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STABILITY AND STABILISATION OF 2D DISCRETE LINEAR SYSTEMS WITH MULTIPLE DELAYS

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ABSTRACT
In this paper, we study the stability and the stabilisation of 2D discrete linear systems with multiple state delays. All of the new results obtained are based on analysis of the Fomasini-Marchesini state space model with delays and the resulting conditions are given in terms of linear matrix inequalities (LMIs). A numerical example is given to illustrate the effectiveness of the overall approach.

1. INTRODUCTION
The analysis of time-delay systems is a very important part of (linear) systems theory and has been a very active research area over the past few decades. The interest in time-delay systems stems from the fact that such delays occur often in, for example, electronic, mechanical, biological, metallurgical and chemical systems — see, for example, [10, 11].

The existence of delays is frequently a source of instability. Much work has been reported on the problem of the stability of standard, termed 1D here, linear systems with delays [4, 12] but relatively little on the stability of 2D (nD) linear systems with delays.

In this paper, we develop stability conditions for 2D linear systems with multiple state delays and then establish some connections between multidimensional delay and delay-free systems. Based on these results conditions for the existence of stabilising controllers are developed. All of these conditions are formulated in terms of linear matrix inequalities (LMIs) [2, 6], where an advantage of using LMIs is the fact that there exist efficient numerical algorithms to solve them as demonstrated by the numerical example which concludes this paper.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I respectively. M > 0 is used to denote the fact that M is a real symmetric positive definite matrix. Also, delays h1, . . . , hq are termed noncommensurate if 3 no integers i1, . . . , iq (not all of them zero) such that ∑iq i=1 lihi = 0. The underlying delay differential system is termed commensurate if q = 1. We will also make extensive use of the following well known result.

Lemma 1 (Schur complement) [2] For matrices Σ1, Σ2, and Σ3 where Σ1 > 0 and Σ3 = Σ3T then

Σ3 + ΣT 2 Σ−1 1 Σ2 < 0

if, and only if,

Σ3

Σ2

Σ1

< 0 or Σ3

Σ2

Σ1

< 0

2. 2D LINEAR SYSTEMS WITH MULTIPLE DELAYS
Consider a 2D linear system with multiple state delays which can be represented by the Fomasini-Marchesini state-space model [5] with delays

\[ x(i+1, j+1) = A_1 x(i+1, j) + A_2 x(i, j+1) + \sum_{k=1}^{s_1} A_{1kd} x(i+1, j - d_{1k}) + \sum_{i=1}^{s_2} A_{2id} x(i, j+1 - d_{2i}) \]

\[ + B_1 u(i+1, j) + B_2 u(i, j+1) + \sum_{k=1}^{s_1} B_{1kd} u(i+1, j - d_{1k}) + \sum_{i=1}^{s_2} B_{2id} u(i, j+1 - d_{2i}) \]

where \( x(i, j) \in R^n, \ u(i, j) \in R^m \) are the state and input vectors respectively, \( i, j \in Z_+ \), where \( Z_+ \) denotes the set of nonnegative integers, \( A_p, B_p \ (p = 1, 2), A_{1kd}, B_{1kd} \ k = 1, \ldots, s_1, A_{2id}, B_{2id} \ l = 1, \ldots, s_2 \) are known constant matrices with compatible dimensions, and \( s_1 \) and \( s_2 \) denote the number of delay terms in each direction respectively. We also assume that \( 0 < d_{11} < d_{12} < \ldots < d_{1s_1} \) and \( 0 < d_{21} < d_{22} < \ldots < d_{2s_2} \).
\[
\begin{align*}
\cdots < d_{1x}, & \text{ and } 0 < d_{21} < d_{22} < \cdots < d_{2x} \\
\text{and in this case the boundary conditions are defined as} & \\
\{ x(i, j) = v_{ij} \}, & \forall i \geq 0; \; j = -d_{1x}, -d_{1x} + 1, \ldots, 0 \\
\{ x(i, j) = w_{ij} \}, & \forall j \geq 0; \; i = -d_{2x}, -d_{2x} + 1, \ldots, 0
\end{align*}
\]

where \( v_{00} = w_{00} \).

