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<tr>
<td>Author(s)</td>
<td>Tam, PHK; Lam, J</td>
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<tr>
<td>Citation</td>
<td>American Control Conference, Philadelphia, USA, 24-26 June 1998, v. 1, p. 97-98</td>
</tr>
<tr>
<td>Issued Date</td>
<td>1998</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10722/46642">http://hdl.handle.net/10722/46642</a></td>
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LMI Solution to Gain-constrained Robust Deadbeat Pole Assignment

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Abstract
A novel optimization approach to robust deadbeat control is proposed. The design problem is cast into a convex programming task in which a special measure of closed-loop eigenvalue sensitivity is minimized. Advantages of the proposed method include: (1) Global optimality is guaranteed when the solution set is non-empty; (2) Constraints on the feedback gain can be catered naturally; (3) Minimum-gain deadbeat control design can be readily treated.

1. Introduction
Consider a linear time-invariant MIMO discrete time system

\[ x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, 2, \ldots \] (1)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}, x_t \in \mathbb{R}^{n \times 1}, u_t \in \mathbb{R}^{q \times 1}. \) Assume \( B \) is of full column rank while \( (A, B) \) is reachable with reachability indices \( k_1 \geq k_2 \geq \ldots \geq k_q. \) By applying a time-invariant state-feedback law

\[ u_t = Kx_t \] (2)

there results the closed-loop system given by

\[ x_{t+1} = (A + BK)x_t \] (3)

The reachability of \( (A, B) \) implies that the spectrum of the closed-loop state matrix \( M := A + BK \) can be assigned to any arbitrary set of self-conjugate complex numbers by proper choice of the feedback gain matrix \( K \in \mathbb{R}^{q \times n}. \) It is often desirable to assign the set of closed-loop poles at the origin of the complex plane. In such case, the closed-loop system exhibits deadbeat characteristics, with which zero-input response of the system dies down to zero in finite time steps. Furthermore, in order for a system under Minimum-Time Deadbeat Control (MTDC), that is, the regulation time to drive every initial state of the closed-loop system to the origin is made shortest, \( M \) is made similar to the block-diagonal matrix

\[ J = \text{diag}(J_1, J_2, \ldots, J_q) \] (4)

where \( J_p \) is a Jordan block of dimension \( k_p \) (for \( k_p = 0, \) \( J_p \) does not appear in \( J \)).

By observing that \( M \) is nilpotent with order of nilpotence equals \( k \) (\( M^k = 0 \)) and generally has a nontrivial Jordan structure, it is especially susceptible to spectral variation as a result of perturbation. As the feedback gain to achieve pole assignment for multi-input systems is in general non-unique, it is particularly meaningful for the non-uniqueness to be exploited in searching for a more robust eigenstructure.

2. Affine Parametrization of MTDC Feedback Gain
The idea of explicitly parametrizing the class of deadbeat regulators proposed by Funahashi et al. [2] is borrowed. Initially, \( (A, B) \) is transformed via \( S \in \mathbb{R}^{n \times n} \) into the Luenberger canonical form

\[ y_{t+1} = SAS^{-1}y_t + SBu_t, \quad y_t = Sz_t \] (5)

where

\[ SAS^{-1} = J + \hat{E} \hat{A}, \quad SB = \hat{E} \hat{B} \] (6)

in which

\[ \hat{E} = \text{block diag}(e_1, e_2, \ldots, e_q) \in \mathbb{R}^{n \times q} \]
\[ e_i = (0 \ldots 01)^T \in \mathbb{R}^{n \times 1} \]

\( \hat{A} \in \mathbb{R}^{n \times n} \) and \( \hat{B} \in \mathbb{R}^{n \times q} \) are determined uniquely by \( (A, B) \). The transformation (6) results in the closed-loop state matrix being transformed into

\[ S(A + BK)S^{-1} = J + \hat{E} \hat{K} \] (7)

where

\[ \hat{K} = \hat{A} + \hat{B} KS^{-1} \in \mathbb{R}^{n \times n} \]

From Funahashi et al. [2], it is not difficult to observe that \( \hat{K} \) can be expressed as

\[ \hat{K} = \sum_{r=2}^{q} \sum_{s=1}^{r-1} k_{r,s} \gamma_{r,s+\sigma(s)} G_{r,s+\sigma(s)} \] (8)

where \( \sigma(s) = \sum_{p=1}^{s-1} k_p \) in which \( G_{r,s+\sigma(s)} \) denotes a matrix in \( \mathbb{R}^{n \times n} \) whose elements equal zero except the \((r, s+\sigma(s))\) element equal to one. It follows from (7) and (8) that

\[ K(w) = K_0 + \sum_{i=1}^{N} w_i K_i, \quad w = (w_1 \ldots w_N)^T \] (9)
where

\[ K_0 = -(\hat{B})^{-1}\hat{A}S \]
\[ K_t = (\hat{B})^{-1}G_{r,v+\sigma(s)}S, \quad w_i = \gamma_{r,v+\sigma(s)}(10) \]

with the indices \( r, s, v, \) and \( i \) being related by

\[ i = \sum_{j=2}^{r-1} \sum_{k=1}^{j-1} (k_i - k_j) + (v + \sigma(s)) - sk_r \]

(11)

\[ N = nq - \sum_{p=1}^{q} (2p - 1)k_p \] is the minimum number of independent free parameters \( w_i \) in parametrizing \( K \).

