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Control Synthesis versus Saturation Compensation for Systems with Rate and Amplitude Constraints

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Abstract
A control synthesis theory was proposed by Horowitz [2] to design a 'three degrees of freedom' controller for rate and amplitude constrained systems. Following anti-reset windup techniques, a saturation compensation structure was proposed to design compensators for given linear controllers [3]. It is shown here that the compensator can be reformulated in terms of the control synthesis theory. Conversely, the 'three degrees of freedom' controller is a special case of the compensator construction. From this analysis, shortcomings of the control synthesis theory are exposed and improvements using the compensator structure are discussed and illustrated by an example.

1 Introduction
Practical control systems often encounter both rate and amplitude constraints from the actuators, reflecting physical bounds on finite power and energy transfers. In the literature, there are a few discussions on systems subject to both rate and amplitude constraints [2,5,7], with rate and amplitude constraints taken into account at the beginning of the controller designs. Alternative approach is to extend general actuator saturation compensators [4] for single constraints to systems subject to both rate and amplitude constraints [3]. Some recent theories of general anti-reset windup (ARW) [6,8] considered multiple saturation nonlinearities in parallel only, instead of in series as in rate and amplitude constrained systems.

A 'three degrees of freedom' controller structure was proposed by Horowitz to handle both rate and amplitude constraints [2]. Quantitative feedback theory (QFT) was then used to design the controller to cater for the constraints and plant uncertainties. The synthesis theory was later extended to unstable plants [7] by restricting the controller outputs to within bounds. To this purpose a supervisory loop was introduced to adjust a nonlinear gain, inserted between the nominal controller and the actuator. Frequency domain techniques were used to design the nominal controller.

The work in [5] considered the structure of amplitude nonlinearity before the rate nonlinearity. It also introduced a supervisory loop to adjust a nonlinear gain, called the error governor, in the closed-loop system. The error governor was inserted before the controller and adjusts the system error according to predetermined bounds.

These works are viewed as control synthesis techniques as they are for controller design or to predetermine the control bounds. A compensator for both rate and amplitude constraints was proposed in [3]. Conditions for stability of the compensated system were derived and some guidelines for designing the compensators discussed. ARW methods are a posteriori techniques in that the compensators are fabricated for given controllers, which were designed assuming absence of saturation. The relationship between these two approaches is not yet available.

In this work, the compensator structure [3] is reviewed in §2. The 'three degrees of freedom' control synthesis [2] is presented and reformulated as compensator structure in §3. It turns out to be a special case of the compensator discussed in §2. Compensator design methods for these two approaches are discussed next in §4, revealing some shortcomings of the synthesis theory. An illustrative example is presented in §5.

2 A Compensator Structure
Let \( G \) be the transfer function of a plant, and the linear controller be described by

\[
v = (T/R)w - (S/R)y \tag{2.1}
\]

where \( y \) is the system output, \( w \) is the reference input and \( v \) is the controller output. \( R, S, T \) are polynomials in Laplace transform variable \( s \) or backward shift operator \( z^{-1} \). \( R \) is monic. The arguments are omitted for convenience. The amplitude and rate constraints for the actuator are

\[
\begin{align*}
\begin{cases}
\u_{\text{max}} & , \quad v \geq \u_{\text{max}} \\
\v_{\min} & , \quad \u_{\min} \leq v \leq \u_{\max} \\
\u_{\min} & , \quad v \leq \u_{\min}
\end{cases}
\end{align*}
\tag{2.2}
\]

\[
\begin{align*}
\begin{cases}
\u_{\text{max}} & , \quad \dot{v} \geq \u_{\text{max}} \\
\v_{\min} & , \quad \u_{\min} \leq v \leq \u_{\max} \\
\u_{\min} & , \quad v \leq \u_{\min}
\end{cases}
\end{align*}
\tag{2.3}
\]

where \( \{\u_{\max}, \u_{\min}\} \) are the amplitude limits, and \( \{\dot{\u}_{\max}, \dot{\u}_{\min}\} \) the rate limits, respectively.

