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Local Reliable Control for Linear Systems with Saturating Actuators*

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Abstract

This paper considers the problem of local reliable control for continuous-time linear systems with saturating actuators and disturbance. The local stability and the performance of the designed closed-loop system is guaranteed not only when all control components are operational, but also in case of actuator outages in the preselected subset of actuators. Linear matrix inequality (LMI) method and iterative LMI (ILMI) method are proposed to design state-feedback controllers. The effectiveness of our methods is shown by an example.

1. Introduction

Reliable control is concerned with the design of a closed-loop system to maintain key properties such as stability and other performances during sensor or actuator failure. In recent years, considerable attention has been paid to the design problems of reliable linear control systems, and a number of design methods have been proposed. For example, Veillette *et al.* [12] presented a methodology for the design of reliable H_∞ control systems by means of the algebraic Riccati equation approach. The reliable linear-quadratic state-feedback control was studied in [11].

In practical systems, limited power supplies manifest as saturating actuators. Their presence may lead to serious degradation of system performance or even instability. Recently, the control problems of linear systems with saturating actuators have been widely studied. However, global and semi-global results can be obtained only when all the eigenvalues of the system matrix have non-positive real parts. When no assumption on the open-loop system stability was made, local stabilization results were obtained in [1, 3, 8, 10]. Recently, Nguyen and Jabbari [7] and Hu *et al.* [5] discussed the analysis and design method for linear systems subject to both actuator saturation and disturbance. However, to our knowledge, there is no results addressing the reliable control for linear systems with saturating actuators.

This paper considers the local reliable control problems for linear systems with saturating actuators. No assumption on the open-loop system stability is made in our study. As in [9], the saturation function considered in this paper is a general one. The problem is to find simultaneously a state-feedback control law and an associated domain of safe admissible initial states such that the local stability of the designed closed-loop system is guaranteed not only when all control components are operational, but also in case of actuators outages in the preselected subset of actuators. A technique similar to that in [6] is used. By investigating the properties of the saturation functions, a linear matrix inequality (LMI) method is presented to solve this problem. An optimal local reliable control problems is also studied. The objective of this problem is to obtain a larger stability region. Moreover, the local reliable H_∞ control problem is studied when there is bounded disturbance. The controller can be obtained by solving two matrix inequalities.

This paper is organized as follows. Section 2 provides the problem statements. The local reliable control problem and the optimal local reliable control problem are studied in Section 3 and Section 4, respectively. In Section 5, the local reliable H_∞ control problem is discussed. An example is given in Section 6 to illustrate our methods. Section 7 concludes the paper.

2. Problem Statements

The following notations are used in this paper. For a real symmetric matrix M, $M > (\geq) 0$ means that M is positive (semi-)definite, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimal and maximal eigenvalues of M, respectively. If not explicitly stated, I and 0 denote the identity matrix and the zero matrix of appropriate dimensions, respectively. Besides, all matrices are assumed to have compatible dimensions.

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Consider the following linear system with saturating actuators

$$\dot{x} = Ax + Bf(u) + B_w w \quad (1)$$
$$z = C_z x \quad (2)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^r$ and $z \in \mathbb{R}^s$ are the state, control input, disturbance and penalty, respectively. The disturbance is assumed to be bounded as $w^T w \leq 1$. $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is an m-dimensional saturation function with $f(u) = [\sigma_1(u_1), \ldots, \sigma_m(u_m)]^T$ and $\sigma_i : \mathbb{R} \longrightarrow \mathbb{R}$ are scalar saturation functions ($i = 1, \ldots, m$). The definition of scalar saturation function is given as follows.

Definition 1 *([9]) A function $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ is called a scalar saturation function if*

(C1) σ is locally Lipschitz,

(C2) $s\sigma(s) > 0, \ \forall s \neq 0$,

(C3) $\min\left\{\lim_{s\to 0^-} \frac{\sigma(s)}{s}, \lim_{s\to 0^+} \frac{\sigma(s)}{s}\right\} > 0$,

(C4) $\liminf_{|s|\to\infty} |\sigma(s)| > 0$.

