$ of $\mathfrak{g}$ can be constructed from each Vogan diagram $v$ for $\mathfrak{g}$ such that $\mathfrak{g}_v \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_v$. The corresponding Poisson structure on $X$ is denoted by $\Pi_v$.

Let $G_v$ be the real form of $G$ corresponding to $\mathfrak{g}_v$, and let $B$ be the Borel subgroup of $G$ with Lie algebra $\mathfrak{b}$. The Poisson structure $\Pi_v$ has the property that each $G_v$-orbit as well as each $B$-orbit in $X$ is a Poisson submanifold. The $B$-orbits in $X$ will be referred to as the Bruhat cells. We compute the rank of $\Pi_v$. In particular, if a $G_v$-orbit $\mathcal{O}$ meets a Bruhat cell $\mathcal{C}$, they intersect transversally, and we find that all the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$ have the same dimension, so $\mathcal{O} \cap \mathcal{C}$ is a regular Poisson manifold. Moreover, we show that all symplectic leaves in each connected component of $\mathcal{O} \cap \mathcal{C}$ are translates of each other by elements of a Cartan subgroup of $G_v$. We also show that the $G_v$-invariant Poisson cohomology for each open $G_v$-orbit in $X$ is isomorphic to the de Rham cohomology of $X$.

Results similar to those presented here for the full flag manifold $X = G/B$ are also valid for a partial flag manifold $G/P$, where $P$ is a parabolic subgroup of $G$. 

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$G$ containing $B$. We will treat these more general cases as well as some further properties of $\Pi_c$ in a future paper.

Throughout this paper, if $V$ is a set and $\sigma$ is an involution on $V$, we will use $V^\sigma$ to denote the fixed point set of $\sigma$ in $V$.

2. Real forms of $g$ and Vogan diagrams

Let $g$ be a complex simple Lie algebra. In this section we recall the classification of real forms of $g$ by Vogan diagrams. Details can be found in [6, Chapter 6].

Suppose that $g_0$ is a real form of $g$ and that $\tau_0$ is the corresponding complex-conjugate linear involution on $g$. Let $\theta_0$ be a Cartan involution of $g_0$, and let $h_0$ be a $\theta_0$-stable maximally compact Cartan subalgebra of $g_0$. Set $t_0 = h_0^{\theta_0}$ and $a_0 = h_0^{-\theta_0}$ so that $h_0 = t_0 + a_0$. Let $\gamma_0$ be the complexification of $\theta_0$. Then the Cartan subalgebra $h = h_0 + ih_0$ of $g$ is $\gamma_0$-stable. Let $\Delta$ be the root system for $(g, h)$. Since $h_0$ is a maximally compact Cartan subalgebra of $g_0$, there exists $x_0 \in h_0$ that is regular for $\Delta$. Define the subset $\Delta^+$ of positive roots in $\Delta$ by $\alpha \in \Delta^+$ if and only if $\alpha(x_0) > 0$. Then $\gamma_0(\Delta^+) = \Delta^+$. Let $\Sigma \subset \Delta^+$ be the set of simple roots in $\Delta^+$. Then $\gamma_0(\Sigma) = \Sigma$, so $\gamma_0$ gives rise to an involutive automorphism of the Dynkin diagram of $g$. Let $I$ be the set of non-compact imaginary simple roots.

The Vogan diagram of $g_0$ associated to the triple $(\theta_0, h_0, \Delta^+)$ is the Dynkin diagram $\mathcal{D}(g)$ of $g$, together with an involutive automorphism $\gamma_0$ on $\mathcal{D}(g)$ and the vertices corresponding to the simple roots in $I$ painted black.

In general, a Vogan diagram for $g$ is defined to be a triple $(\mathcal{D}(g), d, I)$, where $\mathcal{D}(g)$ is the Dynkin diagram of $g$, $d$ is an involutive automorphism of $\mathcal{D}(g)$, and $I$ is a subset of vertices of $\mathcal{D}(g)$ such that $d(\alpha) = \alpha$ for each $\alpha \in I$. Every Vogan diagram for $g$ comes from a real form of $g$ (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in [6]. Every Vogan diagram in the list in [6] is normalized in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of $g$, we now recall the explicit construction of a real form of $g$ from a Vogan diagram [6, Theorem 6.88]. We need to fix the following data for $g$.

Choose a Cartan subalgebra $h$ of $g$ and let $\Delta$ be the root system for $(g, h)$. Fix a choice of positive roots $\Delta^+$ and let $\Sigma$ be the basis of simple roots. Let $\langle \cdot, \cdot \rangle$ be the Killing form of $g$ and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for each $\alpha \in \Delta^+$, where $H_\alpha$ is the unique element of $h$ defined by $\langle H, H_\alpha \rangle = \alpha(H)$ for all $H \in h$, and such that the numbers $m_{\alpha, \beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha, \beta}E_{\alpha + \beta}$ when $\alpha + \beta \in \Delta$ are real. Define a compact real form $t$ of $g$ as

$$t = \text{span}_R \{iH_\alpha, X_\alpha := E_\alpha - E_{-\alpha}, Y_\alpha := i(E_\alpha + E_{-\alpha})\},$$

and let $\theta$ be the complex conjugation of $g$ defining $t$. If $d$ is an involutive automorphism of the Dynkin diagram of $g$, define $\gamma_d$ to be the unique automorphism of $g$ satisfying $\gamma_d(H_\alpha) = H_{d(\alpha)}$ and $\gamma_d(E_\alpha) = E_{d(\alpha)}$ for each simple root $\alpha$.