With \( x_i = \sup \{ \| x(i, j) \| : i + j = r, \; i, j \in \mathbb{Z} \} \), asymptotic stability of the model (3) is defined as follows.

**Definition 1** [8, 5] The 2D linear system with multiple state delays (3) is said to be asymptotically stable if \( \lim_{t \to \infty} X_t = 0 \) for zero input \( u(i, j) = 0 \) and for any bounded boundary conditions of the form (4).

### 2.1. Noncommensurate delays

In the case of noncommensurate delays, the following result characterizes asymptotic stability of the class of systems considered in terms of an LMI condition.

**Theorem 1** The 2D delay system (3) is asymptotically stable if \( \exists \) matrices \( P, Q > 0, U_{11}, \ldots, U_{1x} > 0 \) and \( U_{21}, \ldots, U_{2x} > 0 \) such that the following LMI holds:

\[
\begin{bmatrix}
A^T & A_2 & \Lambda_{1d} & \Lambda_{2d} \\
A_2^T & A_{2d} & \Lambda_{1d} & \Lambda_{2d}
\end{bmatrix}
\begin{bmatrix}
P & \Phi_1 & 0 & 0 \\
0 & P & Q & 0 \\
0 & 0 & \Omega_1 & 0 \\
0 & 0 & 0 & \Omega_2
\end{bmatrix}
\begin{bmatrix}
A^T & A_2 & \Lambda_{1d} & \Lambda_{2d} \\
A_2^T & A_{2d} & \Lambda_{1d} & \Lambda_{2d}
\end{bmatrix}^T < 0
\]

where

\[
\begin{align*}
\Lambda_{1d} &= \begin{bmatrix} A_{11d}, A_{12d}, \ldots, A_{1x_1d} \end{bmatrix}, \\
\Phi_1 &= \sum_{k=1}^{x_1} Q_{1k} \\
\Lambda_{2d} &= \begin{bmatrix} A_{21d}, A_{22d}, \ldots, A_{2x_2d} \end{bmatrix}, \\
\Phi_2 &= \sum_{l=1}^{x_2} Q_{2l} \\
\Omega_1 &= \text{diag}(Q_{11}, Q_{12}, \ldots, Q_{1x_2}), \\
\Omega_2 &= \text{diag}(Q_{21}, Q_{22}, \ldots, Q_{2x_2})
\end{align*}
\]

**Proof:** This is via a Lyapunov-Krasovskii approach. In particular, suppose that \( V(\zeta, \xi) \) denotes a function that express the energy stored at \( x(i + \zeta, j + \xi) \) and consider the particular case when

\[
V(\zeta, \xi) = x^T(i + \zeta, j + \xi)W_{\zeta \xi}x(i + \zeta, j + \xi)
\]

where \( W_{\zeta \xi} > 0 \) is given and \( \zeta, \xi \in \mathbb{Z}_+, \xi \geq -d_{1x}, \xi \geq -d_{2x} \).

Now introduce the following candidate Lyapunov functions for the delayed terms:

\[
\begin{align*}
V_{d1}(\zeta, \xi) &= x^T(i + \zeta, j + \xi)W_{\zeta \xi}x(i + \zeta, j + \xi) \\
&\quad + \sum_{k=1}^{x_1} \sum_{\theta = -d_{2k}}^{x_1 - k - 1} x^T(i + \zeta, j + \theta)U_{1k}x(i + \zeta, j + \theta) \\
V_{d2}(\zeta, \xi) &= x^T(i + \zeta, j + \xi)W_{\zeta \xi}x(i + \zeta, j + \xi) \\
&\quad + \sum_{l=1}^{x_2} \sum_{\theta = -d_{2l}}^{x_2 - l - 1} x^T(i + \theta, j + \xi)U_{2l}x(i + \theta, j + \xi)
\end{align*}
\]

where \( W_{\zeta \xi} > 0 \) and \( U_{1k}, U_{2l} > 0 \) are given and \( \zeta, \xi \in \mathbb{Z}, \xi \geq -d_{1x}, \xi \geq -d_{2x} \).