Remark 1 *It was pointed out in [9] that functions $\sigma(t) = t, \arctan(t), \mathrm{sign}(t)\min\{|t|, 1\}$ are all scalar saturation functions under Definition 1. Moreover, without loss of generality, we may assume that for scalar saturation functions $\sigma_i : \mathbb{R} \longrightarrow \mathbb{R}$, there exist $\Delta_i > 0$, such that*

$$s[\sigma_i(\alpha s) - \mathrm{sign}(s)\min\{|s|, \Delta_i\}] \geq 0, \quad \forall \alpha \geq 1 \ (i = 1, \ldots, m). \quad (3)$$

Suppose that a state-feedback control to be designed must tolerate the outage of certain actuators. Let Ω denote the selected subset of actuators within which outages must be tolerated, and let Ω' denote the complementary subset of actuators, within which actuators outages are not taken into account by the design. Let m_1 denote the dimension of Ω', without loss of generality, suppose $\Omega' = \{1, \ldots, m_1\}$ and $\Omega = \{m_1 + 1, \ldots, m\}$. Let $e^T_{\Omega' i}$ $(i = 1, \ldots, m_1)$ denotes the ith standard basis of the dimension m_1, $e^T_{\Omega j}$ $(j = m_1 + 1, \ldots, m)$ denotes the $(j - m_1)$th standard basis of the dimension $m - m_1$. That is,

$$\left[e^T_{\Omega' 1}, \ldots, e^T_{\Omega' m_1}\right] = I_{m_1},$$
$$\left[e^T_{\Omega(m_1+1)}, \ldots, e^T_{\Omega m}\right] = I_{(m-m_1)}.$$

The matrix B is decomposed as

$$B = \left[\begin{array}{cc} B_{\Omega'} & B_\Omega \end{array}\right].$$

In this paper, we make no assumption on the stability of A and consider the following local reliable control problems.

Local Reliable Control Problem (LRCP): Suppose $w = 0$ in (1). Given $\{A, B, \Delta_i(i = 1, \ldots, m)\}$, find a controller $u = Kx$ and a set $\mathcal{D}_0 \subset \mathbb{R}^n$ with $0 \in \mathcal{D}_0$, such that when $x(0) \in \mathcal{D}_0$, the closed-loop system is asymptotically stable not only when all control components are operational, but also in case of any actuators outages in Ω.

Optimal Local Reliable Control Problem (OLRCP): Suppose $w = 0$ in (1). Given $\Delta_i(i = 1, \ldots, m)$ and A, B, find the maximal $\mathcal{D}_0 \subset \mathbb{R}^n$ together with the controller $u = Kx$ such that LRCP is solved.

Local Reliable H_∞ Control Problem (LRHCP): Given $\{A, B, B_w, C_z, \Delta_i(i = 1, \ldots, m)\}$ and a real number $\gamma > 0$, find a controller $u = Kx$ and two sets $\mathcal{D}_0, \mathcal{D}_\infty \subset \mathbb{R}^n$ with $0 \in \mathcal{D}_\infty \subset \mathcal{D}_0$, such that the following three conditions are satisfied not only when all control components are operational, but also in case of any actuators outages in Ω.

(P1) when $w = 0$ and $x(0) \in \mathcal{D}_0$, the closed-loop system is asymptotically stable.

(P2) when $w \neq 0, w^T w \leq 1$ and $x(0) \in \mathcal{D}_0$, the trajectory of the closed-loop system will arrive $\mathcal{D}_\infty$ and stay in $\mathcal{D}_\infty$.

(P3) the L_2-gain of the closed-loop system is less than γ, that is, for each input $w(\cdot) \in L_2[0, \infty)$ with $w^T w \leq 1$, the response $z(\cdot)$ of the closed-loop system from the initial state $x(0) = 0$ is such that

$$\int_0^\infty \|z(t)\|^2 dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2 dt.$$

3. Local Reliable Control

In this section, we consider the local reliable control problem (LRCP). The following lemma is similar with Lemma 1 in [6] and the proof is omitted here.

Lemma 1 *Suppose $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is the m-dimensional saturation function in (1), then for any $2I \geq W = \mathrm{diag}[w_1, w_2, \ldots, w_m] > 0, R = \mathrm{diag}[r_1, r_2, \ldots, r_m] > 0, N = \mathrm{diag}[n_1, n_2, \ldots, n_m] > 0$, and $y = [y_1, y_2, \ldots, y_m]^T \in \mathbb{R}^m$, if*

$$\forall i \in \{1, \ldots, m\}, \ \left|\frac{y_i}{r_i}\right| \leq \frac{2\Delta_i}{w_i}, \quad (4)$$

then

$$2y^T N f(R^{-1}y) \geq y^T N W R^{-1} y.$$

The following theorem gives a solution of LRCP.