Given a Vogan diagram $v$ for $g$, not necessarily normalized, with the involutive diagram automorphism $d$, let $t_v$ be the unique element in the adjoint group of $g$ such that

$$\text{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha & \text{if } \alpha \text{ is a blank vertex in } v, \\ -E_\alpha & \text{if } \alpha \text{ is a painted vertex in } v. \end{cases}$$
Define a complex conjugate linear involution
\[ \tau_v := \text{Ad}_{t_v} \circ \gamma_d \circ \theta. \]

**Notation 2.1.** We use \( \mathfrak{g}_v = \mathfrak{g}^{\tau_v} \) to denote the real form of \( \mathfrak{g} \) defined by \( \tau_v \). Set \( \theta_v = \theta|_{\mathfrak{g}_v} \). Then \( \theta_v \) is a Cartan involution of \( \mathfrak{g}_v \), and \( \mathfrak{h}^{\tau_v} \) is a \( \theta_v \)-stable maximally compact Cartan subalgebra of \( \mathfrak{g}_v \), with \( \mathfrak{h} = \mathfrak{h}^{\tau_v} + i\mathfrak{h}^{\tau_v} \). The complexification of \( \tau_v \) is
\[ (2.1) \quad \gamma_v := t_v \circ \theta = \text{Ad}_{t_v} \circ \gamma_d. \]

Since \( \gamma_v(\Delta^+) = \Delta^+ \), the Vogan diagram of \( \mathfrak{g}_v \) associated to the triple \( (\theta_v, \mathfrak{h}^{\tau_v}, \Delta^+) \) is \( v \).

One of the advantages of introducing the real form \( \mathfrak{g}_v \) is as follows. We say that a real subalgebra \( \mathfrak{l} \) of \( \mathfrak{g} \) is Lagrangian if its real dimension is equal to the complex dimension of \( \mathfrak{g} \) and if \( \text{Im} \langle x_1, x_2 \rangle = 0 \) for all \( x_1, x_2 \in \mathfrak{l} \). A decomposition \( \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2 \) is called a Lagrangian splitting if both \( \mathfrak{l}_1 \) and \( \mathfrak{l}_2 \) are Lagrangian. Let \( \mathfrak{n} \) be the subalgebra of \( \mathfrak{g} \) spanned by the set of all positive root vectors for \( \Delta^+ \). The following fact is easy to prove.

**Lemma 2.2.** Let \( \mathfrak{l}_d := \mathfrak{h}^{\tau_v} + \mathfrak{n} \). Then \( \mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d \) is a Lagrangian splitting of \( \mathfrak{g} \).

Let \( \mathfrak{a} = \text{span}_\mathbb{R} \{ iH_{\alpha} : \alpha \in \Sigma \} \), and let \( \mathfrak{t} = i\mathfrak{a} \). We note that since
\[ \mathfrak{h}^{\tau_v} = \mathfrak{h}^{\gamma_d \circ \theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{\gamma_d}, \]
the Lagrangian complement \( \mathfrak{l}_d \) of \( \mathfrak{g}_v \) depends only on \( d \), and in the case when \( d = 1 \), we have \( \mathfrak{l}_d = \mathfrak{a} + \mathfrak{n} \). Note that \( \mathfrak{h}^{\tau_v} = \mathfrak{h}^{\gamma_d \circ \theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{\gamma_d} \) also depends only on \( d \).

**Remark 2.3.** Recall [2, Definition 6.10] that two real forms \( \tau_1 \) and \( \tau_2 \) are said to be in the same inner class if there exists \( g \in \text{Int}(\mathfrak{g}) \), the adjoint group of \( \mathfrak{g} \), such that \( \tau_1 = \text{Ad}_g \tau_2 \). Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of \( \mathfrak{g} \) [2, Proposition 6.12]. Let \( d \) be an involutive automorphism of \( D(\mathfrak{g}) \). Then as \( v \) runs over the collection of all Vogan diagrams with \( d \) as the diagram automorphism, the real form \( \mathfrak{g}_v \) runs over all \( \text{Int}(\mathfrak{g}) \)-conjugacy classes of real forms of \( \mathfrak{g} \) in the inner class corresponding to \( d \).

