Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

LMI Solution to Gain-constrained Robust Deadbeat Pole Assignment

Hei Ka Tam James Lam*

Department of Mechanical Engineering
University of Hong Kong
Pokfulam Road, HONG KONG

Abstract

A novel optimization approach to robust deadbeat control is proposed. The design problem is cast into a convex programming task in which a special measure of closed-loop eigenvalue sensitivity is minimized. Advantages of the proposed method include: *(1)* Global optimality is guaranteed when the solution set is nonempty; *(2)* Constraints on the feedback gain can be catered naturally; *(3)* Minimum-gain deadbeat control design can be readily treated.

1. Introduction

Consider a linear time-invariant MIMO discrete time system

$$x_{t+1} = Ax_t + Bu_t \quad , \quad t = 0, 1, 2, \ldots \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $x_t \in \mathbb{R}^{n\times 1}$, $u_t \in \mathbb{R}^{q\times 1}$. Assume B is of full column rank while (A, B) is reachable with reachability indices $k = k_1 \geq k_2 \geq \ldots \geq k_q$. By applying a time-invariant state-feedback law

$$u_t = Kx_t \tag{2}$$

there results the closed-loop system given by

$$x_{t+1} = (A + BK)x_t \tag{3}$$

The reachability of (A, B) implies that the spectrum of the closed-loop state matrix $M \triangleq A + BK$ can be assigned to any arbitrary set of self-conjugate complex numbers by proper choice of the feedback gain matrix $K \in \mathbb{R}^{q\times n}$. It is often desirable to assign the set of closed-loop poles at the origin of the complex plane. In such case, the closed-loop system exhibits deadbeat characteristics, with which zero-input response of the system dies down to zero in finite time steps. Furthermore, in order for a system under *Minimum-time Deadbeat Control (MTDC)*, that is, the regulation time to drive *every* initial state of the closed-loop system to the origin is made shortest, M is made similar to the block-diagonal matrix

$$J = \mathrm{diag}(J_1, J_2, \ldots, J_q) \tag{4}$$

where J_p is a Jordan block of dimension k_p (for $k_p = 0$, J_p does not appear in J).

By observing that M is nilpotent with order of nilpotence equals k ($M^k = 0$) and generally has a nontrivial Jordan structure, it is especially susceptible to spectral variation as a result of perturbation. As the feedback gain to achieve pole assignment for multi-input systems is in general non-unique, it is particularly meaningful for the non-uniqueness to be exploited in searching for a more robust eigenstructure.

2. Affine Parametrization of MTDC Feedback Gain

The idea of explicitly parametrizing the class of deadbeat regulators proposed by Funahashi *et al.* [2] is borrowed. Initially, (A, B) is transformed via $S \in \mathbb{R}^{n\times n}$ into the Luenberger canonical form

$$y_{t+1} = SAS^{-1}y_t + SBu_t \quad , \quad y_t = Sx_t \tag{5}$$

where

$$SAS^{-1} = J + E\hat{A} \quad , \quad SB = E\hat{B} \tag{6}$$

in which

$$\begin{aligned} E &= \mathrm{block\,diag}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q) \in \mathbb{R}^{n\times q} \\ \mathbf{e}_i &= (0\ldots 0\,1)^T \in \mathbb{R}^{k_i\times 1} \end{aligned}$$

$\hat{A} \in \mathbb{R}^{q\times n}$ and $\hat{B} \in \mathbb{R}^{q\times q}$ are determined uniquely by (A, B). The transformation (6) results in the closed-loop state matrix being transformed into

$$S(A + BK)S^{-1} = J + E\hat{K}$$

where

$$\hat{K} = \hat{A} + \hat{B}KS^{-1} \in \mathbb{R}^{q\times n} \tag{7}$$

From Funahashi *et al.* [2] , it is not difficult to observe that $\hat{K}$ can be expressed as

$$\hat{K} = \sum_{r=2}^{q}\sum_{s=1}^{r-1}\sum_{v=k_r+1}^{k_s} \gamma_{r,v+\sigma(s)}G_{r,v+\sigma(s)} \tag{8}$$

where $\sigma(s) = \sum_{p=1}^{s-1} k_p$ in which $G_{r,v+\sigma(s)}$ denotes a matrix in $\mathbb{R}^{q\times n}$ whose elements equal zero except the $(r, v + \sigma(s))$ element equal to one. It follows from (7) and (8) that

$$K(\mathbf{w}) = K_0 + \sum_{i=1}^{N} w_iK_i \quad , \quad \mathbf{w} = (w_1 \ldots w_N)^T \tag{9}$$

* Supported by HKU CRCG Grant

where

$$\begin{aligned} K_0 &= -(\hat{B})^{-1}\hat{A}S \\ K_i &= (\hat{B})^{-1}G_{r,v+\sigma(s)}S \quad , \quad w_i = \gamma_{r,v+\sigma(s)} \end{aligned} \tag{10}$$

with the indices r, s, v, and i being related by

$$i = \sum_{j=2}^{r-1}\sum_{l=1}^{j-1}(k_l - k_j) + (v + \sigma(s)) - sk_r \tag{11}$$

$N = nq - \sum_{p=1}^{q}(2p-1)k_p$ is the minimum number of independent free parameters w_i in parametrizing K.

3. Measure of Robustness

Recently, Lam *et al.* [3] have shown that the spectral norm of M serves as a special measure of eigenvalue sensitivity for deadbeat control systems. In the following, $||M||_2$ and $||M||_{\max}$ denotes respectively the spectral norm and the maximum norm of M (maximum absolute value of the entries in M).