Theorem 1 *Suppose $(A, B_{\Omega'})$ is stabilizable and $B_{\Omega'}$ has no zero column. If there exist $\hat{P} \in \mathbb{R}^{n\times n} > 0$ and diagonal matrix $E \in \mathbb{R}^{m_1\times m_1} > 0$ such that the following linear matrix inequality (LMI) holds*

$$A\hat{P} + \hat{P}A^T - B_{\Omega'} E B^T_{\Omega'} < 0, \quad (5)$$

then the controller

$$u = -\left[\begin{array}{cc} R^{-1}_{\Omega'} & 0 \\ 0 & R^{-1}_{\Omega} \end{array}\right] B^T \hat{P}^{-1} x \quad (6)$$

and the set

$$\mathcal{D}_0 = \left\{x \in \mathbb{R}^n : x^T \hat{P}^{-1} x \leq \lambda^2_{\min}(\Gamma_{\hat{P}^{-1}} E^{-1})\right\} \quad (7)$$

is a solution of LRCP, where

$$\Gamma_{\hat{P}^{-1}} = \text{diag}\left[\frac{2\Delta_i}{\sqrt{e_{\Omega' i}^T B_{\Omega'}^T \hat{P}^{-1} B_{\Omega'} e_{\Omega' i}}}\right] \quad (8)$$

and $R_{\Omega'} > 0$ *and* $R_\Omega > 0$ *are arbitrarily chosen diagonal matrices such that* $2E^{-1} \geq R_{\Omega'}$.

Proof. Denote

$$P = \hat{P}^{-1}, \quad R = \begin{bmatrix} R_{\Omega'} & 0 \\ 0 & R_\Omega \end{bmatrix},$$

then the condition (5) becomes

$$PA + A^T P - PB_{\Omega'} E B_{\Omega'}^T P < 0, \quad (9)$$

and the controller (6) becomes

$$u = -R^{-1} B^T P x.$$

Since actuator outages may occur in Ω, the closed-loop system can be expressed as

$$\dot{x} = Ax + BNf(-R^{-1}B^T P x). \quad (10)$$

where $N = \text{diag}[I, N_\Omega]$ denotes the gain matrix that may be inserted into the feedback paths, and N_Ω with $0 \leq N_\Omega \leq I$ is a diagonal matrix.
Consider the Lyapunov function

$$V(x) = x^T P x,$$

the derivative of $V(x)$ along the trajectories of system (10) is

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T(PA + A^T P)x + 2x^T PBNf(-R^{-1}B^T P x) \end{aligned}$$

Define

$$W_{\Omega'} = ER_{\Omega'}$$

and choose diagonal matrix W_Ω with $2I \geq W_\Omega > 0$ and small enough such that

$$\begin{aligned} &\min_{j\in\Omega} \left\{\min\left\{x^T P x : x \in \mathbb{R}^n, \left|e_{\Omega j}^T W_\Omega R_\Omega^{-1} B_\Omega^T P x\right| = 2\Delta_j\right\}\right\} \\ &\geq \max_{i\in\Omega'} \left\{\min\left\{x^T P x : x \in \mathbb{R}^n, \left|e_{\Omega' i}^T E B_{\Omega'}^T P x\right| = 2\Delta_i\right\}\right\}. \end{aligned} \quad (11)$$

Denote

$$W = \begin{bmatrix} W_{\Omega'} & 0 \\ 0 & W_\Omega \end{bmatrix}.$$

Since $2E^{-1} \geq R_{\Omega'} > 0$, we have $2I \geq W_{\Omega'} > 0$ and hence $2I \geq W > 0$. From Lemma 1, if

$$\begin{aligned} &\forall i \in \Omega', \left|e_{\Omega' i}^T E B_{\Omega'}^T P x\right| = \left|e_{\Omega' i}^T W_{\Omega'} R_{\Omega'}^{-1} B_{\Omega'}^T P x\right| \leq 2\Delta_i; \\ &\forall j \in \Omega, \left|e_{\Omega j}^T W_\Omega R_\Omega^{-1} B_\Omega^T P x\right| \leq 2\Delta_j, \end{aligned} \quad (12)$$