3. The Poisson Structure \( \Pi_v \) on \( X \)

Let \( \mathfrak{g} \) be a complex semi-simple Lie algebra, and let \( X \) be the variety of all Borel subalgebras of \( \mathfrak{g} \). We keep the notation from Section 2. Let \( v \) be a Vogan diagram for \( \mathfrak{g} \) and let \( \mathfrak{g}_v = \mathfrak{g}^{\tau_v} \) be the real form of \( \mathfrak{g} \) constructed in Section 2. Let \( G \) be the connected and simply-connected Lie group with Lie algebra \( \mathfrak{g} \). Without any risk of confusion, we shall also denote by \( \tau_v \) the lift of \( \tau_v \) from \( \mathfrak{g} \) to \( G \), and we set \( G_v = G^{\tau_v} \). It follows from [3, Theorem 8.2, p. 320] that the group \( G_v \) is connected.

In this section, we will start with a Vogan diagram \( v \) for \( \mathfrak{g} \) and define a Poisson structure \( \Pi_v \) on \( X \) such that every \( G_v \)-orbit in \( X \) is a Poisson submanifold. This Poisson structure comes from an identification of \( X \) with the \( G \)-orbit through \( t + \mathfrak{n} \) inside the variety \( L \) of Lagrangian subalgebras of \( \mathfrak{g} \), which was studied in [3]. We now recall the relevant details.

Set \( n = \text{dim} \mathfrak{g} \) and let \( \text{Gr}_\mathbb{R}(n, \mathfrak{g}) \) be the Grassmannian of real \( n \)-dimensional subspaces of \( \mathfrak{g} \). The set \( L \) of all Lagrangian subalgebras of \( \mathfrak{g} \) is naturally a real subvariety of \( \text{Gr}_\mathbb{R}(n, \mathfrak{g}) \). The natural action of \( G \) on \( \text{Gr}_\mathbb{R}(n, \mathfrak{g}) \) gives rise to a Lie algebra anti-homomorphism \( \kappa \) from \( \mathfrak{g} \) to the Lie algebra of vector fields on \( \text{Gr}_\mathbb{R}(n, \mathfrak{g}) \),
whose extension from $\wedge^2 g$ to the space of bi-vector fields on $Gr(n, g)$ will also be denoted by $\kappa$. Given a Lagrangian splitting $g = l_1 + l_2$, we define the element $R_{l_1, l_2} \in \wedge^2 g$ by

\[(3.1) \langle R_{l_1, l_2}, (x_1 + \xi_1) \wedge (x_2 + \xi_2) \rangle = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, \quad x_1, x_2 \in l_1, \xi_1, \xi_2 \in l_2,\]

where $\langle \cdot, \cdot \rangle = \text{Im}(\langle \cdot, \cdot \rangle)$. Set $\Pi_{l_1, l_2} = \frac{1}{2}\kappa(R_{l_1, l_2})$. Clearly, $\Pi_{l_1, l_2}$ is tangent to every $G$-orbit in $Gr(n, g)$, so it is tangent to $L$.

**Theorem 3.1** ([3, Theorems 2.14 and 2.18]). The bi-vector field $\Pi_{l_1, l_2}$ restricts to a Poisson structure on $L$. If $L_1$ and $L_2$ are the connected subgroups of $G$ with Lie algebras $l_1$ and $l_2$ respectively, then all the $L_1$- as well as $L_2$-orbits in $L$ are Poisson submanifolds with respect to $\Pi_{l_1, l_2}$.

For $l \in L$, let $n(l)$ be the normalizer subalgebra of $l$ in $l_1$. Let $m(l)$ be the annihilator of $n(l)$ in $l$, i.e. $m(l) = \{x \in l : \langle x, y \rangle = 0 \forall y \in n(l)\} \subseteq l$, and let $V(l) = n(l) + m(l)$.

**Proposition 3.2** ([3, Theorem 2.21], [3, Corollary 7.3]). For each $l \in L$, the space $V(l)$ is a Lagrangian subalgebra of $g$. The co-dimension of the symplectic leaf of $\Pi_{l_1, l_2}$ through $l$ is equal to $\dim(V(l) \cap l_2)$.

**Notation 3.3.** Let $v$ be a Vogan diagram for $g$. We denote by $\Pi_v$, the Poisson structure on $L$ defined by the Lagrangian splitting $g = g_v + l_d$ in Lemma 2.2. Let $H$, $N$, and $B$ be respectively the connected subgroups of $G$ with Lie algebras $h$, $n$, and $b = h + n$, so $B = HN$. Identify the $G$-orbit through $t + n \in L$ with $G/B \cong X$. The induced Poisson structure on $X$ will also be denoted by $\Pi_v$. Let $H^{-\gamma \circ \theta} = \{h \in H : \gamma \circ \theta(h) = h^{-1}\}$ and let $L_d = H^{-\gamma \circ \theta}N$. By the Bruhat lemma, orbits of $L_d$ in $X \cong G/B$, which are the same as the $N$-orbits in $X$, are labeled by the elements in the Weyl group $W$ of $\Delta$. We refer to these $N$-orbits as the Bruhat cells in $X$.