Due to difficulties in measuring the actuator velocity, an actuator model is inserted after the controller and before the actuator (c.f. [5,7]) with known bounds of \( \{\u_{\max}, \u_{\min}\} \) and \( \{\dot{\u}_{\max}, \dot{\u}_{\min}\} \). In this manner, both the velocity and the amplitude outputs of the actuator model, \( (\u, \dot{\u}) \), are within the rate and amplitude bounds of the actuator. Saturation compensation is carried out within the actuator model.

Extending an ARW compensation framework for single-nonlinearity systems [4], rate and amplitude constrained system is compensated by two linear blocks: rate saturation by \( P_r \) and amplitude saturation by \( P_a \) [Fig.1]. \( R_2 \) models a differentiator \( R_2(s)=s \). The separation of rate and amplitude constraints allows independent
consideration and compensation of saturation effects.

From Fig. 1, the output of the actuator model is

\[ v = \frac{T}{R} w - \frac{S}{R} y + (1 + P_c) R_2 \delta_a + P_c \delta_a \]  

\[ u = \frac{T}{R} w - \frac{S}{R} y + (1 + P_c) R_2 \delta_a + (1 + P_c) \delta_a \]  

and

\[ \delta_a(t) \Delta \delta_a(t) = v(t) \Delta v(t) \]  

\[ \delta_r(t) \Delta \delta_r(t) = v(t) \Delta v(t) \]  

For convenience, let \( G_c = \frac{GS}{R_c} \) and \( G_c = \frac{GS}{R_c} \).

When there is no compensation, i.e., \( R_2 = 0 \), then the closed-loop actuator output is

\[ u = u_0 + \Delta u_r + \Delta u_a \]

and the system output is

\[ y = Gu \Delta y_0 + \Delta y_r + \Delta y_a \]

\[ y = Gu_0 + \Delta y_r + \Delta y_a = G \Delta u_r \]

\[(\delta_0, \delta_a) \] are the unconstrained linear system responses. \( \Delta y_r \) and \( \Delta y_a \) arise from amplitude saturation, \( \Delta y_r \) and \( \delta_r \) from rate saturation. When there is no compensation, i.e., \( R_2 = 0 \), in (2.8) is still affected by \( \Delta y_r \) and \( \Delta y_a \), as \( \delta_r \) and \( \delta_a \) are nonzero. The fact that output \( y \) is a linear function of \( u \) is the realizability condition, ensuring that the equivalent blocks are physically implementable.

To study the stability of rate and amplitude saturation compensated systems, equivalent systems representing Fig. 1 are derived first. From (2.4)-(2.6), the following closed-loop expressions are obtained:

\[ v = u_0 + \Delta u_r + \Delta u_a \]

\[ u_1 = u_0 + \Delta u_r + \Delta u_a \]

\[ v_1 = u_0 + \Delta u_r + \Delta u_a \]

where

\[ \Delta u_r = \frac{1}{1 + (1 + P_c) R_2} u \]

\[ \Delta u_a = \frac{1}{1 + (1 + P_c) R_2} \delta_a \]

\[ \delta_a = \frac{1}{1 + (1 + P_c) R_2} \delta_a \]

\[ \delta_r = \frac{1}{1 + (1 + P_c) R_2} \delta_r \]

From (2.7a) and (2.9), \( \delta_r \) and \( \delta_a \) are reconstructed from observations of \( u \) and \( u_1 \) as

\[ \delta_a = \frac{1}{1 + (1 + P_c) R_2} \delta_a \]

\[ \delta_r = \frac{1}{1 + (1 + P_c) R_2} \delta_r \]

Using (2.10), (2.9) simplifies to

\[ v_1 = F_a w + G_u u_1 - \delta_a \]

\[ v = G_c u_1 + G_a u \]

where the equivalent blocks are

\[ G_1 = \frac{1}{1 + P_c} \]

\[ G_2 = \frac{GS}{R_c} \]

\[ G_3 = \frac{1}{1 + P_c} \]

\[ G_4 = \frac{1}{1 + P_c} \]

\[ F_a = \frac{1}{1 + P_c} \]