then

$$2x^T PBNf(R^{-1}B^T P x) \geq x^T PBNWR^{-1}B^T P x.$$

So under condition (12), we have

$$\begin{aligned} \dot{V}(x) &\leq x^T(PA + A^T P)x - x^T PBNWR^{-1}B^T P x \\ &= x^T(PA + A^T P - PB_{\Omega'} W_{\Omega'} R_{\Omega'}^{-1} B_{\Omega'}^T P \\ &\quad - PB_\Omega N_\Omega W_\Omega R_\Omega^{-1} B_\Omega^T P)x \\ &\leq x^T(PA + A^T P - PB_{\Omega'} W_{\Omega'} R_{\Omega'}^{-1} B_{\Omega'}^T P)x \\ &= x^T(PA + A^T P - PB_{\Omega'} E B_{\Omega'}^T P)x \end{aligned}$$

By condition (9), $x \neq 0 \Longrightarrow \dot{V}(x) < 0$.
Because of (11) and (12), system (10) is asymptotically stable when the initial condition $x(0)$ is in the set

$$\mathcal{D}_0 = \left\{x \in \mathbb{R}^n : x^T P x \leq V_0\right\}$$

where

$$V_0 = \min_{i\in\Omega'} V_i$$

and

$$V_i = \min\left\{x^T P x : x \in \mathbb{R}^n, \left|e_{\Omega' i}^T E B_{\Omega'}^T P x\right| = 2\Delta_i\right\}.$$

By the properties of quadratic forms (see e.g. Lemma 1 in [4]), we have

$$\begin{aligned} V_i &= \min\left\{x^T P x : x \in \mathbb{R}^n,\ e_{\Omega' i}^T E B_{\Omega'}^T P x = 2\Delta_i\right\} \\ &= \frac{(2\Delta_i)^2}{(e_{\Omega' i}^T E B_{\Omega'}^T P) P^{-1} (P B_{\Omega'} E e_{\Omega' i})} \\ &= \frac{4\Delta_i^2}{e_{\Omega' i}^T E B_{\Omega'}^T P B_{\Omega'} E e_{\Omega' i}}. \end{aligned}$$

Denote

$$\Gamma_P = \text{diag}\left[\frac{2\Delta_i}{\sqrt{e_{\Omega' i}^T B_{\Omega'}^T P B_{\Omega'} e_{\Omega' i}}}\right]. \quad (13)$$

Since E is diagonal, we have

$$V_0 = \min_{i\in\Omega'} V_i = \lambda_{\min}^2(\Gamma_P E^{-1}) = \lambda_{\min}^2(\Gamma_{\hat{P}^{-1}} E^{-1}).$$

The proof is completed. ∎

When the eigenvalues of A have non-positive real parts, a reliable *semi-global stabilization* ([9]) result can be obtained, as shown in the following corollary.

Corollary 1 *Suppose the eigenvalues of A have non-positive real parts, $(A, B_{\Omega'})$ is stabilizable and $B_{\Omega'}$ has no zero column, then for any bounded subset $\mathcal{B}_0 \subset \mathbb{R}^n$, there exists a controller $u = Kx$ such that $\mathcal{B}_0$ is contained in the stability region of the closed-loop system, not only when all control components are operational, but also in case of any actuators outages in Ω.*

Proof. Since $(A, B_{\Omega'})$ is stabilizable and the eigenvalues of A have non-positive real parts, from Lemma 1 in [9], the unique solution P_ε of Riccati equation

$$P_\varepsilon A + A^T P_\varepsilon - P_\varepsilon B_{\Omega'} B_{\Omega'}^T P_\varepsilon + \varepsilon I = 0,\ \varepsilon > 0$$

satisfies

$$\lim_{\varepsilon\to 0} P_\varepsilon = 0.$$

Because P_ε and $E = I$ satisfy condition (9), from Theorem 1, if we choose the controller as

$$u = -\begin{bmatrix} R_{\Omega'}^{-1} & 0 \\ 0 & R_\Omega^{-1} \end{bmatrix} B^T P_\varepsilon x,$$

where $R_{\Omega'} > 0$ and $R_\Omega > 0$ are arbitrarily chosen diagonal matrices such that $2I \geq R_{\Omega'}$, then

$$\mathcal{D}_{0\varepsilon} = \left\{x \in \mathbb{R}^n : x^T P_\varepsilon x \leq \lambda_{\min}^2(\Gamma_{P_\varepsilon})\right\}$$

is contained in the stability region. From (13),

$$\lim_{\varepsilon\to 0} \lambda_{\min}(\Gamma_{P_\varepsilon}) = \infty.$$

Hence, for any bounded subset $\mathcal{B}_0 \subset \mathbb{R}^n$, we can choose

