Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

Lyapunov Equations and Riccati Equations for Descriptor Systems *

Q.L. Zhang G.Z. Dai
Department of Automatic Control
Northwestern Polytechnical University
Xian, Shanxi 710072, P.R.C.

J. Lam
Department of Mechanical Engineering
University of Hong Kong
Pokfulam Road, H.K.

ABSTRACT: In this paper, two new types of Lyapunov and Riccati equations are presented for linear time-invariant descriptor systems. The two equations play key roles in asymptotic stability analysis and control synthesis for this class of systems. Fundamental properties of the two equations are investigated and interesting results are obtained.

1 Introduction

Descriptor systems arise in many applications such as electrical networks, economic systems, and biochemical engineering systems. Recently, the Lyapunov methods have been extended to the descriptor systems systems [1]. However, an issue remains to be resolved is to relate asymptotic stability and stabilizability of descriptor systems with impulses using Lyapunov and Riccati equations in a way similar to those of normal systems. Preliminary studies were conducted by Syrmos *et al.* [4] and Zhang *et al.* [5,6] in the impulse-free case. Unfortunately, the results in these works cannot be directly utilized to analyze the asymptotic stability and stabilizability of descriptor systems with impulses. The purpose of this paper is to study the properties of Lyapunov and Riccati equations associated with descriptor systems (with or without impulses). In particular, we relate the two equations to the asymptotic stability and the asymptotical stabilizability of descriptor systems.

2 System Descriptions

Consider a linear time-invariant descriptor system given by

$$E\frac{dx}{dt} = Ax + Bu \quad , \quad y = Cx \tag{1}$$

where x, u and y are respectively the state, input and output; E, A, B and C are real matrices. The descriptor system in (1) will be identified by the quadruple (E, A, B, C). Whenever an argument, E, A, B, or C, of a realization is of no consequence in the development, we may replace it by a $*$. (E, A, B, C) is assumed to be *regular*, that is $\det(sE - A) \not\equiv 0$ and it is said to be *asymptotically stable* if the finite roots of $\det(sE - A) = 0$ lie in the open LHP. System concepts related to descriptor systems such as R-controllability and R-observability may be found in [1]. Since there exists s_0 such that $s_0E - A$ is invertible, we define

$$\overline{E} = (s_0E - A)^{-1}E, \quad \overline{A} = (s_0E - A)^{-1}A, \quad \overline{B} = (s_0E - A)^{-1}B,$$

we have $\overline{E}\,\overline{A} = \overline{A}\,\overline{E}$, $\overline{A} = s_0\overline{E} - I$ and (E, A, B, C) is *restricted system equivalent (r.s.e.)* to system $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$ where $\overline{C} = C$. Systems under restricted system equivalence have the same "stability" and "stabilizability" properties.

There exists a real invertible matrix T such that

$$T\overline{E}T^{-1} = \operatorname{diag}\left(\overline{E}_1, \overline{E}_2\right), \tag{2}$$

$$\begin{aligned} T\overline{A}T^{-1} &= \operatorname{diag}\left(\overline{A}_1, \overline{A}_2\right) \\ &= \operatorname{diag}\left(s_0\overline{E}_1 - I, s_0\overline{E}_2 - I\right), \end{aligned} \tag{3}$$

with $\overline{E}_1$ invertible, $\overline{E}_2$ nilpotent with nilpotent index equals h, i.e. $\overline{E}_2^h \neq 0$, $\overline{E}_2^{h+1} = 0$. Note that $\overline{A}_2$ is an invertible matrix. (E, A, B, C) is r.s.e. to $(T\overline{E}T^{-1}, T\overline{A}T^{-1}, T\overline{B}, \overline{C}T^{-1})$, which may be decomposed into the slow and fast subsystems,

$$\overline{E}_1\frac{dx_1}{dt} = \overline{A}_1x_1 + \overline{B}_1u, \quad y_1 = \overline{C}_1x_1, \tag{4}$$

$$\overline{E}_2\frac{dx_2}{dt} = \overline{A}_2x_2 + \overline{B}_2u, \quad y_2 = \overline{C}_2x_2, \tag{5}$$

$$y = y_1 + y_2, \tag{6}$$

Asymptotical stability of (E, A, B, C) is equivalent to that of the slow subsystem (4). Unfortunately, the decomposition sometimes suffers from numerical problems [3].

