Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

A Gradient Flow Approach to Robust Pole Assignment in Second-Order Systems

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Abstract

Robust pole assignment for second-order systems is considered. It is shown that assignment can be achieved by solving a linear matrix equation or linear system. An objective function measuring the robustness of the closed-loop spectrum is minimized via gradient flow.

1 Introduction

Consider a time-invariant, second-order system

$$M\ddot{q} + D\dot{q} + Kq = f \tag{1}$$

where $q, f \in \mathbb{R}^n$ and $M, D, K \in \mathbb{R}^{n\times n}$. Responses of (1) can be altered by applying a feedback control force $f = Bu$ to (1) with $B \in \mathbb{R}^{n\times m}$ denoting the input matrix and $u \in \mathbb{R}^m$ the control vector. The state feedback control law is defined as $u = -F_K q - F_D \dot{q}$ where F_D and F_K are derivative and proportional feedback matrices respectively. Hence, the resulting closed-loop system is

$$M\ddot{q} + (D + BF_D)\dot{q} + (K + BF_K)q = 0. \tag{2}$$

In this note, M is assumed to be invertible and (2) can be written in the familiar first-order form

$$\dot{z} = \bar{A}z + \bar{B}u$$

where

$$\bar{A} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}.$$

For complete assignability, the pair $(\bar{A}, \bar{B})$ is assumed to be completely controllable.

2 Pole Assignment

Let Λ be a real pseudo-diagonal matrix containing all the assigned poles, in which 1×1 blocks represent real poles and 2×2 blocks of the form

$$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \tag{3}$$

represent complex conjugate pairs $\sigma \pm i\omega$. Note that (3) is unitary equivalent to a diagonal matrix, that is

$$U^* \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} U = \begin{bmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{bmatrix}$$

where $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ is a unitary matrix.

The idea of state feedback pole assignment is to find $[F_K \;\; F_D]$ such that

$$\bar{A} + \bar{B}[F_K \;\; F_D] = T\Lambda T^{-1} \text{ or } \bar{A} + \bar{B}G = T\Lambda \tag{4}$$

where $G = [F_K \;\; F_D]T$ and T is some nonsingular matrix. In fact, (4) corresponds to the following linear matrix equation

$$\tilde{A}T\tilde{B} - \tilde{C}T\tilde{D} = \tilde{E} \tag{5}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \quad \tilde{B} = I, \quad \tilde{C} = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix},$$

$$\tilde{D} = \Lambda, \quad \tilde{E} = \begin{bmatrix} 0 \\ B \end{bmatrix} [F_K \;\; F_D]T = \begin{bmatrix} 0 \\ B \end{bmatrix} G.$$

Since the pencils $\tilde{A} - \lambda\tilde{B}$ and $\tilde{D} - \lambda\tilde{B}$ are regular and the spectra $\rho\left(\tilde{A}, \tilde{C}\right)$ and $\rho\left(\tilde{D}, \tilde{B}\right)$ are assumed to be disjoint, i.e. no common closed-loop and open-loop poles, (5) is uniquely solvable for a given G [1, 3]. However, by exploiting the special structures

0-7803-3590-2/96 $5.00 © 1996 IEEE

of the coefficient matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$, we can express (5) into linear systems of equations which can then be solved by some effective algorithms [4]. Define $\hat{U} = \text{diag}\left\{\hat{U}_1, \ldots, \hat{U}_{n+\frac{p}{2}}\right\}$ where p, an even number, is the number of real poles of (2) and

$$\begin{cases} \hat{U}_j = 1, & j = 1, \ldots, p, \\ \hat{U}_j = U, & j = p+1, \ldots, n + \frac{p}{2}. \end{cases}$$

Note that $\hat{U} = U \bigotimes I_n$ if $p = 0$ and $\hat{U} = I_{2n}$ if $p = 2n$. By comparing both sides of (5), we eventually obtain the linear system

$$(\lambda_j^2 M + \lambda_j D + K)\, w_j = -B\hat{g}_j, \quad j = 1, \ldots, 2n. \quad (6)$$

where $\hat{g}$ is the j-th column of $\hat{G} = G\hat{U}$. Then for a given G, we can solve for $W = [w_1, \ldots, w_{2n}]$ from (6) and $T = \begin{bmatrix} W\hat{U}^* \\ W\hat{U}^*\Lambda \end{bmatrix}$. Consequently,

$$[F_K \quad F_D] = GT^{-1} \quad (7)$$

which is the augmented derivative and proportional feedback matrix such that the closed-loop system (2) have the assigned spectrum.