ε sufficiently small such that $\mathcal{B}_0 \subset \mathcal{D}_{0\varepsilon}$, the proof is completed. ■

4. Optimal Local Reliable Control

In this section, we consider the optimal local reliable control problem (OLRCP).
By Theorem 1, OLRCP is to maximize

$$\begin{aligned} \mathcal{D}_0 &= \{x \in \mathbb{R}^n : x^T P x \leq \lambda^2_{\min}(\Gamma_P E^{-1})\} \\ &= \left\{ x \in \mathbb{R}^n : x^T P x \leq \frac{1}{\lambda^2_{\max}(E\Gamma_P^{-1})} \right\} \\ &= \{x \in \mathbb{R}^n : x^T \left[\lambda^2_{\max}(E\Gamma_P^{-1})P\right] x \leq 1\} \end{aligned}$$

subject to (9).
Since P, E satisfy (9) if and only if for any $a > 0$, $\tilde{P} = aP$, $\tilde{E} = \frac{1}{a}E$ also satisfy (9), and

$$\begin{aligned} \lambda^2_{\max}(E\Gamma_P^{-1}) &= \tfrac{1}{a}\lambda^2_{\max}(\tilde{E}\Gamma_{\tilde{P}}^{-1}), \\ \lambda^2_{\max}(E\Gamma_P^{-1})P &= \lambda^2_{\max}(\tilde{E}\Gamma_{\tilde{P}}^{-1})\tilde{P}, \end{aligned} \qquad (14)$$

without loss of generality, we can assume $\lambda^2_{\max}(E\Gamma_P^{-1}) = 1$. Now OLRCP is to maximize

$$\mathcal{D}_0 = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$$

subject to (9) and

$$E \leq \Gamma_P. \qquad (15)$$

If the problem to maximize the largest ball that contained in $\mathcal{D}_0$ is considered, then we should minimize $\lambda_{\max}(P)$ and the following result can be obtained.

Theorem 2 *Suppose $\hat{P} \in \mathbb{R}^{n\times n} > 0$, diagonal matrix $E \in \mathbb{R}^{m_1\times m_1} > 0$, $\lambda \in \mathbb{R} > 0$ are the solutions of the following matrix inequality (MI) problem*

$$\max \lambda \text{ subject to (5) and}$$

$$\hat{P} \geq \lambda I \qquad (16)$$

$$E \leq \Gamma_{\hat{P}^{-1}}. \qquad (17)$$

Then the largest ball contained in the stability region can be obtained by

$$\mathcal{B}_{\max} = \{x \in \mathbb{R}^n : x^T x \leq \lambda\}. \qquad (18)$$

The corresponding controller is given by (6) where $R_{\Omega'} > 0$ and $R_\Omega > 0$ are arbitrarily chosen diagonal matrices such that $2E^{-1} \geq R_{\Omega'}$.

Proof. If we denote $\hat{P} = P^{-1}$, then (9) and (15) become (5) and (17). To minimize $\lambda_{\max}(P)$ is equivalent to maximize λ that satisfies (16). Since $\mathcal{D}_0 = \{x \in \mathbb{R}^n : x^T \hat{P}^{-1} x \leq 1\}$, the largest ball $\mathcal{B}_{\max}$ can be obtained by (18). The proof is completed. ■
The MI problem in Theorem 2 is not an LMI problem because $\Gamma_{\hat{P}^{-1}}$ in (17) is not linear in $\hat{P}$. So we suggest using the following iterative LMI (ILMI) algorithm to design the controller.

Algorithm ILMI:

Step 1 Choose $\hat{P}_0 \in \mathbb{R}^{n\times n} > 0$, $\lambda_0 \in \mathbb{R} > 0$ and a tolerance $\eta \in \mathbb{R} > 0$.

Step 2 Compute $\Gamma_{\hat{P}_0^{-1}} = \operatorname{diag}\left[\frac{2\Delta_i}{\sqrt{e^T_{\Omega' i} B^T_{\Omega'} \hat{P}_0^{-1} B_{\Omega'} e_{\Omega' i}}}\right]$.

Step 3 Solve LMI problem

$$\max \lambda \text{ subject to (5), (16) and}$$

$$\hat{P} \geq \hat{P}_0 \qquad (19)$$

$$E \leq \Gamma_{\hat{P}_0^{-1}} \qquad (20)$$

to obtain $\hat{P}$.

