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LMI based stability analysis and controller design for a class of 2D discrete linear systems

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Abstract

Discrete linear repetitive processes are a distinct class of 2D linear systems with applications in areas ranging from long-wall coal cutting through to iterative learning control schemes. The main feature which makes them distinct from other classes of 2D linear systems is that information propagation in one of the two distinct directions only occurs over a finite duration. In this paper we give an LMI based interpretation of stability for the sub-class of so-called discrete linear repetitive processes, both open loop and closed loop under a well defined practically relevant control law, and then apply this theory to solve currently open problems relating to robustness and stability margins for these processes. Also it is shown that the LMI approach to the computation of the stability margins for these processes can be combined with the recently developed concept of a pole for them to link these margins to expected performance - a key feature which is missing from the analysis of stability margins currently available in the 2D systems literature.

1 Introduction

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction.

To introduce a formal definition, let $\alpha < \infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p)$, $0 \leq p \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the next pass profile $y_{k+1}(p)$, $0 \leq p \leq \alpha$, $k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations [1]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications of repetitive processes include classes of iterative learning control (ILC) schemes [2] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [3].

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass to pass and along a given pass. In seeking a rigorous foundation on which to develop a control theory for these processes it is natural to attempt to exploit structural links which exist between, in particular, the sub-class of so-called discrete linear repetitive processes and 2D linear systems described by the extensively studied Roesser [4] or Fornasini Marchesini [5] state space models. Discrete linear repetitive processes are distinct from such 2D linear systems in the sense that information propagation in one of the two separate directions (along the pass) only occurs over a finite duration.

A rigorous stability theory for linear repetitive processes has been developed. This theory [6] is based on an abstract model in a Banach space setting which includes all such processes as special cases. Also the results of applying this theory to a wide range of cases have been reported, including the processes considered here. This has resulted in stability tests that can be implemented by direct application of well known 1D linear systems tests.

Despite this progress, there remains much work to be done for these processes before a 'mature' control systems theory for them can be achieved. In this paper, we give an LMI based interpretation of stability for the processes considered, both open loop and closed loop under a well defined practically relevant control law, and then apply this theory to solve currently open problems relating to robustness and stability margins for them. Also it is shown that the LMI approach to the computation of the stability margins for these processes can be combined with the recently developed concept of a pole for them to link these margins to expected performance - a key feature which is missing from the analysis of stability margins currently available in the 2D systems literature see, for example, [8]. We begin in the next section by giving the

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necessary background results.

2 Background

Following Rogers and Owens [6] the state-space model of a discrete non-unit memory linear repetitive process has the following form over $0 \leq p \leq \alpha$, $k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{1}$$

Here on pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ vector pass profile, and $u_k(p)$ is the $r \times 1$ vector of control inputs. To complete the process description, it is necessary to specify the 'initial conditions' - termed the boundary conditions here, i.e. the state initial vector on each pass and the initial pass profile. Here we assume these to be of the form $x_{k+1}(0) = 0$, $k \geq 0$, and $y_0(p) = f(p)$, where $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

The the abstract model based stability theory for linear constant pass length repetitive processes can be found in [6]. This consists of two distinct concepts termed asymptotic stability and stability along the pass respectively. Of these, the former is a necessary condition for the latter which, in effect, demands that bounded input sequences produce bounded sequences of pass profiles independent of the pass length. (Here bounded is defined in terms of the norm on the underlying function space.)

In the case of processes defined by (1) (with the assumed boundary conditions), several equivalent sets of necessary and sufficient conditions for stability along the pass have been reported (see, for example, [6]) but here it is the following set which is required.

Theorem 1 *Suppose that the pair $\{C, A\}$ is observable and the pair $\{A, B_0\}$ is controllable. Then discrete linear repetitive processes defined by (1) are stable along the pass if, and only if, the 2D characteristic polynomial*

$$C(z_1, z_2) := \det \begin{bmatrix} I_n - z_1 A & -z_1 B_0 \\ -z_2 C & I_m - z_2 D_0 \end{bmatrix} \tag{2}$$

satisfies

$$C(z_1, z_2) \neq 0, \forall (z_1, z_2) \in \overline{U}^2 \tag{3}$$

where

$$\overline{U}^2 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\} \tag{4}$$

Note that (3) in this theorem gives the necessary conditions $r(D_0) < 1$ and $r(A) < 1$ which (given their relative simplicity) should be tested before proceeding further with any stability analysis.