3 Lyapunov Equations and Stability

To motivate the use of a new Lyapunov equation, we note that $\overline{E}^{h+1}x \neq 0 \Leftrightarrow x_1 \neq 0$. Thus one can construct a Lyapunov function of (E, A, B, C) as

$$\mathcal{V}\left(\overline{E}^{h+1}x\right) = x^T\left(\overline{E}^{h+1}\right)^T V\overline{E}^{h+1}x$$

where $V \geq 0$, with the property that $\mathcal{V}\left(\overline{E}^{h+1}x\right) > 0$ for $\overline{E}^{h+1}x \neq 0$, and $\mathcal{V}(0) = 0$ for $\overline{E}^{h+1}x = 0$. The Lyapunov equation associated to (E, A, B, C) and $\mathcal{V}$ is given by

$$\overline{A}^T\left(\overline{E}^h\right)^T V\overline{E}^{h+1} + \left(\overline{E}^{h+1}\right)^T V\overline{E}^h\overline{A} = -\left(\overline{E}^{h+1}\right)^T W\overline{E}^{h+1} \tag{7}$$

where $W \geq 0$. When $E = I$, (7) is the usual Lyapunov equation. The Lyapunov equation with $h = 0$ was considered in [4,2]. From (2) and (3), Lyapunov equation (7) becomes $\overline{A}_2^T\left(\overline{E}_2^h\right)^T V_2^T\overline{E}_1^{h+1} = 0$ and

$$\overline{A}_1^T\left(\overline{E}_1^h\right)^T V_1\overline{E}_1^{h+1} + \left(\overline{E}_1^{h+1}\right)^T V_1\overline{E}_1^h\overline{A}_1 = -\left(\overline{E}_1^{h+1}\right)^T W_1\overline{E}_1^{h+1} \tag{8}$$

where

$$T^{-T}VT^{-1} = \begin{bmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{bmatrix}, \quad T^{-T}WT^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} \tag{9}$$

*The research was supported by the NNSFC under grant No.69574004, NSFLP under grant No.95168 and ITDC, contract No.105-82150, respectively

such that the partitions are conformal to the dimensions of $\overline{E}_1$ and $\overline{E}_2$. Let $\overline{V}_1 = \left(\overline{E}_1^h\right)^T V_1 \overline{E}_1^h$, $\overline{V}_2 = \left(\overline{E}_2^h\right)^T V_2 \overline{E}_1^h$, $\overline{W}_1 = \left(\overline{E}_1^h\right)^T W_1 \overline{E}_1^h$, then we have $\overline{V}_2 = 0$ and

$$\overline{A}_1^T \overline{V}_1 \overline{E}_1 + \overline{E}_1^T \overline{V}_1 \overline{A}_1 = -\overline{E}_1^T \overline{W}_1 \overline{E}_1 \tag{10}$$

since $\overline{E}_1$ and $\overline{A}_2$ are invertible. Also, by defining $\overline{V} = \left(\overline{E}^h\right)^T V \overline{E}^h$ and $\overline{W} = \left(\overline{E}^h\right)^T W \overline{E}^h$, (7) becomes

$$\boxed{\overline{A}^T \overline{V}\,\overline{E} + \overline{E}^T \overline{V}\,\overline{A} = -\overline{E}^T \overline{W}\,\overline{E}} \tag{11}$$

Similar to the partition in (9) for V and W, we have

$$T^{-T}\overline{V}T^{-1} = \begin{bmatrix} \overline{V}_1 & \overline{V}_2 \\ \overline{V}_2^T & \overline{V}_3 \end{bmatrix}, \quad T^{-T}\overline{W}T^{-1} = \begin{bmatrix} \overline{W}_1 & \overline{W}_2 \\ \overline{W}_2^T & \overline{W}_3 \end{bmatrix}. \tag{12}$$

Theorem 1 *Descriptor system $(\overline{E}, \overline{A}, *, *)$ is asymptotically stable if and only if Lyapunov equation (11) has solution $\overline{V} \geq 0$, with rank$\left(\overline{E}^T \overline{V}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$ for $\overline{W} \geq 0$, with rank$\left(\overline{E}^T \overline{W}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$. Moreover, $\overline{V} \geq 0$ with rank$\left(\overline{E}^T \overline{V}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$ =rank$\left(\overline{V}\right)$ is unique.*

Similar to normal systems, asymptotic stability and observability are related to the solution of a Lyapunov equation.