3. Measure of Robustness

Recently, Lam et al. [3] have shown that the spectral norm of \( M \) serves as a special measure of eigenvalue sensitivity for deadbeat control systems. In the following, \( ||M||_2 \) and \( ||M||_{\text{max}} \) denotes respectively the spectral norm and the maximum norm of \( M \) (maximum absolute value of the entries in \( M \)).

**Theorem 1** Suppose \( M \) is subjected to a perturbation \( \Delta \) where \( \lambda \) is an eigenvalue of \( M + \Delta \), then we have: (I)

\[ ||\lambda|| \leq \max_{i=0,\ldots,k-1} \left\{ ||M^i\Delta||_2 \right\} \]

(12)

and (II)

\[ ||\lambda|| \leq (1 + \epsilon)||M^{k-1}||_2 \frac{||\Delta||_2}{2} \leq (1 + \epsilon)||M||_2^{k-1}||\Delta||_2 \]

(13)

where \( \epsilon = O(||\lambda||) \rightarrow 0 \) as \( ||\Delta||_2 \rightarrow 0 \).

It turns out that variations on the perturbed closed-loop eigenvalues would be small if \( ||M||_2 \) is minimized.

4. Robust Pole Assignment

The Gain-constrained Robust Deadbeat Pole Assignment (GCRDPA) problem is now formulated as finding a state feedback gain \( K \) of constrained size to assign the set of closed-loop poles into the origin of the \( z \)-plane and at the same time minimize the sensitivity of the closed-loop zero eigenvalues. The problem is cast into a constrained minimization task:

\[ \min_{w \in \mathbb{R}^{nq}} \phi(w) = ||A + BK||_2 \]

(14)

where

\[ \mathcal{P} \triangleq \{ w \mid ||K||_2 < \alpha \} \]

(15)

and

\[ \mathcal{Q} \triangleq \{ w \mid ||K||_{\text{max}} < \beta \} \]

(16)

Since \( \phi(w) \) together with \( \mathcal{P}, \mathcal{Q}, \) and \( \mathcal{P} \cap \mathcal{Q} \) are all convex in \( w \), (14) can be readily solved by state-of-the-art convex optimization routines. Moreover, (14) can be recast into a Semidefinite Programming (SDP) task in which a functional \( z \) is minimized subject to a set of Linear Matrix Inequality (LMI) constraints [1]:

\[ \min_{w} z \]

subject to

\[ \begin{pmatrix} zI_n \quad M(w) \\ M(w)^T \quad zI_n \end{pmatrix} > 0, \]

(18)

\[ \begin{pmatrix} \alpha I_n \quad K(w) \\ K(w)^T \quad \alpha I_n \end{pmatrix} > 0, \]

(19)

and

\[ -\beta \zeta < h + Fw < \beta \zeta \]

(20)

where

\[ F = (\text{vec}(K_1) \quad \text{vec}(K_2) \quad \ldots \quad \text{vec}(K_N) \quad) \in \mathbb{R}^{(q \times n) \times N}, \]

\[ h = \text{vec}(K_0) \in \mathbb{R}^{q \times n}, \quad \zeta = (1 \ldots 1)^T \in \mathbb{R}^{q \times n}, \]

and \( \text{vec}(\cdot) \) denotes the lexicographical ordering of the elements of a matrix. Notice that (20) is a pair of polyhedron constraints which can be represented as

\[ \text{diag}(\beta \zeta - h - Fw) > 0 \]

(21)

and

\[ \text{diag}(\beta \zeta + h + Fw) > 0 \]

(22)

It turns out that if (18), (19), and (20) are feasible (i.e. \( \mathcal{P} \cap \mathcal{Q} \neq \emptyset \)), the solution produced by (17) will be globally optimal.

**Remark 1** In (15) and (16), the greatest lower bound for \( \alpha \) and \( \beta \) to achieve deadbeat pole assignment can be computed using the following SDP settings.

To determine \( \inf_{\alpha} \alpha \):

\[ \min_{w} \alpha \]

subject to

\[ \begin{pmatrix} \alpha I_n \quad K(w) \\ K(w)^T \quad \alpha I_n \end{pmatrix} > 0 \]

(24)

To determine \( \inf_{\beta} \beta \):

\[ \min_{w} \beta \]

subject to

\[ \text{diag}(\beta \zeta - h - Fw) > 0 \]

(26)

and

\[ \text{diag}(\beta \zeta + h + Fw) > 0 \]

(27)

Notice that (23) and (25) by themselves serve as computational procedures for solving the Minimum-gain Deadbeat Pole Assignment (MGDPA) problem.

References