By [3, Theorem 2.18], we have

**Proposition 3.4.** Each $G_v$-orbit in $X$ as well as each Bruhat cell in $X$ is a Poisson submanifold with respect to $\Pi_v$.

When $v$ is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $g_v = t$. The Poisson structure $\Pi_v$ in this case was first introduced in [11] and [13], and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name “Bruhat Poisson structure” in [11]). In [3] and [10] this Poisson structure was related to some earlier work of Kostant [7] and of Kostant-Kumar [8] on the Schubert calculus on $X$.

The splitting $g = g_v + l_d$ naturally defines a Lie bialgebra structure on $g_v$ and therefore a Poisson Lie group structure on $G_v$ [11]. All the $G_v$-orbits in $L$ become $G_v$-Poisson homogeneous spaces [3, 9]. We remark that in [11], Andruskiewitsch and Jancsa classified non-triangular Lie bialgebra structures on $g_v$ using Belavin-Drinfeld triples. The one defined by the splitting $g = g_v + l_d$ comes from the standard Belavin-Drinfeld triple. We refer to [1] for details.

**Example.** Here we take $g = sl(2, \mathbb{C})$ and

\[g_v = su(1, 1) = \left\{ \begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.\]
Then $d = 1$ and $I_d = a + n$ consists of upper triangular matrices in $\mathfrak{sl}(2, \mathbb{C})$ with real diagonal entries. Identify $G/B$ with $\mathbb{P}^1$ via the action
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [w_0 : w_1] = [aw_0 + bw_1 : cw_0 + dw_1]
\]
of $G$ on $\mathbb{P}^1$ and by taking $[1 : 0] \in \mathbb{P}^1$ as the basepoint. There are two Bruhat cells: the zero-dimensional basepoint $[1 : 0]$, and the other being the rest:
\[U_1 = \mathbb{P}^1 \setminus \{ [1 : 0] \} = \{ [w_0 : w_1], \ w_1 \neq 0 \}.
\]
In terms of the holomorphic coordinate $z$ on $U_1$ given by $z = w_0/w_1$, the Poisson structure $\Pi_v$, up to a scalar multiple, is given by
\[\Pi_v = i(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \overline{z}}.
\]
Setting $u = 1/z$, we see that in the $u$-coordinate on the open set
\[U_0 = \{ [w_0 : w_1] \in \mathbb{P}^1, w_0 \neq 0 \} = \{ [1 : u], u \in \mathbb{C} \},
\]
we have
\[\Pi_v = i(|u|^2 - 1)|u|^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \overline{u}}.
\]
Thus $\Pi_v$ vanishes precisely at the basepoint $[1 : 0]$ and at every point of the form $[z : 1]$ with $|z| = 1$. If we identify $\mathbb{P}^1$ with the unit sphere $S^2$ in $\mathbb{R}^3$ via
\[(3.2) \quad \mathbb{P}^1 \rightarrow S^2 : [w_0, w_1] \mapsto \left( \frac{2\text{Re}(w_0 \overline{w_1})}{|w_0|^2 + |w_1|^2}, \frac{2\text{Im}(w_0 \overline{w_1})}{|w_0|^2 + |w_1|^2}, \frac{|w_0|^2 - |w_1|^2}{|w_0|^2 + |w_1|^2} \right)
\]
then we see that $\Pi_v$ vanishes at the “North pole” $(0, 0, 1)$ and at every point on the Equator $x_3 = 0$. Under this identification, there are exactly three orbits of $\text{SU}(1, 1)$ on $S^2$: the Northern hemisphere, the Equator, and the Southern hemisphere. Each of these three orbits is clearly a Poisson submanifold.

4. SYMPLECTIC LEAVES OF $\Pi_v$ IN $X$

Suppose that $\mathcal{O}$ is a $G_v$-orbit in $X$ and $\mathcal{C}$ is a Bruhat cell such that $\mathcal{O} \cap \mathcal{C} \neq \emptyset$. Since $g = g_v + I_d$, $\mathcal{O}$ and $\mathcal{C}$ intersect transversally. By Proposition 3.5, $\mathcal{O} \cap \mathcal{C}$ is a Poisson submanifold of $\Pi_v$. In this section, we show that $(\mathcal{O} \cap \mathcal{C}, \Pi_v)$ is a regular Poisson manifold, and we compute the dimension of its symplectic leaves.

It is well known [13] that there are only finitely many $G_v$-orbits in $X$. We first recall from [12, Section 6] some facts about these orbits.

Let $N_G(\mathfrak{h})$ be the normalizer subgroup of $\mathfrak{h}$ in $G$. Set
\[Z = \{ g \in G : g^{-1} \tau_v(g) \in N_G(\mathfrak{h}) \}.
\]
Then $H$ acts on $Z$ from the right by right multiplication, and $G_v$ acts on $Z$ from the left by left multiplication. Let $Z$ be the double coset space
\[Z = G_v \backslash Z / H.
\]
For each $z \in Z$, choose any $g_z \in Z$ in the double coset $z$ and define $\mathcal{O}_z$ to be the $G_v$-orbit in $X$ through $g_zB \in X \cong G/B$. Clearly, $\mathcal{O}_z$ is independent of the choice of $g_z$.