3 LMI based stability analysis and controller design

In this work, a crucial result is the following whose proof is well known.

Lemma 1 *Given constant matrices W, L, V of appropriate dimensions where $W = W^T$ and $V = V^T > 0$, then*

$$W + L^T V L < 0 \tag{5}$$

if, and only if,

$$\begin{bmatrix} W & L^T \\ L & -V^{-1} \end{bmatrix} < 0 \tag{6}$$

or, equivalently,

$$\begin{bmatrix} -V^{-1} & L \\ L^T & W \end{bmatrix} < 0 \tag{7}$$

The matrix $W + L^T V L$ is known as the Schur complement of V.

Now define the following matrices from the state space model (1).

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix} \tag{8}$$

Then from [7] we have the following sufficient condition for stability along the pass of processes defined by (1).

Theorem 2 *Suppose that the controllability and observability assumptions of Theorem 1 hold. Then discrete linear repetitive processes defined by (1) are stable along the pass if $\exists$ matrices $P = P^T > 0$ and $Q = Q^T > 0$ satisfying the following LMI*

$$\begin{bmatrix} \widehat{A}_1^T P \widehat{A}_1 + Q - P & \widehat{A}_1^T P \widehat{A}_2 \\ \widehat{A}_2^T P \widehat{A}_1 & \widehat{A}_2^T P \widehat{A}_2 - Q \end{bmatrix} < 0 \tag{9}$$

In terms of the design of control schemes for discrete linear repetitive processes, most work has been done in the ILC area. Here it has become clear that a very power class of control laws comes from using feedback action on the current pass augmented by feedforward action from the previous pass. Here we consider a control law of the form over $0 \leq p \leq \alpha$, $k \geq 0$

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) = K \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix} \tag{10}$$

This results in the following sufficient condition for closed loop stability along the pass

$$C_c(z_1, z_2) \neq 0, \forall (z_1, z_2) \in \overline{U}^2 \tag{11}$$

where

$$C_c(z_1, z_2) := \det \begin{bmatrix} I_n - z_1 \hat{A} & -z_1 \hat{B}_0 \\ -z_2 \hat{C} & I_m - z_2 \hat{D}_0 \end{bmatrix} \tag{12}$$

where $\hat{A} = A + BK_1$, $\hat{B}_0 = B_0 + BK_2$, $\hat{C} = C + DK_1$, $\hat{D}_0 = D_0 + DK_2$.

Now introduce the matrices

$$\hat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} 0 \\ D \end{bmatrix} \tag{13}$$

Then again from [7] we have the following result.

Theorem 3 *Suppose that a discrete linear repetitive process defined by (1) is subjected to a control law of the form (10). Then (assuming the controllability and observability assumptions of Theorem 1 hold closed-loop) the resulting closed-loop system is stable along the pass if* $\exists$ *matrices* $Y = Y^T > 0$, $Z = Z^T > 0$, *and* N *such that*

$$\begin{bmatrix} Z-Y & 0 & \hat{W}_{13} \\ 0 & -Z & \hat{W}_{23} \\ \hat{W}_{13}^T & \hat{W}_{23}^T & -Y \end{bmatrix} < 0 \tag{14}$$

where

$$\hat{W}_{13} = Y\hat{A}_1^T + N^T\hat{B}_1^T,\ \hat{W}_{23} = Y\hat{A}_2^T + N^T\hat{B}_2^T$$

Also a stabilizing K *for the control law (10) is given by*

$$K = NY^{-1} \tag{15}$$

Proof: The proof is based on first interpreting (9) for the process state space model obtained from applying the control law. Then applying the Schur complement, the congruence transform defined by $diag(P^{-1}, P^{-1}, P^{-1})$, and finally the substitutions

$$Y = P^{-1},\ Z = P^{-1}QP^{-1} \tag{16}$$

yields the condition stated in the theorem. ∎

4 Robustness

In this section, we develop an LMI approach to stability analysis in the presence of uncertainty in the process definition. In particular, introduce the so-called augmented process and input matrices respectively as

$$\Phi = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix},\ \Psi = \begin{bmatrix} B \\ D \end{bmatrix} \tag{17}$$