Step 4 If $\frac{|\lambda-\lambda_0|}{\lambda_0} < \eta$, go to Step 6, else go to Step 5.

Step 5 Let $\hat{P}_0 = \hat{P}$, $\lambda_0 = \lambda$, go to Step 2.

Step 6 Obtain the controller and the largest ball by (6) and (18).

Remark 2 *By (14), a choice of the initial $\hat{P}_0$ to guarantee the feasibility of the LMI problem in Step 3 is as follows. Solve LMI (5) to obtain $\hat{P}$ and E, then let*

$$\hat{P}_0 = \lambda^2_{\max}(E\Gamma^{-1}_{\hat{P}^{-1}})\hat{P}.$$

Remark 3 *Since $\hat{P} \geq \hat{P}_0$ implies $\Gamma_{\hat{P}^{-1}} \geq \Gamma_{\hat{P}_0^{-1}}$, (19) and (20) imply (17). Hence $\hat{P}, E$ and λ obtained in Step 3 satisfies the MIs in Theorem 2. Moreover, once the LMI problem in Step 3 is feasible for a $\hat{P}_0$, it is also feasible for the next iterate defined in Step 5. Furthermore, the positive sequence $\{\lambda\}$ obtained in Algorithm ILMI is non-decreasing. If it is bounded, then the algorithm converges. If $\lambda \to +\infty$, then the system is reliably semi-global stabilizable and the algorithm can be stopped when a required λ is obtained.*

Remark 4 *If the problem to maximize the volume of $\mathcal{D}_0$ is considered, then we should minimize $\log\det(P)$ instead of $\lambda_{\max}(P)$ and similar results as in Theorem 2 can be obtained.*

5. Local Reliable H_∞ Control

In this section, we consider the local reliable H_∞ control problem (LRHCP). The following theorem gives a solution of the problem.

Theorem 3 *Suppose $(A, B_{\Omega'})$ is stabilizable and $B_{\Omega'}$ has no zero column. For a given scalar $\gamma > 0$, if there exist $P \in \mathbb{R}^{n\times n} > 0$, diagonal matrix $E \in \mathbb{R}^{m_1\times m_1} > 0$ and scalars $\beta_1, \beta_2 > 0, 1 > \varepsilon > 0$ such that the following matrix inequalities hold*

$$\begin{aligned} &PA + A^T P - PB_{\Omega'} E B^T_{\Omega'} P \\ &+ \tfrac{\beta_1}{\gamma^2} PB_w B_w^T P + \tfrac{1}{\beta_1} C_z^T C_z < 0, \end{aligned} \qquad (21)$$

$$\begin{aligned} &PA + A^T P - PB_{\Omega'} E B^T_{\Omega'} P \\ &+ \beta_2 PB_w B_w^T P + \tfrac{1}{\varepsilon \beta_2 \lambda^2_{\min}(\Gamma_P E^{-1})} P < 0, \end{aligned} \qquad (22)$$

where Γ_P is defined in (13), then the controller

$$u = -\begin{bmatrix} R_{\Omega'}^{-1} & 0 \\ 0 & R_{\Omega}^{-1} \end{bmatrix} B^T P x \quad (23)$$

and the sets

$$\mathcal{D}_0 = \{x \in \mathbb{R}^n : x^T P x \le \lambda_{\min}^2(\Gamma_P E^{-1})\} \quad (24)$$
$$\mathcal{D}_\infty = \{x \in \mathbb{R}^n : x^T P x \le \varepsilon\lambda_{\min}^2(\Gamma_P E^{-1})\} \quad (25)$$

is a solution of LRHCP, where $R_{\Omega'} > 0$ and $R_\Omega > 0$ are arbitrarily chosen diagonal matrices such that $2E^{-1} \ge R_{\Omega'}$.