According to [12, Theorem 6.1.4], the map $z \mapsto \mathcal{O}_z$ is a one-to-one correspondence between the set $Z$ and the set of $G_v$-orbits in $X$. Let $W = N_G(\mathfrak{h})/H$ be the Weyl group. Thus we also have the map
\[\varphi : Z \rightarrow W : z = G_vg_zH \mapsto g_z^{-1} \tau_v(g_z)H \in W.
\]
According to [12, Theorem 6.4.2], the codimension of the $G_\mathfrak{z}$-orbit $\mathcal{O}_z$ in $X$ equals $l(\varphi(z))$, where $l$ is the length function on the Weyl group $W$. We also introduce the map

$$\sigma_z = \varphi(z)\tau_v : \mathfrak{h} \rightarrow \mathfrak{h}.$$ 

For any $g_z$ in the double coset $z$, we also have $\sigma_z = \text{Ad}^{-1}_{g_z} \circ \tau_v \circ \text{Ad}_{g_z}$, so $\sigma_z$ is an involution.

Assume now that $z \in Z$ and $w \in W$ are such that $O_z \cap C_w \neq \emptyset$, where $C_w$ is the Bruhat cell in $X$ corresponding to $w$, i.e. the $N$-orbit through $w \in G/B$. Then $\dim_{\mathbb{R}}C_w = 2l(w)$, and since $O_z$ and $C_w$ intersect transversally, we have

$$\dim(O_z \cap C_w) = 2l(w) - l(\varphi(z)).$$

Now define

$$\delta_{z,w} = \dim(\mathfrak{h}^{l\sigma_z\mathfrak{w}^{-1}} \cap \mathfrak{h}^{\mathfrak{t}v}).$$

**Theorem 4.1.** Each symplectic leaf in the intersection $O_z \cap C_w$ has dimension equal to

$$\dim(O_z \cap C_w) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.$$ 

**Proof.** We use Proposition [12] to compute dimensions of the symplectic leaves in $O_z \cap C_w$. Let $x = g_z B \in X$ be a point in $O_z \cap C_w$, where $g_z \in Z$ lies in the double coset $z$. Let $t_z = \text{Ad}_{g_z}(t + n) \in L$. Let $n(t_z) = g_v \cap \text{Ad}_{g_z}(\mathfrak{h} + n)$ be the normalizer subalgebra of $t_z$ in $g_v$, let $m(t_z)$ be the annihilator subspace of $n(t_z)$ in $t_z$, and let $V(t_z) = n(t_z) + m(t_z)$. We claim that $V(t_z) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + n)$. Indeed, it follows from the definition of $\sigma_z$ that

$$\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z}) \subset g_v \cap \text{Ad}_{g_z}(\mathfrak{h} + n) = n(t_z).$$

It is also clear that $\text{Ad}_{g_z}n \subset m(t_z)$, so

$$\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + n) \subset n(t_z) + m(t_z) = V(t_z).$$

Since both $\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + n)$ and $V(t_z)$ have the same dimension, they must coincide.

Now let $S_x$ be the symplectic leaf of $\Pi_x$ in $X$ through $x$. By Proposition [3,2] the codimension of $S_x$ in $O_z$ is equal to $\dim(V(t_z) \cap l_d)$. Let $\bar{w} \in N_G(\mathfrak{h})$ be a representative of $\bar{w}$ in $K$. Since $x \in C_w$, there exist $n \in N$ and $b \in B$ such that $g_z = n\bar{w}b$. Then we have

$$V(t_z) \cap l_d = (\text{Ad}_{n\bar{w}b}(\mathfrak{h}^{\sigma_z} + n)) \cap (\mathfrak{h}^{-\mathfrak{t}v} + n)$$

$$= \text{Ad}_n ((\text{Ad}_{\bar{w}}(\mathfrak{h}^{\sigma_z} + n)) \cap (\mathfrak{h}^{-\mathfrak{t}v} + n))$$

$$= \text{Ad}_n (\mathfrak{h}^{l\sigma_z\mathfrak{w}^{-1}} \cap \mathfrak{h}^{-\mathfrak{t}v} + (\text{Ad}_{\bar{w}} n) \cap n),$$

where in the last line we have the direct sum of vector spaces. Since $\dim(\text{Ad}_{\bar{w}} n) \cap n = \dim_{\mathbb{R}}X - \dim_{\mathbb{R}}C_w$, we have

$$\dim(V(t_z) \cap l_d) = \delta_{z,w} + \dim_{\mathbb{R}}X - \dim_{\mathbb{R}}C_w,$$