Then here we treat the case when these matrices are subject to additive perturbations defined as follows

$$\Phi_p \ := \ \Phi + \Delta\Phi \tag{18}$$

$$\Psi_p \ := \ \Psi + \Delta\Psi \tag{19}$$

where

$$\Delta\Phi = \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix},\ \Delta\Psi = \begin{bmatrix} \Delta B \\ \Delta D \end{bmatrix} \tag{20}$$

Also we assume that the uncertainties here have the following typical structure

$$\begin{bmatrix} \Delta\Phi & \Delta\Psi \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F \begin{bmatrix} E_1 & E_2 \end{bmatrix} \tag{21}$$

where the matrices on the right-hand side are of compatible dimensions and also $F^TF \leq I$.

Now introduce the following matrices.

$$\begin{array}{ll} \widehat{\Delta\Phi}_1 = \begin{bmatrix} \Delta A & \Delta B_0 \\ 0 & 0 \end{bmatrix} & \widehat{\Delta\Phi}_2 = \begin{bmatrix} 0 & 0 \\ \Delta C & \Delta D_0 \end{bmatrix} \\ \widehat{\Delta\Psi}_1 = \begin{bmatrix} \Delta B \\ \Delta 0 \end{bmatrix} & \widehat{\Delta\Psi}_2 = \begin{bmatrix} 0 \\ \Delta D \end{bmatrix} \end{array} \tag{22}$$

Then we can write $\Delta\Phi$ and $\Delta\Psi$ in the form

$$\Delta\Phi = \widehat{\Delta\Phi}_1 + \widehat{\Delta\Phi}_2 = \hat{H}_1FE_1 + \hat{H}_2FE_1 \tag{23}$$

$$\Delta\Psi = \widehat{\Delta\Psi}_1 + \widehat{\Delta\Psi}_2 = \hat{H}_1FE_2 + \hat{H}_2FE_2 \tag{24}$$

where

$$\hat{H}_1 = \begin{bmatrix} H_1 \\ 0 \end{bmatrix},\ \hat{H}_2 = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \tag{25}$$

The LMI sufficient condition for stability along the pass given in Theorem 2 is equivalent to the existence of matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$\hat{A}^TP\hat{A} + \hat{Q} < 0 \tag{26}$$

where

$$\hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \end{bmatrix},\ \hat{Q} = \begin{bmatrix} P-Q & 0 \\ 0 & -Q \end{bmatrix} \tag{27}$$

and we now have the following result.

Theorem 4 *Discrete linear repetitive processes of the form (1) with the uncertainty structure defined above is stable along the pass if* $\exists P = P^T > 0,\ Q = Q^T > 0$ *such that*

$$(\hat{A} + \hat{H}\hat{F}\hat{E}_1)^T P (\hat{A} + \hat{H}\hat{E}\hat{E}_1) + \hat{Q} < 0 \tag{28}$$

where

$$\hat{H} = \begin{bmatrix} \hat{H}_1 & \hat{H}_2 \end{bmatrix},\ \hat{F} = I_2 \otimes F,\ \hat{E}_1 = I_2 \otimes E_1 \tag{29}$$

where $\otimes$ *denotes the matrix Kronecker product.*

Also it can be shown that, for any choice of Q, $\exists P = P^T > 0$ such that (28) holds if, and only if, $\exists$ a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} -P^{-1} + \varepsilon\hat{H}\hat{H}^T & \hat{A} \\ \hat{A}^T & \varepsilon^{-1}\hat{E}_1^T\hat{E}_1 + \hat{Q} \end{bmatrix} < 0 \tag{30}$$

Now we have the following result which gives a sufficient condition expressed in terms of an LMI, for stability along the pass under the uncertainty structure defined above.