Proof. Denote

$$R = \begin{bmatrix} R_{\Omega'} & 0 \\ 0 & R_{\Omega} \end{bmatrix},$$

then the controller (23) becomes

$$u = -R^{-1}B^T P x.$$

and the closed-loop system can be expressed as

$$\dot{x} = Ax + BNf(-R^{-1}B^T Px) + B_w w \quad (26)$$
$$z = C_z x \quad (27)$$

Notice that (21) is stronger than (9), when $w = 0$, closed-loop stability is ensured by Theorem 1.
Now we consider the trajectory of the closed-loop system when $w \ne 0$. The derivative of $V(x)$ along the trajectories of system (26) is

$$\dot{V}(x) = x^T(PA + A^T P)x + 2x^T PB_w w + 2x^T PBNf(-R^{-1}B^T Px).$$

$\forall x \in \mathcal{D}_0$, (12) holds and $\forall \beta_2 > 0$,

$$\begin{aligned}
&\dot{V}(x) - \frac{1}{\beta_2} w^T w \\
&= x^T(PA + A^T P)x + 2x^T PBNf(-R^{-1}B^T Px) \\
&\quad +2x^T PB_w w - \frac{1}{\beta_2} w^T w \\
&\le x^T\left(PA + A^T P\right)x - x^T PBNWR^{-1}B^T Px \\
&\quad +2x^T PB_w w - \frac{1}{\beta_2} w^T w \\
&= x^T(PA + A^T P - PB_{\Omega'}W_{\Omega'}R_{\Omega'}^{-1}B_{\Omega'}^T P \\
&\quad -PB_\Omega N_\Omega W_\Omega R_\Omega^{-1}B_\Omega^T P)x \\
&\quad -\left(\sqrt{\beta_2}B_w^T Px - \frac{1}{\sqrt{\beta_2}}w\right)^T\left(\sqrt{\beta_2}B_w^T Px - \frac{1}{\sqrt{\beta_2}}w\right) \\
&\quad +\beta_2 x^T PB_w B_w^T Px \\
&\le x^T\left(PA + A^T P + \beta_2 PB_w B_w^T P\right)x \\
&\quad -x^T(PB_{\Omega'}W_{\Omega'}R_{\Omega'}^{-1}B_{\Omega'}^T P + PB_\Omega N_\Omega W_\Omega R_\Omega^{-1}B_\Omega^T P)x \\
&\le x^T\left(PA + A^T P - PB_{\Omega'}EB_{\Omega'}^T P + \beta_2 PB_w B_w^T P\right)x
\end{aligned}$$

Now from (22),

$$\dot{V}(x) - \frac{1}{\beta_2}w^T w \le -\frac{1}{\varepsilon\beta_2\lambda_{\min}^2(\Gamma_P E^{-1})}x^T Px$$

Since $w^T w \le 1$, we have

$$\dot{V}(x) \le \frac{1}{\beta_2} - \frac{1}{\varepsilon\beta_2\lambda_{\min}^2(\Gamma_P E^{-1})}x^T Px.$$

If $x^T Px > \varepsilon\lambda_{\min}^2(\Gamma_P E^{-1})$, i.e. $x \notin \mathcal{D}_\infty$, then $\dot{V}(x) < 0$.
Now we have proved that

$$\forall x \in \mathcal{D}_0 \backslash \mathcal{D}_\infty, \quad \dot{V}(x) < 0.$$

So when $x(0) \in \mathcal{D}_0$, the trajectory of the closed-loop system will arrive $\mathcal{D}_\infty$ and stay in $\mathcal{D}_\infty$.
Now consider the L_2-gain of the closed-loop system. Similar to the above, $\forall x \in \mathcal{D}_0, \forall \beta_1 > 0$,

$$\begin{aligned}
&\dot{V}(x) - \frac{\gamma^2}{\beta_1}w^T w + \frac{1}{\beta_1}z^T z \\
&= x^T(PA + A^T P)x + 2x^T PBNf(-R^{-1}B^T Px) \\
&\quad +2x^T PB_w w - \frac{\gamma^2}{\beta_1}w^T w + \frac{1}{\beta_1}x^T C_z^T C_z x \\
&\le x^T(PA + A^T P - PB_{\Omega'}EB_{\Omega'}^T P \\
&\quad +\frac{\beta_1}{\gamma^2}PB_w B_w^T P + \frac{1}{\beta_1}C_z^T C_z)x.
\end{aligned}$$

By condition (21), $\forall t \in \mathbb{R}$,

$$\frac{dV(x(t))}{dt} + \frac{1}{\beta_1}\|z(t)\|^2 - \frac{\gamma^2}{\beta_1}\|w(t)\|^2 \le 0.$$

Integrating the above inequality on the interval $[0, T]$ yields

$$V(x(T)) - V(x(0)) + \tfrac{1}{\beta_1}\int_0^T \|z(t)\|^2\,dt - \tfrac{\gamma^2}{\beta_1}\int_0^T \|w(t)\|^2\,dt \le 0.$$