and thus

$$\dim S_x = \dim O_z - \dim(V(t_z) \cap l_d) = \dim(O_z \cap C_w) - \delta_{z,w}. \square$$
Note that the number $\delta_{z,w}$ depends only on $d$ and the two Weyl group elements $\varphi(z)$ and $w$. Define $d : W \to W$ by $d(w) = \gamma_d w \gamma_d$. Following [12], we say that $w \in W$ is a $d$-twisted involution if $d(w) = w^{-1}$. Denote by $\mathcal{I}_d$ the set of all $d$-twisted involutions in $W$. Since for every $g \in G$ we have $\tau_v(g^{-1} \tau_v(g)) = (g^{-1} \tau_v(g))^{-1}$, every $\varphi(z)$ is in $\mathcal{I}_d$. The Weyl group $W$ acts on $\mathcal{I}_d$ by

$$w_1 * w = w_1 w d(w_1^{-1})$$

for $w_1 \in W$, and $w \in \mathcal{I}_d$, and the set $\varphi(Z) \subset \mathcal{I}_d$ is $W$-invariant. In fact, the $W$-action on $G/H$, given by $w \cdot y H = gw^{-1} H$, commutes with the left action of $G_v$ by left multiplication, and thus induces a left action of $W$ on $Z$, which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is also easy to see that $\varphi : Z \to W$ is $W$-equivariant, i.e. $\varphi(w \cdot z) = w \cdot \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v : G \to G$ gives rise to an involution on $Z$ which depends only on $d$. Denote this involution by $z \to d(z)$. Then we also have $\varphi(d(z)) = d \varphi(z) = \varphi(z)^{-1}$. As maps on $h$, we see that $w \sigma_z w^{-1} = (w \cdot \varphi(z)) \tau_v$. Thus we also have

$$\delta_{z,w} = \dim(\{ h \cdot \varphi(z) \} \cap h^{-1} \tau_v).$$

**Corollary 4.2.** 1) When $w \cdot \varphi(z) = 1$, symplectic leaves of $\Pi_v$ in $\mathcal{O}_z \cap \mathcal{C}_w$ are precisely its connected components.

2) Every open orbit $\mathcal{O}_z$ has an open symplectic leaf $\mathcal{O}_z \cap \mathcal{C}_{w_0}$, where $w_0$ is the longest element in $W$.

3) If $d = 1$, symplectic leaves in an open orbit $\mathcal{O}_z$ are precisely the connected components of intersections of Bruhat cells with $\mathcal{O}_z$.

**Proof.** 1) When $w \cdot \varphi(z) = 1$, we have $\delta_{z,w} = 0$, so every symplectic leaf in $\mathcal{O}_z \cap \mathcal{C}_w$ is open in $\mathcal{O}_z \cap \mathcal{C}_w$.

2) Since $\mathcal{C}_{w_0}$ is dense in $X$, it intersects with every open orbit $\mathcal{O}_z$. Since an orbit $\mathcal{O}_z$ is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that $w_0$ commutes with $d$. The fact that $\mathcal{C}_{w_0} \cap \mathcal{O}_z$ is connected follows from the observation that $\mathcal{O}_z$ is a connected open complex submanifold of $X$ and thus $\mathcal{O}_z \cap (X \setminus \mathcal{C}_{w_0})$ is a divisor in $\mathcal{O}_z$.

3) follows directly from 1). \hfill \Box

Now consider the group $H^{r_v} = H \cap G_v$. Since the centralizer of $h^{r_v}$ in $G_v$ also centralizes $h$, we see that $H^{r_v}$ is the Cartan subgroup of $G_v$ corresponding to the Cartan subalgebra $h^{r_v}$. Then according to [3] Proposition 7.90 the group $H^{r_v}$ is connected.

The Poisson structure $\Pi_v$ on $X$ is $H^{r_v}$-invariant. Indeed, let $R \in \bigwedge^2 \mathfrak{g}$ be the element given in [3] for $I_1 = \mathfrak{g}_v$ and $I_2 = I_d$. We can also represent $R$ as $R = \sum_i \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\xi_i\}$ is the dual basis of $I_d$ with respect to the pairing between $\mathfrak{g}_v$ and $I_d$ given by $\langle , \rangle$, the imaginary part of the Killing form on $\mathfrak{g}$. If $h \in H^{r_v}$, then $\{\text{Ad}_h y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\text{Ad}_h \xi_i\}$ is its dual basis. Thus $\text{Ad}_h R = R$.