In order to determine the change of the energy in the both sides of (3) consider the increment \( \Delta V(i, j) \) where

\[
\Delta V(i, j) = V_{11}(i, j) - V_{11}(1, 0) - V_{22}(0, 1)
\]

Now consider the result of substituting (7) and (8) into (9) and define the augmented state vector as

\[
\dot{x} = \begin{bmatrix} x^T(i + 1, j) & x^T(i, j + 1) & x^T(i + 1, j - d_{1x}) & \ldots & x^T(i + 1, j - d_{1x_1}) \ldots & x^T(i - d_{2x_2}, j + 1) & \ldots & x^T(i - d_{2x_2}, j + 1) \end{bmatrix}^T
\]

where \( x_k \) includes all states from \( x(i + 1, j - 1) \ldots x(i + 1, j - d_{1x_1}) \) excluding those defined before and \( x_k \) includes all states from \( x(i - 1, j + 1) \ldots x(i - 2x_2, j + 1) \) but also excluding those defined before. Then (9) can be rewritten as (using the same notation as in (6))

\[
\Delta V(i, j) = \dot{x}^T(\Theta^T W_{11} \Theta - \Xi) \dot{x}
\]

where

\[
\begin{align*}
\Xi &= \text{diag}(\Omega_{11}, \Omega_{12}, \ldots, \Omega_{1x_2}, \Omega_{21}, \ldots, \Omega_{2x_2}) \\
\Theta &= \begin{bmatrix} A_1 & A_2 & \Lambda_{1d} & \Lambda_{2d} & 0 \end{bmatrix}, \\
\Omega_{11} &= \text{diag}(Q_{11}, Q_{12}, \ldots, Q_{1x_2}) \\
\Omega_{21} &= \text{diag}(Q_{21}, Q_{22}, \ldots, Q_{2x_2})
\end{align*}
\]

Now, if \( \Delta V(i, j) < 0 \) for \( \dot{x} \neq 0 \), then the 2D discrete linear system considered here is asymptotically stable. In order to guarantee that this stability condition holds, it is clear that \( \Pi < 0 \) must hold. Also the last two rows and columns in this matrix (i.e. those which only consist of \(-\Omega_3 \) and \(-\Omega_4 \) in (10)) can be omitted because they only contribute terms that are guaranteed to be negative definite. By again making use of (7) and (8) it is easily seen that to guarantee the dissipative property \( \Delta V(i, j) < 0 \) we can choose

\[
\begin{align*}
W_{11} &= P, \\
Q_k &= \sum_{\theta = -d_{2k}}^{x_2 - k - 1} U_{2\theta}, \\
W_{10} &= P - Q - Q_{11} - \ldots - Q_{1k} - Q_{21} - \ldots - Q_{2l},
\end{align*}
\]

**Remark 1** It was shown in [9] (see also [3] and [11]) that there exist connections between (linear) 2D delay-free systems and 1D time-delay systems. These arise because the delayed signal in the 1D case can be viewed as a signal transmitted through another dimension in the 2D framework. Theorem 1 here shows that the same result can be established for 2D linear systems with multiple delays. Hence, asymptotic stability of a 2D linear system with \( m_1 \) and \( m_2 \) delayed terms in each direction respectively is equivalent to asymptotic stability of an mD linear system where \( m = m_1 + m_2 + 2 \).
2.2. Commensurate Delays

In what follows, we show that if all delays present in (5) are commensurate then investigation of the stability properties of 2D delay system can be equivalently treated as the stability investigation of a 4D delay free system. The key to establishing this fact is the Elementary Operation Algorithm (EOA) developed by Gałkowski [7].

In general case, the notation associated with this area is very cumbersome and hence for ease of presentation only we consider the particular case of a 2D linear system of the form (3) with two delays in each direction, i.e., we restrict attention to \( m_1 = m_2 = 2 \). In which case it is clear that the associated characteristic polynomial for stability is given by the determinant of the following 2D polynomial matrix

\[
I - A_1 z_1^{-1} - A_2 z_2^{-1} - A_3 z_3^{-1} - A_4 z_4^{-1} - A_5 z_5^{-1} - A_6 z_6^{-1}.
\]

where \( k, l \in \mathbb{R}^+ \) and \( h_1, h_2, p_1, p_2 \) are natural numbers. Now introduce the new variables \( z_1 = z_3 \), \( z_2 = z_4 \) and then rewrite (12) as

\[
I - A_1 z_1^{-1} - A_2 z_2^{-1} - A_3 z_3^{-1} - A_4 z_4^{-2}.
\]