Since $w(\cdot) \in L_2[0, \infty)$, we have $z(\cdot) \in L_2[0, \infty)$. Taking $x(0) = 0$ and letting $T \to \infty$ yields

$$\int_0^\infty \|z(t)\|^2\,dt \le \gamma^2 \int_0^\infty \|w(t)\|^2\,dt.$$

The proof is completed. ■

Remark 5 *MI (22) contains $\lambda_{\min}^2(\Gamma_P E^{-1})$ which is nonlinear in P and E. However, as in Section 4, we can assume $\lambda_{\min}(\Gamma_P E^{-1}) = 1$ and the problem can be solved by an iterative LMI method similar to Algorithm ILMI.*

Remark 6 *When $\Omega' = \{1, \ldots, m\}$ (no possible actuator failures), the problems considered in this paper are similar with that in [5]. However, the methods used in [5] may not be suitable for reliable control problems.*

6. Example

Consider the system in (1) and (2) with

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ -2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 20.$$

The open-loop system is unstable because $\mathrm{spec}(A) = \{1.8563 \pm 0.8918i, -1.3563 \pm 0.9949i\} \not\subseteq \mathbb{C}^-$. Suppose

$$\Omega' = \{1,2,3\}\,,\ \Omega = \{4\}\,.$$

It is easy to see that $B_{\Omega'}$ has no zero column. Moreover, it is easy to test that $(A, B_{\Omega'})$ is stabilizable.
Consider optimal reliable control problem, using Algorithm ILMI we obtain

$$\mathcal{B}_{\max} = \{x \in \mathbb{R}^n : x^T x \le \lambda = 170.7389\}$$

and the corresponding controller can be chosen as

$$u = \begin{bmatrix} -0.9026 & 0.0999 & 0.0060 & 0.0233 \\ -0.7605 & 0.0507 & -0.1226 & -0.1323 \\ 0.8034 & -0.2820 & -0.7432 & -0.8982 \\ 0.0007 & -0.0002 & -0.0005 & -0.0006 \end{bmatrix} x.$$

For the local reliable H_∞ control problem,

$$\begin{aligned} P &= \begin{bmatrix} 0.0180 & 0.0002 & -0.0051 & -0.0038 \\ 0.0002 & 0.0033 & 0.0007 & 0.0002 \\ -0.0051 & 0.0007 & 0.0054 & 0.0029 \\ -0.0038 & 0.0002 & 0.0029 & 0.0066 \end{bmatrix}, \\ E &= \mathrm{diag}\,[485.5409, 710.1207, 531.7968]\,, \\ \beta_1 &= 428.2413, \quad \beta_2 = 10.0213, \quad \varepsilon = 0.5367 \end{aligned}$$

is a feasible solution of the MIs (21) and (22). Choose

$$R_{\Omega'} = \mathrm{diag}\,[0.0041, 0.0028, 0.0038]\,, \quad R_\Omega = 1,$$

we obtain the controller

$$u = \begin{bmatrix} -3.1317 & -0.2185 & -0.0728 & 0.2185 \\ -5.0419 & -0.1420 & 0.7811 & -0.9942 \\ 2.3133 & -1.1168 & -2.3931 & -2.5792 \\ -0.0002 & -0.0033 & -0.0007 & -0.0002 \end{bmatrix} x$$

and the two sets

$$\begin{aligned} \mathcal{D}_0 &= \{x \in \mathbb{R}^4 : x^T P x \le 0.1866\}\,, \\ \mathcal{D}_\infty &= \{x \in \mathbb{R}^4 : x^T P x \le 0.1002\}\,. \end{aligned}$$

7. Conclusion

Local reliable control for continuous-time linear systems with saturating actuators were studied in this paper. The state-feedback control law and estimated stability region of the closed-loop system can be obtained simultaneously by solving an LMI problem. An iterative LMI method is given to design state feedback controllers such that the stability region of the closed-loop system is enlarged. The local reliable H_∞ control problem is also studied and it is related with the solution of two matrix inequalities. It should be noted that the number of LMIs (MIs) will not increase when the dimension of the system increases, which is different from the results in [2], [3] and [5]. Thus our method is likely to be more useful to high dimensional systems. Further research works will be focused on the optimal local reliable H_∞ control problems for linear systems with saturating actuators.

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