Theorem 2 *If any two of the following three statements are true, then the other statement is true.*

(i) *Descriptor system $(\overline{E}, \overline{A}, *, \overline{C})$ is asymptotically stable.*

(ii) *Lyapunov equation (11) has solution $\overline{V} \geq 0$, satisfying rank$\left(\overline{E}^T \overline{V}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$ for $\overline{W} = \left(\overline{E}^h\right)^T \overline{C}^T \overline{C}\,\overline{E}^h$.*

(iii) *Descriptor system $(\overline{E}, \overline{A}, *, \overline{C})$ is R-observable.*

4 Riccati Equations and Stabilizability

For some $R > 0$, we define the Riccati equation associated with the $(\overline{E}, \overline{A}, \overline{B}, C, *)$ as

$$\boxed{\overline{E}^T \overline{V}\,\overline{A} + \overline{A}^T \overline{V}\,\overline{E} - \overline{E}^T \overline{V}\,\overline{B} R^{-1} \overline{B}^T \overline{V}\,\overline{E} = -\overline{E}^T \overline{W}\,\overline{E}} \tag{13}$$

Riccati equation (13) can be rewritten as

$$\overline{E}^T \overline{V}\left(\overline{A} - \overline{B}R^{-1}\overline{B}^T\overline{V}\,\overline{E}\right) + \left(\overline{A} - \overline{B}R^{-1}\overline{B}^T\overline{V}\,\overline{E}\right)^T \overline{V}\,\overline{E}$$
$$= -\overline{E}^T\left(\overline{W} + \overline{V}\,\overline{B}R^{-1}\overline{B}^T\overline{V}\right)\overline{E} \tag{14}$$

which may be considered as the Lyapunov equation associated with descriptor system

$$\overline{E}\frac{dx}{dt} = \left(\overline{A} - \overline{B}R^{-1}\overline{B}^T\overline{V}\,\overline{E}\right)x + v \tag{15}$$

resulting from closed-loop control of $(\overline{E}, \overline{A}, \overline{B}, *)$ with state feedback given by $u = -R^{-1}\overline{B}^T\overline{V}\,\overline{E}x$. When $\overline{V}$ is such that (15) is asymptotically stable then we refer $\overline{V}$ to as a *stabilizing* solution of (13). From (4), (5), and (9), we have (13) reduced to $\overline{V}_2 = 0$ and

$$\overline{E}_1^T \overline{V}_1 \overline{A}_1 + \overline{A}_1^T \overline{V}_1 \overline{E}_1 - \overline{E}_1^T \overline{V}_1 \overline{B}_1 R^{-1} \overline{B}_1^T \overline{V}_1 \overline{E}_1 = -\overline{E}_1^T \overline{W}_1 \overline{E}_1 \tag{16}$$

on noting that $\overline{E}_1$ and $\overline{A}_2$ are invertible while $\overline{V}_3$ is real symmetric. In other words, Riccati equation (13) is solvable if and only if $\overline{V}_2 = 0$ and Riccati equation (16) is solvable.

Theorem 3 *If descriptor system $(\overline{E}, \overline{A}, \overline{B}, *)$ is stabilizable, then for $\overline{W} \geq 0$, with rank$\left(\overline{E}^T \overline{W}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$, Riccati equation (13) has stabilizing solution $\overline{V} \geq 0$ with rank$\left(\overline{E}^T \overline{V}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$. Moreover, there is a unique solution with rank$\left(\overline{E}^T \overline{V}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$ =rank$\left(\overline{V}\right)$.*

In the following, we relate the stabilizability and detectability of $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$ to the solution of (13).

Theorem 4 *Descriptor system $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$ is stabilizable and detectable if and only if for $\overline{W} = \left(\overline{E}^h\right)^T \overline{C}^T \overline{C}\,\overline{E}^h \geq 0$, Riccati equation (13) has solution $\overline{V} \geq 0$ with rank$\left(\overline{E}^T \overline{V}\,\overline{E}\right)$ =rank$\left(\overline{E}^{h+1}\right)$ so that descriptor system $(\overline{E}, \overline{A} - \overline{B}R^{-1}\overline{B}^T\overline{V}\,\overline{E}, *, *)$ is asymptotically stable.*

5 Conclusion

In this paper, two new types of Lyapunov and Riccati equations are developed for descriptor systems. They are related to the asymptotic stability and stabilizability of descriptor systems, which may have impulses. In this way, the results unify the use of Lyapunov methods to tackle a variety of control problems for descriptor systems.

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