(2.11) describes an equivalent configuration of Fig. 1, shown in Fig. 2, where \( N_r \) represents the rate constraint nonlinearity and \( N_a \) the amplitude constraint nonlinearity. As \( \{G_1, G_2, G_3, G_4\} \) in (2.12) have to be open-loop stable for the compensated system to be globally stabilizable, \( \{P_1, P_2, P_3, P_4\} \) must satisfy the additional condition:

\[ \frac{1}{1 + k_1 P_3} + \frac{1}{1 + k_2 P_4} = 0 \]

(3.1)

If \( k_1 = k_2 = 1 \), then (3.1) degenerates to the unconstrained linear system: \( 1 + G_a = 0 \). Furthermore, the equivalent system \( G_{sa} \) as seen by the amplitude constraint nonlinearity, is obtained by writing (2.13) in the form of

\[ 1 + k_a G_{sa} = 0 \]

\[ G_{sa} = \frac{(k_1 G_2 P_3)}{(1 + (1 - k_1) P_3 P_4)} \]

(3.2)

Asymmetry of nonlinear stability of the rate and amplitude constrained systems using (3.1)-(3.2) can be found in [3].

3 A Control Synthesis Theory

The control synthesis theory proposed by Horowitz [2] is now reviewed. To compare the two approaches, the synthesis theory is reformulated into the structure of §2.

The system configuration used by Horowitz [2] is reproduced in Fig. 3. Without loss of generality, noise disturbances are omitted here and the same notations as in Fig. 1 are used. Plant constituents \( \{P_1, P_2, P_3, P_4\} \) are assumed known prior to the control synthesis process. \( \{F_m, G_m, H_m\} \) are the 'three degrees of freedom' of the controller and are to be synthesized. From Fig. 3,

\[ v_1 = P_1 [P_2, G_m (F_m w - y) - H_m P_4 u] \]

(3.1)

and from (2.6),

\[ u = v + \delta_a \]

(3.2)

(3.1) can be written as

\[ v_1 = \frac{P_3 P_4 G_m}{1 + H_a P_3 P_4} \]

\[ v = \frac{1}{1 + H_a P_3 P_4} \]

\[ \delta_a = \frac{-H_a P_3 P_4}{1 + H_a P_3 P_4} \]

(3.3)
\[
\nu = \frac{p_3 p_4 g_4}{1 + h_4 p_2 p_3 / r_2} (f_w - y) + \frac{1 / r_2}{1 + h_4 p_2 p_3 / r_2} s_r - \frac{h_4 p_3 / r_2}{1 + h_4 p_2 p_3 / r_2} s_u
\]

Comparing (3.4) with the compensated controller in (2.4), the following relationships are obtained:

\[
\begin{align*}
\frac{S}{R} & \Delta \frac{p_3 p_4 g_4 / r_2}{1 + h_4 p_2 p_3 / r_2} ; \quad \frac{T}{R} \Delta \frac{SF_R}{R} \\
\frac{1 + p_3 / r_2}{1 + h_4 p_2 p_3 / r_2} & \Delta \frac{1 / r_2}{1 + h_4 p_2 p_3 / r_2} ; \quad \frac{P_a}{1 + h_4 p_2 p_3 / r_2} \Delta -\frac{-h_4 p_3 / r_2}{1 + h_4 p_2 p_3 / r_2}
\end{align*}
\]

From (3.6):

\[
P_a = P_r
\]

Using (3.7) and (3.8), compensator design by Horowitz [2] is to simultaneously define a common compensator \( P_a \) or \( P_r \) for both rate and amplitude saturations. Any modifications to the direct control synthesis by Horowitz [2] is to satisfy the rate and amplitude constraints, as the compensators can perform well under either rate or amplitude constraints only. Since the system under consideration is rate constrained first, it is logical to expect that performance of the rate compensator \( P_r \) would have a more pronounced effect on the system performance, than that by the amplitude compensator \( P_a \). This was implicitly utilized by Horowitz's synthesis. A design procedure to select \( \{P_a, P_r\} \) is discussed below.