Assume also that \( h_1 = 1, h_2 = 2, p_1 = 1, p_2 = 2 \) which yields

\[
I - A_1 z_1^{-1} - A_2 z_2^{-1} - A_3 z_3^{-1} - A_4 z_4^{-2}.
\]

Application of the EOA to this last 4D polynomial matrix now gives

\[
\begin{bmatrix}
I & 0 & z_1^{-1} A_6 \\
0 & I & z_2^{-1} A_4 \\
z_1^{-1} A_1 & A_2 & -A_3 z_3^{-1} - A_4 z_4^{-2}.
\end{bmatrix}
\]

which is equivalent to

\[
I - \tilde{A}_1 z_1^{-1} - \tilde{A}_2 z_2^{-1} - \tilde{A}_3 z_3^{-1} - \tilde{A}_4 z_4^{-1}.
\]

where

\[
\tilde{A}_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_1
\end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & A_2
\end{bmatrix},
\]

\[
\tilde{A}_3 = \begin{bmatrix}
0 & 0 \\
0 & -A_3 \\
-1 & A_3
\end{bmatrix}, \quad \tilde{A}_4 = \begin{bmatrix}
0 & 0 \\
0 & -A_4 \\
0 & 0
\end{bmatrix}.
\]

Here only elementary operations that preserve the matrix determinant are used and hence it is straightforward to see that (13) and (15) have the same determinant and hence the stability property for both system descriptions is the same. This result is easily extended to the partially commensurate case. In particular, assume that each delay \( d_{i_1}, v = 1, \ldots, m_1 \) is a multiple of one of basic noncommensurate delays \( k_1, k_2, \ldots, k_n \), and similarly for \( d_{i_2}, h = 1, \ldots, m_2 \) of \( l_1, l_2, \ldots, l_2 \). Then the previous method exploiting this fact requires the investigation of an \( nD \) linear system, where \( n = t_1 + t_2 + 2 \) whereas the method of Theorem 1 here requires the investigation of an \( mD \) linear system with \( m = m_1 + m_2 + 2 \).

3. STABILIZATION 2D LINEAR SYSTEM WITH MULTIPLE DELAYS

Consider a 2D linear system with multiple state and input delays described by (3) and assume that the state feedback control law

\[
u(i, j) = K x(i, j)
\]

is used. The corresponding closed-loop system is

\[
x(i+1, j+1) = (A_1 + B_1 K) x(i+1, j) + (A_2 + B_2 K) x(i, j+1) + \sum_{k=1}^{m_1} (A_{1kd} + B_{1kd} K) x(i, j) + \sum_{l=1}^{m_2} (A_{2ld} + B_{2ld} K) x(i, j)
\]

Definition 2 If there exists \( K \) such that (17) is asymptotically stable, then the 2D delay system (3) is said to be stabilisable.

Theorem 2 The 2D delay system (3) is asymptotically stable if \( \exists \) matrices \( W, Z > 0 \), \( Z_1, \ldots, Z_{t_1} > 0 \), \( Z_{s_1}, \ldots, Z_{s_2} > 0 \) and any \( N \) such that the following LMI holds:

\[
\begin{bmatrix}
-W & A_1 W + B_1 N \\
WA_1^T + NT B_1^T & -T \\
WA_2^T + NT B_2^T & 0 \\
\mathbf{0} & \mathbf{0} \\
-A_2 W + B_2 N & T_{1d} & T_{2d} \\
0 & 0 & 0 \\
-Z & 0 & 0 \\
0 & -Z_{1d} & 0 \\
0 & 0 & -Z_{2d}
\end{bmatrix} < 0
\]

where

\[
T = W - Z - R_{11} - \ldots - R_{1s_1} - R_{21} - \ldots - R_{2s_2}
\]

\[
T_{1d} = \begin{bmatrix}
A_{11d} W + B_{11d} N \\
\ldots \\
A_{1s_1 d} W + B_{1s_1 d} N
\end{bmatrix},
\]