Theorem 5 *Discrete linear repetitive processes of the form defined by (1) with the uncertainty structure defined above are stable along the pass if* $\exists Y = Y^T > 0$ *and* $Z = Z^T > 0$ *such that the following LMI holds*

$$\begin{bmatrix} -Y & \hat{A}_1Y & \hat{A}_2Y & \varepsilon\hat{H}_1 & \varepsilon\hat{H}_2 & 0 & 0 \\ Y\hat{A}_1^T & Z-Y & 0 & 0 & 0 & YE_1^T & 0 \\ Y\hat{A}_2^T & 0 & -Z & 0 & 0 & 0 & YE_1^T \\ \varepsilon\hat{H}_1^T & 0 & 0 & -\varepsilon I & 0 & 0 & 0 \\ \varepsilon\hat{H}_2^T & 0 & 0 & 0 & -\varepsilon I & 0 & 0 \\ 0 & E_1Y & 0 & 0 & 0 & -\varepsilon I & 0 \\ 0 & 0 & E_1Y & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{31}$$

It is also possible to derive a solution of the stabilization problem under the uncertainty structure considered here. This can again be found in [7].

5 Stability Margins

In the design of discrete linear repetitive processes it is also clearly of interest to determine if a stable along the pass example can retain this property in presence of process parameter variations. As for 2D discrete linear systems described by the Roesser and Fornasini Marchesini state space models (see, for example, [8]) the stability margin for discrete linear repetitive processes has been defined [6] as the shortest distance between a singularity of the process and the stability along the pass limit which is the boundary of the unit bidisc ($\overline{U}^2$). Hence, the stability margin is a measure of the degree to which the process will remain stable under variations.

The so-called generalized stability margin for discrete linear repetitive processes of the form (1) is defined as follows.

Definition 1 *The generalized stability margin σ_β for discrete linear repetitive processes of the form defined by (1) is defined as the largest bidisc in which the 2D characteristic polynomial of (2) satisfies*

$$C(z_2,z_2) \neq 0 \text{ in } \overline{U}^2_{\sigma_\beta} \tag{32}$$

where

$$\overline{U}^2_{\sigma_\beta} = \{(z_1,z_2) : |z_1| \leq (1-\sigma_\beta), |z_2| \leq 1+\beta\sigma_\beta\} \tag{33}$$

and $0 \leq \beta \leq 1$.

Note that when $\beta = 0, 1$, and 0.5 respectively, the set $\overline{U}^2_{\sigma_\beta}$ here reduces to

$$\overline{U}^2_{\sigma_1} = \{(z_1,z_2) : |z_1| \leq 1+\sigma_1, |z_2| \leq 1\} \tag{34}$$

$$\overline{U}^2_{\sigma_2} = \{(z_1,z_2) : |z_1| \leq 1, |z_2| \leq 1+\sigma_2\} \tag{35}$$

$$\overline{U}^2_{\sigma} = \{(z_1,z_2) : |z_1| \leq 1+\sigma, |z_2| \leq 1+\sigma\} \tag{36}$$

introduced and studied in, for example, [8] for 2D discrete linear systems described by the Roesser state space model. (Note that $\sigma_{0.5} = 2\sigma$ in (36).) In particular, $(1-\beta)\sigma_\beta$ and $\beta\sigma_\beta$ give the stability margins corresponding to z_1 and z_2 respectively, i.e. along the pass and pass-to-pass respectively. We will also need the following easily proven lemma in what follows.

Lemma 2 *Given $q_i \in \mathbb{R}$, $q_i > 0$, $i = 1,2$ such that*

$$\hat{C}(z_1,z_2) = \det\begin{bmatrix} I_{n_1} - \bar{q}_1 z_1 A_{11} & -\bar{q}_1 z_1 A_{12} \\ -\bar{q}_2 z_2 A_{21} & I_{n_2} - \bar{q}_2 z_2 A_{22} \end{bmatrix} \neq 0 \tag{37}$$

where $\bar{q}_1 = (1+q_1)$ and $\bar{q}_2 = (1+q_2)$, in $\overline{U}^2$, then

$$C(z_1',z_2') = \det\begin{bmatrix} I_{n_1} - z_1' A_{11} & -z_1' A_{12} \\ -z_2' A_{21} & I_{n_2} - z_2' A_{22} \end{bmatrix} \neq 0 \tag{38}$$

in $\overline{U}^2_{q}$, where

$$\overline{U}^2_q = \{(z_1',z_2') : |z_i'| \leq 1+q_i, i = 1,2\} \tag{39}$$

Theorem 6 *For a given $\hat{\alpha}$ such that $0 \leq \hat{\alpha} \leq 1$, a lower bound for the generalized stability margin $\sigma_{\hat{\alpha}}$ is given by the solution of the following quasi-convex optimization problem:*