Theorem 1 *Suppose M is subjected to a perturbation Δ where λ is an eigenvalue of $M+\Delta$, then we have: (I)*

$$\begin{aligned} |\lambda| &\leq \max_{i=0,\ldots,k-1}\{(k||M^i\Delta||_2)^{\frac{1}{i+1}}\} , \\ |\lambda| &\leq \max_{i=0,\ldots,k-1}\{(k||M^i||_2||\Delta||_2)^{\frac{1}{i+1}}\} , \\ |\lambda| &\leq \max_{i=0,\ldots,k-1}\{(k||M||_2^i||\Delta||_2)^{\frac{1}{i+1}}\} ; \end{aligned} \tag{12}$$

and (II)

$$|\lambda| \leq (1+\epsilon)||M^{k-1}||_2^{\frac{1}{k}}||\Delta||_2^{\frac{1}{k}} \leq (1+\epsilon)||M||_2^{\frac{k-1}{k}}||\Delta||_2^{\frac{1}{k}} \tag{13}$$

where $\epsilon = \mathcal{O}(|\lambda|) \to 0$ as $||\Delta||_2 \to 0$. ■

It turns out that variations on the perturbed closed-loop eigenvalues would be small if $||M||_2$ is minimized.

4. Robust Pole Assignment

The *Gain-constrained Robust Deadbeat Pole Assignment (GCRDPA)* problem is now formulated as finding a state feedback gain K of constrained size to assign the set of closed-loop poles into the origin of the z-plane and at the same time minimize the sensitivity of the closed-loop zero eigenvalues. The problem is cast into a constrained minimization task:

$$\underset{\mathbf{w}\in\mathcal{P}\cap\mathcal{Q}}{\text{minimize}}\ \phi(\mathbf{w}) = ||A + BK(\mathbf{w})||_2 \tag{14}$$

where

$$\mathcal{P} \triangleq \{\mathbf{w} \mid ||K(\mathbf{w})||_2 < \alpha\} \tag{15}$$

and

$$\mathcal{Q} \triangleq \{\mathbf{w} \mid ||K(\mathbf{w})||_{\max} < \beta\} \tag{16}$$

Since $\phi(\mathbf{w})$ together with $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{P}\cap\mathcal{Q}$ are all convex in $\mathbf{w}$, (14) can be readily solved by state-of-the-art convex optimization routines. Moreover, (14) can be recast into a *Semidefinite Programming (SDP)* task in which a functional z is minimized subject to a set of *Linear Matrix Inequality (LMI)* constraints [1] :

$$\underset{\mathbf{w}}{\text{minimize}}\ z \tag{17}$$

subject to

$$\begin{pmatrix} zI_n & M(\mathbf{w}) \\ M(\mathbf{w})^T & zI_n \end{pmatrix} > 0, \tag{18}$$

$$\begin{pmatrix} \alpha I_q & K(\mathbf{w}) \\ K(\mathbf{w})^T & \alpha I_n \end{pmatrix} > 0, \tag{19}$$

and

$$-\beta\boldsymbol{\zeta} < \mathbf{h} + F\mathbf{w} < \beta\boldsymbol{\zeta} \tag{20}$$

where

$F = (\ \text{vec}(K_1) \quad \text{vec}(K_2) \quad \cdots \quad \text{vec}(K_N)\ ) \in \mathbb{R}^{(q\times n)\times N}$, $\mathbf{h} = \text{vec}(K_0) \in \mathbb{R}^{(q\times n)\times 1}$, $\boldsymbol{\zeta} = (1\ldots 1)^T \in \mathbb{R}^{(q\times n)\times 1}$, and vec(.) denotes the lexicographical ordering of the elements of a matrix. Notice that (20) is a pair of polyhedron constraints which can be represented as

$$\text{diag}(\beta\boldsymbol{\zeta} - \mathbf{h} - F\mathbf{w}) > 0 \tag{21}$$

and

$$\text{diag}(\beta\boldsymbol{\zeta} + \mathbf{h} + F\mathbf{w}) > 0 \tag{22}$$

It turns out that if (18), (19), and (20) are feasible (i.e. $\mathcal{P}\cap\mathcal{Q} \neq \emptyset$), the solution produced by (17) will be globally optimal.

Remark 1 *In (15) and (16), the greatest lower bound for α and β to achieve deadbeat pole assignment can be computed using the following SDP settings.*

To determine $\inf_{\mathbf{w}}\alpha$:

$$\underset{\mathbf{w}}{\text{minimize}}\ \alpha \tag{23}$$

subject to

$$\begin{pmatrix} \alpha I_q & K(\mathbf{w}) \\ K(\mathbf{w})^T & \alpha I_n \end{pmatrix} > 0 \tag{24}$$

To determine $\inf_{\mathbf{w}}\beta$:

$$\underset{\mathbf{w}}{\text{minimize}}\ \beta \tag{25}$$

subject to

$$\text{diag}(\beta\boldsymbol{\zeta} - \mathbf{h} - F\mathbf{w}) > 0 \tag{26}$$

and

$$\text{diag}(\beta\boldsymbol{\zeta} + \mathbf{h} + F\mathbf{w}) > 0 \tag{27}$$

Notice that (23) and (25) by themselves serve as computational procedures for solving the Minimum-gain Deadbeat Pole Assignment (MGDPA) problem. ■

References

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