\[
T_{2d} = \begin{bmatrix}
A_{21d} W + B_{21d} N \\
\ldots \\
A_{2s_2 d} W + B_{2s_2 d} N
\end{bmatrix}
\]

\[
Z_{1d} = \text{diag} (R_{11}, R_{12}, \ldots, R_{1s_1}),
\]

\[
Z_{2d} = \text{diag} (R_{21}, R_{22}, \ldots, R_{2s_2})
\]

\[
Z = W Q W, \quad R_{1k} = \sum_{\theta = 1}^{s_1 - k + 1} Z_{1\theta} = \sum_{\theta = 1}^{s_1 - k + 1} W U_{1\theta} W,
\]

\[
R_{2k} = \sum_{\theta = 1}^{s_2 - k + 1} Z_{2\theta} = \sum_{\theta = 1}^{s_2 - k + 1} W U_{2\theta} W.
\]

If this condition holds, then the system is stabilised by feedback of \( K = NW^{-1} \).

Proof: Using (5) and (16) and applying the Shur's complement (2), the closed-loop system is asymptotically stable if \( \exists \) matrices
Now set $P = W^{-1}$ and apply the congruence transformation defined by $\text{diag}(W, \ldots, W, \ldots, W)$. Then employ the notation (19) to obtain (18).

### 4. A NUMERICAL EXAMPLE

We illustrate the results developed in this paper via one example where the computations involved have been undertaken using the LMI Control Toolbox [6].

Consider the following 2D system of the type (3) with 4 delays (for simplicity, we assume 2 delays in each direction) described by

\[
\begin{align*}
A_1 & = \begin{bmatrix} 0.2 & 0.17 \\ 0.4 & 0.9 \end{bmatrix}, & A_2 & = \begin{bmatrix} 0.4 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}, & B_1 & = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.8 \end{bmatrix}, \\
B_2 & = \begin{bmatrix} 0.5 & 0.2 \\ 0.6 & 0.3 \end{bmatrix}, & A_{11d} & = \begin{bmatrix} 0.7 & 0.4 \\ 0.6 & 0.5 \end{bmatrix}, & A_{12d} & = \begin{bmatrix} 0.7 & 0.6 \\ 0.1 & 0.1 \end{bmatrix}, \\
A_{21d} & = \begin{bmatrix} 0.4 & 0.9 \\ 0.1 & 0.1 \end{bmatrix}, & A_{22d} & = \begin{bmatrix} 0.9 & 0.7 \\ 0.3 & 0.4 \end{bmatrix}, & B_{11d} & = \begin{bmatrix} 0.4 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}, \\
B_{12d} & = \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{bmatrix}, & B_{21d} & = \begin{bmatrix} 0.7 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}, & B_{22d} & = \begin{bmatrix} 0.6 & 0.8 \\ 0.2 & 0.8 \end{bmatrix}.
\end{align*}
\]

Note that this particular example is unstable. Also the following matrices solve the LMI condition (18) in this case

\[
W = \begin{bmatrix} 1.7828 & -0.3422 \\ -0.3422 & 2.1173 \end{bmatrix}, \quad Z = \begin{bmatrix} 3.9047 & -0.0398 \\ -0.0398 & 3.9436 \end{bmatrix},
\]

\[
Z_{11} = \begin{bmatrix} 2.3288 & 0.0159 \\ 0.0159 & 2.3133 \end{bmatrix}, \quad Z_{22} = \begin{bmatrix} 3.0958 & 0.0478 \\ 0.0478 & 3.0491 \end{bmatrix},
\]

\[
Z_{21} = \begin{bmatrix} 2.3288 & 0.0159 \\ 0.0159 & 2.3133 \end{bmatrix}, \quad Z_{22} = \begin{bmatrix} 3.0958 & 0.0478 \\ 0.0478 & 3.0491 \end{bmatrix},
\]

and

\[
N = \begin{bmatrix} -0.6356 & -1.4445 \\ -1.4445 & -0.7135 \end{bmatrix}.
\]

Hence the system (4) is asymptotically stable independent of the delays under the control law (16) with

\[
K = \begin{bmatrix} -0.5028 & -0.7635 \\ -0.2784 & -0.3820 \end{bmatrix}.
\]