Maximize $\sigma_{\hat{\alpha}}$ subject to $P = P^T > 0$, $Q = Q^T > 0$, $\sigma_{\hat{\alpha}} > 0$ and the LMI

$$\begin{bmatrix} Q-P & 0 & R_{13} \\ 0 & -Q & R_{23} \\ R_{13}^T & R_{23}^T & -P \end{bmatrix} < 0 \tag{40}$$

where

$$R_{13} = (1+(1-\hat{\alpha})\sigma_{\hat{\alpha}})\widehat{A}_1^T P, \; R_{23} = (1+\hat{\alpha}\sigma_{\hat{\alpha}})\widehat{A}_2^T P$$

Proof: This is immediate on applying Lemma 1 with

$$W = \begin{bmatrix} Q-P & 0 \\ 0 & -Q \end{bmatrix}, \quad V = P \tag{41}$$

and

$$L = \begin{bmatrix} (1+\sigma_1)P\widehat{A}_1 & (1+\sigma_2)P\widehat{A}_2 \end{bmatrix} \tag{42}$$

and then making use of Lemma 2. ∎

It is now possible to consider controller design with prescribed lower bound on the stability margins σ_1 and σ_2. Here we denote such bounds by σ_1^* and σ_2^* respectively and we have the following result.

Theorem 7 *Discrete linear repetitive processes are stable along the pass under control laws of the form (10) with prescribed lower bounds on the stability margins σ_1^*, σ_2^*, corresponding to z_1 and z_2 respectively, if $\exists$ matrices $Y = Y^T > 0$, $Z = Z^T > 0$, and N, such that*

$$\begin{bmatrix} Z-Y & 0 & \hat{R}_{13} \\ 0 & -Z & \hat{R}_{23} \\ \hat{R}_{13}^T & \hat{R}_{23}^T & -Y \end{bmatrix} < 0 \tag{43}$$

where

$$\hat{R}_{13} = (1+\sigma_1^*)\left(Y\widehat{A}_1^T + N^T\widehat{B}_1^T\right)$$

$$\hat{R}_{23} = (1+\sigma_2^*)\left(Y\widehat{A}_2^T + N^T\widehat{B}_2^T\right)$$

are feasible. Then a stabilizing K for the control law (10) is given by

$$K = NY^{-1} \tag{44}$$

Proof: The proof of this result is immediate from a similar argument to that used in establishing Theorem 3. Hence the details are omitted here. ∎

One major defect of this (and all other currently available see, for example, [8]) stability margins for 2D linear systems/discrete linear repetitive processes is (unlike the classical gain and phase margins for 1D linear systems) the lack of a link to the trajectories (or dynamic response) of the system/process. In the next section, we specialize some results from [9], which uses a behavioural setting to develop a trajectory based characterization for the poles of nD linear systems, to produce the first results on this key aspect for discrete linear repetitive processes.

6 Poles and Relative Stability

Since the state in pass 0 plays no role, it is convenient to relabel the state trajectories $x_{k+1}(t) \mapsto x_k(t)$ (keeping of course the same interpretation). The repetitive process (1), with $D = 0$ for simplicity, is now described by the kernel representation

$$\begin{pmatrix} I_n - z_1 A & -B & -z_1 B_0 \\ -z_2 C & 0 & I_m - z_2 D_0 \end{pmatrix} \begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0, \quad (45)$$

where here z_1 and z_2 denote the shift operators along the pass and from pass-to-pass respectively, i.e. $x_k(t)$ as follows:

$$x_k(p) := z_1 x_k(p+1), \qquad x_k(p) := z_2 x_{k+1}(p) \quad (46)$$

The components of the solutions of the system can be considered as functions from $\mathbb{N}^2$ to $\mathbb{R}$, though for purposes of interpretation they are cut off in one dimension at the pass length α.