Let $z \in Z$ and $w \in W$ be such that $\mathcal{O}_z$ and $\mathcal{C}_w$ have a non-empty intersection, and let $x \in \mathcal{O}_z \cap \mathcal{C}_w$. Clearly, $H^{r_v}$ leaves $\mathcal{O}_z \cap \mathcal{C}_w$ invariant. Since the Poisson structure $\Pi_v$ is $H^{r_v}$-invariant, if $S_x$ is the symplectic leaf of $\Pi_v$ through $x$, then $hS_x := \{hx_1 : x_1 \in S_x\}$ is the symplectic leaf of $\Pi_v$ through $hx$. Define

$$F_x := \bigcup_{h \in H^{r_v}} hS_x.$$

**Proposition 4.3.** For any $x \in X$, the set $F_x$ is a connected component of $\mathcal{O}_z \cap \mathcal{C}_w$. 


Proof. It is easy to see that if $F_{x_1} \cap F_{x_2} \neq \emptyset$, then $F_{x_1} = F_{x_2}$. The statement would follow once we prove that $F_x$ is an open subset of $\mathcal{O}_z \cap \mathcal{C}_w$ for each $x$.

Let $x = g_x B \in \mathcal{O}_z \cap \mathcal{C}_w$ with $g_x \in \mathcal{Z}$ in the double coset $z$. For $y \in \mathfrak{h}^{\tau_v}$, let $X_y$ be the vector field on $X$ generating the action of $\exp(t y) \in H^{\tau_v}$ on $X$. We claim that $X_y(x) \in T_x S_x$ if and only if $y \in p(\mathfrak{h}^{(w \ast \varphi(z))\tau_v})$, where $p \colon \mathfrak{h} \to \mathfrak{h}^{\tau_v}$ is the projection with respect to the decomposition $\mathfrak{h} = \mathfrak{h}^{\tau_v} + \mathfrak{h}^{-\tau_v}$. Assume the claim. Then since the kernel of the map $p \colon \mathfrak{h}^{(w \ast \varphi(z))\tau_v} \to \mathfrak{h}^{\tau_v}$ has dimension $\dim(\mathfrak{h}^{(w \ast \varphi(z))\tau_v} \cap \mathfrak{h}^{-\tau_v}) = \delta_{x,w}$, the image of the map

$$J_x : \mathfrak{h}^{\tau_v} \longrightarrow T_x \mathcal{O}_z / T_x S_x : \ y \longmapsto X_y(x) + T_x S_x$$

has dimension equal to $\dim(\mathfrak{h}^{\tau_v}) - \dim(\mathfrak{h}^{(w \ast \varphi(z))\tau_v}) + \delta_{x,w} = \delta_{x,w}$. Thus $J_x$ is onto, and the $H^{\tau_v}$-orbit in $\mathcal{O}_z \cap \mathcal{C}_w$ through $x$ is transversal to the symplectic leaf $S_x$. It follows that $F_x$ is open in $\mathcal{O}_z \cap \mathcal{C}_w$.

It remains to prove the claim. Also denote by $p : \mathfrak{g} \to \mathfrak{g}_w$, the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_w + \mathfrak{u}_d$, and let $q$ be the projection $q : \mathfrak{g}_w \to \mathfrak{g}_w / \mathfrak{g}_w \cap \text{Ad}_{g_x} \mathfrak{b} \cong T_x \mathcal{O}_z$. Then by [3] Corollary 7.3, we have $T_x S_x = (q \circ p)(\mathcal{V}(I_z))$, where, as in the proof of Theorem 4.1, $\mathcal{V}(I_z) = \text{Ad}_{g_x}(\mathfrak{h}^{\tau_v} + \mathfrak{n})$. Let $y \in \mathfrak{h}^{\tau_v}$. If $X_y(x) \in T_x S_x$, then there exist $y_1 \in \mathfrak{u}_d$ and $y_2 \in \mathfrak{g}_w$ with $y_1 + y_2 \in \mathcal{V}(I_z)$ such that $y = y_2 \in \mathfrak{g}_w \cap \text{Ad}_{g_x} \mathfrak{b} \subset \mathcal{V}(I_z)$. Thus $y + y_1 = y - y_2 + y_1 + y_2 \in \mathcal{V}(I_z)$. Write $y_1 = \xi_1 + u_1$, where $\xi_1 \in \mathfrak{h}^{\tau_v}$ and $u_1 \in \mathfrak{n}$. Then there exist $\xi_2 \in \mathfrak{h}^{\tau_v}$ and $u_2 \in \mathfrak{n}$ such that $y + \xi_1 + u_1 = \text{Ad}_{g_x}(\xi_2 + u_2)$. Write $g_x = n \omega b$, where $n \in N, b \in B$, and $\omega$ is a representative of $\omega$ in $K$. Write $\text{Ad}_{n^{-1}}(y + \xi_1 + u_1) = y + \xi_1 + u_1'$ and $\text{Ad}_{n}(\xi_2 + u_2) = \xi_2 + u_2'$, where $u_1', u_2' \in \mathfrak{n}$. Then we have