3 Robust Pole Assignment

In measuring the robustness of the closed-loop spectrum, an objective function is defined as

$$\phi(T(G)) = \|T(G)\|_F^2 + \|T(G)^{-1}\|_F^2. \quad (8)$$

It can be shown that [5] the objective function (8) has to be minimized in order to obtain a robust closed-loop system. For notational simplicity, we write $\phi(T(G))$ as $\phi(G)$ throughout.

For a given G, T is solved via the linear matrix equation (5) or linear system (6) at each minimization iteration for $\phi(G)$. When an optimum solution, say G^*, is reached such that $\phi(G^*)$ attains its minimum, the required augmented feedback matrix $F = [F_K \quad F_D] = G^*T^{*-1}$ is recovered from (7), where G^* and T^* satisfy (4).

With gradient flow analysis, this minimization problem comes down to solving the following system of ordinary differential equations

$$\begin{aligned} \dot{G}(t) &= 2\left[\text{trace}\left\{\tfrac{\partial T}{\partial g_{jk}}\left(T^{-1}T^{-T}T^{-1} - T^T\right)\right\}\right]_{m\times 2n} \\ G(0) &= G_0 \in \mathcal{G}. \end{aligned} \quad (9)$$

where $\mathcal{G} = \{G \mid T$ is a nonsingular solution of (4)$\}$ is open and dense in $\mathbb{R}^{m\times 2n}$. The solution, say G^*, to (9) will be a minimum of $\phi(G)$ in (8). Important issues on the existence and convergence of the solution to (9) on $[0, \infty)$ are discussed in [5].

In the ODE (12), $T = S\hat{U}^*$ where

$$s_l = -\begin{bmatrix} I \\ \lambda_l I \end{bmatrix} \mathcal{C}_l^{-1} B\hat{g}_l$$

with $\mathcal{C}_l = \lambda_l^2 M + \lambda_l D + K$. And also $\dfrac{\partial T}{\partial g_{jk}} = \dfrac{\partial S}{\partial g_{jk}}\hat{U}^*$ where

$$\frac{\partial s_l}{\partial g_{jk}} = \begin{cases} -\delta_{lk}\begin{bmatrix} I \\ \lambda_l I \end{bmatrix}\mathcal{C}_l^{-1}b_j, \\ \quad k = 1, \ldots, p, \\ -\frac{\delta_{lk}+\delta_{l(k+1)}}{\sqrt{2}}\begin{bmatrix} I \\ \lambda_l I \end{bmatrix}\mathcal{C}_l^{-1}b_j, \\ \quad k = p+1, p+3, \ldots, 2n-1, \\ -i\frac{\delta_{l(k-1)}-\delta_{lk}}{\sqrt{2}}\begin{bmatrix} I \\ \lambda_l I \end{bmatrix}\mathcal{C}_l^{-1}b_j, \\ \quad k = p+2, p+4, \ldots, 2n \end{cases}$$

where δ_{jk} the Kronecker delta. It is important to realize that $\dfrac{\partial T}{\partial g_{jk}}$ is a constant matrix and is only required to evaluate once in the computation process.

4 Conclusion

In this note, the problem of robust pole assignment for second-order systems by state feedback is examined. It has been shown that pole assignment can be achieved by either solving a linear matrix equation or linear system. The assigned spectrum is made optimally robust by minimizing an objective function via gradient flow which involves the solution of a system of ordinary differential equation.

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