The **poles** of the system are essentially the 2D frequencies which can arise in the state and output when the input vanishes. The behavior of all trajectories with $u = 0$ is described by the matrix

$$\begin{pmatrix} I_n - z_1 A & -z_1 B_0 \\ -z_2 C & I_m - z_2 D_0 \end{pmatrix} \quad (47)$$

and applying Theorem/Definition 4.4 from [9] we can formally define a pole as a point where (47) loses rank. In other words, the poles are given by the set

$$\mathcal{V} := \{(a_1, a_2) \in \mathbb{C}^2 \mid C(a_1, a_2) = 0\}, \quad (48)$$

where $C(z_1, z_2)$ is the polynomial given in (2). The set $\mathcal{V}$ is called the **pole variety** of the system.

Since in this case the pole variety is given by the vanishing of a single 2D non-unit polynomial, it is guaranteed to be a one-dimensional geometric set in 2D complex space, that is, a union of curves. In particular, the pole variety cannot be a finite set. Note also that the pole variety is a complex variety, even though the entries of the matrices A, B_0, C and D_0 are real. This is essential in order to capture the full exponential-type dynamics of the system.

Poles can be interpreted in terms of exponential trajectories [9], which in the case of repetitive processes have a clear physical interpretation. Take therefore a point $(a_1 = \frac{1}{r_1} e^{i\theta_1}, a_2 = \frac{1}{r} e^{i\theta}) \in \mathbb{C}^2$. Then (a_1, a_2) is a pole of the system if and only if there exists an 'exponential trajectory' in the system having the form

$$\begin{aligned} x_k'(p) &= x_{00}^1 (\frac{1}{r_1})^p (\frac{1}{r})^k \cos(\theta_1 p + \theta k) \\ &+ x_{00}^2 (\frac{1}{r_1})^p (\frac{1}{r})^k \sin(\theta_1 p + \theta k) \quad (49) \\ y_k'(p) &= y_{00}^1 (\frac{1}{r_1})^p (\frac{1}{r})^k \cos(\theta_1 p + \theta k) \\ &+ y_{00}^2 (\frac{1}{r_1})^p (\frac{1}{r})^k \sin(\theta_1 p + \theta k) \quad (50) \\ u_k'(p) &= 0 \quad (51) \end{aligned}$$

where $x_{00}^1, x_{00}^2 \in \mathbb{R}^n$, $y_{00}^1, y_{00}^2 \in \mathbb{R}^m$.

In the case of a pole $(a_1, a_2) \in \mathbb{R}^2$, it is straightforward to construct such a trajectory. Take a_1 and a_2 to be real numbers satisfying $C(a_1, a_2) = 0$. There must then exist $x_{00} \in \mathbb{R}^n$, $y_{00} \in \mathbb{R}^m$ (at least one of which is not the zero vector) satisfying

$$\begin{pmatrix} I_n - a_1 A & -a_1 B_0 \\ -a_2 C & I_m - a_2 D_0 \end{pmatrix} \begin{pmatrix} x_{00} \\ y_{00} \end{pmatrix} = 0 \quad (52)$$

Now extend (x_{00}, y_{00}) to a system trajectory by

$$\begin{aligned} x_k'(p) &= x_{00} (\frac{1}{a_1})^p (\frac{1}{a_2})^k \quad (53) \\ y_k'(p) &= y_{00} (\frac{1}{a_1})^p (\frac{1}{a_2})^k \quad (54) \\ u_k'(p) &= 0 \quad (55) \end{aligned}$$

It is easy to check that (53)–(55) is indeed a solution of the system.

Returning to the general case (49)–(51), we see that if $|a_2| = r < 1$ then we have a non-zero exponential (or sinusoidal) state-output trajectory in the system, which tends towards infinity as the pass number increases (but may remain stable along any given pass). Conversely, if $|a_2| = r \geq 1$ for all poles (a_1, a_2), then no trajectory tends to infinity for a given value of p as the pass number increases, but there may be trajectories tending to infinity along the pass. Thus we again run up against the distinction between asymptotic stability and stability along the pass. In order to avoid having trajectories of the form (49)–(51) which are unstable either along the pass or in the k-direction, we also need to avoid poles (a_1, a_2) with $|a_1| < 1$. Equivalently, with zero input there should be no exponential/ sinusoidal state-output trajectories which tend to infinity either in the pass-to-pass direction or along the pass.