\begin{align*}
\Delta v_1 & = R_2 [\frac{p_r g_c}{1 + g_c} (1 + g_r) s_t - R_2 p_r s_a], \\
\Delta v_1 & = -R_2 g_e s_t - R_2 p_a s_a
\end{align*}

Comparing (3.18) with (3.11), the design basis in [2] was for the special condition: \( R_2 p_a s_a = 0 \) (3.19).

\[
\Delta v_1 = -R_2 g_e s_t - R_2 p_a s_a
\]
\[ 1 + G_E(s) = F \left[ 1 + G_E(s) \right] = \frac{F_1 + G_E(s)}{1 + P_o(s)} = \frac{1 + G_E(s)}{1 + P_o(s)} \]  

in which the later equalities are from (4.1). The modified compensator is thus given by

\[ P_o(s) = \left[ 1 + P_o(s) \right] / F_1 \quad (4.4) \]

If \( P_o(s) = 0 \), then (4.4) amounts to the direct design of \( P_o(s) \). A feasible lead filter design for \( F(s) \) is

\[ F(s) = \frac{s + 1/\beta}{s + 1/\alpha \beta} ; \quad \alpha = \frac{1 + \sin \phi}{\sin \phi} , \quad \beta = \frac{1}{\omega_m \sqrt{\alpha}} \]  

so that \( F(s) \) introduces a phase shift \( \phi \) at frequency \( \omega_m \). The digital equivalent of (4.5) was shown in [4]. \( P_o(s) \) is checked to satisfy (P1)-(P3). \( \{ \phi, \omega_m \} \) are determined as follows.

Assuming the \( G_{1d}(j\omega) \)-plot as in Fig.4, interception of \( G_{1d}(j\omega) \)-plot with the \(-\)ve real-axis implies local stability only [1]. To improve the stability behaviours, point \( A \) as shown in Fig.4 may be identified. Let its frequency be \( \omega_0 \) and \( \angle ACB=\omega_m \) choose

\[ \omega_m = \omega_0 \quad \text{and} \quad \phi = \phi_0 + \Delta \phi \]  

with \( \Delta \phi \approx 0.5^\circ \sim 1\% \), filter \( F(s) \) is thus completely defined and so is the modified compensator (4.4). Exact amount of \( \Delta \phi \) may be fitted when necessary.

**Design Procedure**

1. Choose \( P_o = 0 \) or \( P_o \neq 0 \).
2. Plot \( G_{1d}(j\omega, k_o) \) of (2.14a) for \( 0 \leq k_o \leq 1 \), or \( G_{1d}(j\omega, k_o) \) of (2.14b) for \( 0 \leq k_o \leq 1 \).
3. If none of Nyquist curves intercept to the left of \(-1+j0 \) on the real axis, and the compensated system response is satisfactory, exit design loop.
4. If intersection exists, identify the appropriate phase angle and frequency such as Fig.4.
5. Calculate filter (4.5) and compensator (4.4).
6. If the compensated system response is satisfactory, exit design loop. Else return to Step (2).

The above procedure is first used to design the rate compensator \( P_r(s) \) and then the amplitude compensator \( P_r(s) \). Finally the above procedure is used, if necessary, to select \( P_o(s) \) and simultaneously for modifications of the preliminary compensators, in order to ensure that necessary conditions for global stability are satisfied [1].

Apart from choosing \( P_o = 0 \) and \( P_r = 0 \) in Step (1), they can be initialized using existing compensation schemes [4,6,8]. In such case the above Design Procedure can be used to ensure fulfillment of necessary and/sufficient conditions for global asymptotic stability.