$$y + \xi_1 + u_1' = \text{Ad}_{n}(\xi_2 + u_2') .$$

Since $y + \xi_1, \text{Ad}_{n} \xi_2 \in \mathfrak{h}$ and $u_1', \text{Ad}_{n}u_2' \in \mathfrak{n} + \mathfrak{n}_-$, where $\mathfrak{n}_- = \theta(\mathfrak{n})$, we have $y + \xi_1 = \text{Ad}_{n} \xi_2 \in \mathfrak{h}^{(w \ast \varphi(z))\tau_v}$. Thus $y \in p(\mathfrak{h}^{(w \ast \varphi(z))\tau_v})$. Conversely, if $y \in \mathfrak{h}^{\tau_v}$ is such that $y + \xi_1 \in \mathfrak{h}^{(w \ast \varphi(z))\tau_v} = \text{Ad}_{n} \mathfrak{h}^{\tau_v}$ for some $\xi_1 \in \mathfrak{h}^{-\tau_v}$, write $y + \xi_1 = \text{Ad}_{n} \xi_2$ for $\xi_2 \in \mathfrak{h}^{\tau_v}$. Let $\text{Ad}_{n^{-1}} \xi_2 = \xi_2 + u_2$ for some $u_2 \in \mathfrak{n}$. We then have

$$\text{Ad}_n(y + \xi_1) = \text{Ad}_{n \omega b}(\xi_2 + u_2) \in \mathcal{V}(I_z).$$

On the other hand, let $\text{Ad}_n(y + \xi_1) = y + \xi_1 + u_1$ with $u_1 \in \mathfrak{n}$. We see that $y = p(\text{Ad}_n(y + \xi_1))$ so $X_y(x) \in T_x S_x$. \qed

5. Invariant Poisson cohomology of open orbits

Let $\mathcal{O}_z$ be a $G_v$-orbit in $X$ equipped with the Poisson structure $\Pi_v$. Then $(\mathcal{O}_z, \Pi_v)$ is a Poisson homogeneous space for the Poisson Lie group $G_v$. The $G_v$-invariant Poisson cohomology of $(\mathcal{O}_z, \Pi_v)$, denoted by $H^*_\Pi_v G_v(\mathcal{O}_z)$, is defined as the cohomology of the cochain complex $(\chi^*(\mathcal{O}_z)^{G_v}, \partial_{\Pi_v})$, where $\chi^*(\mathcal{O}_z)^{G_v}$ is the space of all $G_v$-invariant complex multi-vector fields on $\mathcal{O}_z$, $d_{\Pi_v}(V) = [\Pi_v, V]$, and $[\cdot, \cdot]$ is the Schouten bracket of the multi-vector fields.

Proposition 5.1. When $\mathcal{O}_z$ is an open $G_v$-orbit in $X$, the $G_v$-invariant Poisson cohomology $H^*_\Pi_v G_v(\mathcal{O}_z)$ is isomorphic to the de Rham cohomology of $X$.

Proof. As in the proof of Theorem 4.1, let $x = g_x B \in X$ be an arbitrary point in $\mathcal{O}_z$, where $g_x \in \mathcal{Z}$ is in the coset $z$, and let $\mathcal{V}(I_z) = \text{Ad}_{g_x}(\mathfrak{h}^{\tau_v} + \mathfrak{n})$. Since $\mathcal{O}_z$ is open, the stabilizer subalgebra of $g_x$ at $x$ is $\mathfrak{g}_w \cap \mathcal{V}(I_z) = \text{Ad}_{g_x}(\mathfrak{h}^{\tau_v})$. By [3] Theorem 7.5, the $G_v$-invariant Poisson cohomology $H^*_\Pi_v G_v(\mathcal{O}_z)$ is isomorphic to the relative Lie algebra cohomology of the Lie algebra $\mathcal{V}(I_z) \otimes \mathbb{C}$ relative to the subalgebra
Thus the $G_v$-invariant Poisson cohomology is isomorphic to the $h$-invariant part of the Lie algebra cohomology of the direct sum Lie algebra $\mathfrak{n} \oplus \mathfrak{n}$ with coefficients in $\mathbb{C}$, which by Kostant’s theorem [7], is isomorphic to the de Rham cohomology of $X$. □

6. Remarks

We have constructed a Poisson structure $\Pi_v$ on $X$ for each Vogan diagram $v$ for $\mathfrak{g}$ (which is not necessarily normalized). In particular, each Bruhat cell $C_w$ in $X$ carries the Poisson structure $\Pi_v$. It would be interesting to study connections between the Poisson structures for different $v$. Especially interesting are the properties of $\Pi_v$ that depend only on the inner class $d$ of the real form $\mathfrak{g}_v$. We also remark that the Poisson structure $\Pi_v$ is defined on the whole variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g}$. We have only been looking at the restriction of $\Pi_v$ to a particular $G$-orbit, namely the $G$-orbit through the Lagrangian subalgebra $\mathfrak{t} + \mathfrak{n}$. There are many other interesting $G$-orbits in $\mathcal{L}$, such as the $G$-orbit through a given real form of $\mathfrak{g}$. It would be interesting to study the properties of the Poisson structure $\Pi_v$ on these orbits as well as on their closures, with respect to both the classical topology and the Zariski topology.

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