Given these results, suppose that the example under consideration is stable along the pass. Suppose also that the LMI based sufficient condition of Theorem 2 holds and the stability margins have been computed from Theorem 6. Then the analysis of this section gives information of the form of the trajectories which can arise in this example. Current work is aiming to place a full interpretation on what this means for onward analysis and controller design for these processes.

7 Conclusions

Discrete linear repetitive processes arise in a number of areas of practical and theoretic interest. Their essential unique characteristic means that they cannot be studied and controlled using either existing 1D linear systems theory or that for 2D discrete linear systems described by the well known and extensively studied Roesser and Fornasini Marchesini state space models. Instead, a distinct systems theory must be developed for them.

Previous work has established a rigorous stability theory for linear repetitive processes based on an abstract model in a Banach space setting which includes all such examples as special cases. Application of this theory to discrete linear repetitive processes has resulted in stability tests which can be implemented by direct application (suitably modified in some cases) of 1D linear systems tests. Even the 'Nyquist-like' tests which can be applied here, however, do not supply useful indicators as to expected performance either open loop or closed loop under suitable control action, e.g. the equivalents of gain and phase margins in the single-input single-output case.

Recently, the LMI approach has emerged as a potentially very powerful tool for, in particular, the design of control schemes for discrete linear repetitive processes. In this paper we have started from an LMI based interpretation of stability for them and then used this setting to solve currently open problems relating to robustness and stability margins. Also it is shown that this approach to these processes provides a (potentially very powerful) method of extracting information as to expected performance by means of the recently developed concept of a pole for nD linear systems (specialized to the processes considered here). (This feature is not present in the LMI approach to the analysis of other classes of 2D linear systems.) Finally, some numerical case studies (and associated simulation studies) on the application of the results of this paper can be found in [10].

References

[1] Edwards, J. B.: 'Stability problems in the control of multipass processes', 1974, Proceedings of The Institution of Electrical Engineers, **121**(11), pp. 1425-1431.

[2] Amann, N., Owens, D. H., and Rogers, E.: 'Predictive optimal iterative learning control', International Journal of Control, 1998, **69**(2), pp. 203-206.

[3] Roberts, P. D.: 'Stability analysis of iterative optimal control algorithms modelled as linear repetitive processes', 2000, Proceedings of The Institution of Electrical Engineers, Part D, **147**(3), pp. 229-238.

[4] Roesser, R. P.: 'A discrete state space model for linear image processing', 1975, IEEE Transactions on Automatic Control, **AC-20**(1), pp. 1-10.

[5] Fornasini, E., and Marchesini, G.: 'Doubly-indexed dynamical systems: state space models and structural properties', 1978, Mathematical Systems Theory, **12**, pp. 59-72, 1978.

[6] Rogers, E., and Owens, D. H. 'Stability Analysis for Linear Repetitive Processes', (Springer-Verlag: Berlin, 1992), Lecture Notes in Control and Information Sciences Series, Vol **175**.

[7] Rogers, E., Galkowski, K., and Owens, D. H.: 'Control Systems Theory and Applications for Linear Repetitive Processes', (Springer-Verlag: Berlin 2002), Lecture Notes in Control and Information Sciences Series, to appear.

[8] Agathoklis, P. 'Lower bounds for the stability margin of discrete two-dimensional systems based on the two-dimensional Lyapunov equation', 1988, IEEE Transactions on Circuits and Systems, **35**, pp. 745-749.

[9] Wood, J. Oberst, U., Rogers, E., and Owens, D. H.: A behavioural approach to the pole structure of one-dimensional and multidimensional linear systems, 2000, SIAM Journal on Control and Optimization, **38**(2), pp. 627-661.

[10] Paske, W., Sulikowski, B., Galkowski, K., Rogers, E., and Owens, D. H.: 'Numerical studies for LMI based control of differential and discrete linear repetitive processes', Technical Report, University of Zielona Gora, Poland, 2001. (Available via the 3rd author.)