**5 Example**

The following example was studied by Horowitz [2] and will be compared here between the control synthesis and the compensator design technique. The improved performance of the compensated system with simpler designs supports the compensator approach.

From [2], plant \( G(s) = 1/s \), and the controller is

\[ S(s) = \frac{T(s)}{R(s)} = \frac{13(s + 5)^3}{(s + 0.7)^4(s/250 + 1)} \]  

The uncompensated system limit-cycles when subject to constraints \( \{ \theta_{lim} = \pm 1 \}, \{ \theta_{lim} = \pm 0.5 \} \), as shown in phase diagrams [\( \theta_f = \theta \delta \theta/\delta t \) of Fig.5 for unit step inputs.

Using (3.12) with \( \xi = 1 \) and \( \omega_n = 16 \), the control synthesis gives the following high order controllers of \( G_n \) and \( H_n \) from (3.13) \( F_{n-1} \) as

\[ G_n = \frac{3250(s+5)(s+16)^2}{D_n} \]  

\[ H_n = \{ 32(s^3 + 158.5s^4 + 1019.8s^5 + 3037.6s^6)^2 \} - 9666.8s + 686 \]  

\[ D_n = s^5 + 252.1s^4 + 3776.5s^3 + 49117.8s^2 + 243835.75s + 406250 \]  

Clearly such complex transfer functions are highly unsatisfactory from implementation considerations. The equivalent lead filter \( F_r(s) \) from (3.14) and (4.2)] is

\[ F_r(s) = (s + 0.7)^2(s + 16)^2(s + 250)/s D_n \]  

To design the rate and amplitude compensators, a \( G_r(j\omega) \)-plot reveals that \( \phi = 57.7^\circ \omega_m = 1.9 \) [Fig.4]. Thus \( \phi \) may take \( 60^\circ \) and both \( G_r(j\omega) \) and \( G_a(j\omega) \)-plots are to advance this amount by the compensators. From (4.4)-(4.5), the amplitude saturation compensator \( P_a \) is designed as

\[ P_a(s) = 6.6122 \]  

To design the rate compensator \( P_r(s) \), notice that (5.1) does not have an integrator and so, in order to satisfy condition (P2), \( F_r(s) \) must include one which however introduces a \( 90^\circ \) phase lag into \( F_r(j\omega) \). Consequently a cascade of (4.5) is required to shift \( 150^\circ \) in \( F_r(j\omega) \). \( 90^\circ + 60^\circ = 257.5^\circ \). Choose two lead filters of \( 75^\circ \) each, and then replace one pole by an integrator to satisfy (P1-P2), giving

\[ F_r(s) = (s + 0.25)^2/s(s + 14.93) \]  

\[ P_r(s) = (13.93s - 0.0625)/(s + 0.25)^2 \]  

The compensators (5.4), (5.6) are applied and the results shown in Figs.6-7. Improvements in the system responses are clearly observed in comparison with those obtained by the control synthesis using (5.2). This system is dominated by velocity constraints and so, compared to the equivalent structure [\( \theta_f = \theta \delta \theta/\delta t \) of Fig.5]. For more severe saturation levels, it will be necessary to further advance \( F_r(s) \) and then higher order compensators cannot be avoided. Incidentally, the phase shift of (5.3) is \( 71.3^\circ \) at \( \omega_m = 1.9 \), and its maximum is \( 72.7^\circ \) at \( \omega_m = 2.36 \).

**6 Conclusion**

This paper compares a control synthesis theory proposed by Horowitz for designing controllers for systems subject to rate and amplitude constraints, and a compensator structure previously proposed for given linear controllers. It was shown that the control synthesis theory is a special case of the compensator construction. In a limited scope, it shows the equivalence between \textit{a priori} control synthesis and a
posteriori compensator designs. It also shows that the design selection proposed by Horowitz was only appropriate for low-order saturation systems dominated by velocity constraints, whereas the compensator structure applies to any combination of the severity of rate and amplitude constraints. Results were demonstrated with an illustrative example.

References