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Pole Assignment with Optimally Conditioned Eigenstructure

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ABSTRACT: This paper presents a novel method for pole assignment with robustness measured in terms of the spectral condition number of the closed-loop eigenvector matrix. It is established that the spectral condition number can be minimized asymptotically via a sequence of unconstrained minimizations on some auxiliary objective functions. Moreover, the sequence of minimizers converges to a minimizer of the spectral condition number. A numerical algorithm with analytical formulas of the gradient of the auxiliary objective functions is provided. The efficiency and effectiveness of the approach is demonstrated via an example.

Keywords: Pole assignment, robustness, optimal, condition number.

1 Introduction

For a completely state controllable system, it is well known that the closed-loop poles via state-feedback can be assigned at any set of self-conjugate complex numbers (Petkov, Christov, and Konstantinov 1991). The state-feedback gain matrix, except in the single-input case, is in general nonunique for a given set of desired closed-loop poles. In the past decade, many methods have been proposed on the choice of the state-feedback gain matrix which, in certain sense, leads to a well-conditioned or robust closed-loop system matrix (Varga 1981; Kautsky, Nichols, and van Dooren 1985). Different measures of robustness on the closed-loop system matrix led to different robust pole assignment methods (Kautsky, Nichols, and van Dooren 1985; Owens and O'Reilly 1989; Byers and Nash 1989; Jiang 1991), the spectral condition number of the eigenvector matrix of the closed-loop system matrix still remains as the most widely accepted measure of robustness. This is because by the Bauer-Fike Theorem, the spectral variation of the closed-loop system matrix A_c due to an unstructured perturbation Δ in A_c is bounded by $||T||_2||T^{-1}||_2||\Delta||_2$ where $||\cdot||_2$ denotes the spectral norm of a matrix and T is a nonsingular eigenvector matrix of A_c. For this reason, the spectral condition number $\kappa_2(T) \triangleq ||T||_2||T^{-1}||_2$ provides a meaningful measure on the sensitivity of the closed-loop eigenvalues due to unstructured perturbations in A_c (Stewart 1973; Horn and Johnson 1985). In other words, the smallness of the spectral condition number leads to the smallness in the sensitivity of the eigenvalues.

One major difficulty in using the spectral condition number as the objective function for optimization in a state-feedback closed-loop system is nonsmoothness of the spectral norm. This led to the consideration of the Frobenius condition number $\kappa_F(T) \triangleq ||T||_F||T^{-1}||_F$ which is a more conservative robustness measure (Byers and Nash 1989; Lam and Yan 1995). So far, very little effort has been made by researchers on the computation of state-feedback gain based on the minimization of the spectral condition number. In the present work, we propose a numerical procedure for computing such a feedback gain. The main advantage is that the computation of the optimal feedback gain is achieved through a sequence of unconstrained smooth optimization problems with solutions approaching to a minimum point of $\kappa_2(T)$. Furthermore, analytic gradient formulas are available for efficient implementation.

This paper is divided into five sections. A formulation of the robust pole assignment problem with background materials is given in Section 2. The minimization of the spectral condition number of the closed-loop eigenvector matrix via a sequence of unconstrained minimization is discussed in Section 3. Analytic formulas for minimization are also derived. A schematic algorithm for solving the robust pole assignment problem is also presented. In Section 4, we use a numerical example to demonstrate the effectiveness of computational procedure. We will also compared our results with others. Finally, concluding remarks are given in Section 5.

2 Problem Formulation

Throughout this paper, we use $\|\cdot\|_2$ and $\|\cdot\|_F$ to denote the spectral norm and the Frobenius norm respectively.

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times q}$ $(q > 1)$. Suppose (A, B) is a completely controllable pair and B has full column rank. Then there exists a $K \in \mathbb{R}^{q\times n}$ such that $A + BK$ has spectrum equal to a given self-conjugate set of complex numbers of cardinality n . In other words, there exists an invertible $T \in \mathbb{R}^{n\times n}$ such that

$$(A + BK)T = T\Lambda \tag{1}$$

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where Λ is a real pseudo-diagonal matrix given by

$$\Lambda = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \dots & \dots & \dots & 0 \\ -\beta_1 & \alpha_1 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \alpha_2 & \beta_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\beta_2 & \alpha_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \gamma_1 & \vdots \\ 0 & \dots & \dots & \dots & 0 & \ddots & \ddots \end{pmatrix}$$

with the eigenvalues of Λ as the desired closed-loop poles,

$$\alpha_i \pm \beta_i j \quad , \quad i = 1, \dots, n_1, \quad \gamma_k \quad , \quad k = 1, \dots, n_1$$

The matrix Λ is real normal and diagonalizable with a unitary similarity transformation. It will be assumed that

$$\operatorname{spec}(A) \cap \operatorname{spec}(\Lambda) = \emptyset$$

In order that the closed-loop poles to have good robustness against perturbation, we seek to determine a K that minimizes the spectral condition number of the eigenvector matrix of the closed-loop system matrix $A + BK$. Although T is not an eigenvector matrix, there exists a unitary U such that TU is an eigenvector matrix. Hence, the spectral condition number defined via

$$\kappa_2(T) := \|T\|_2 \; \|T^{-1}\|_2$$

measures the spectral conditioning of the closed-loop state matrix. Also, any re-ordering of the diagonal blocks does not affect the value of $\kappa_2(T)$. Although the spectral condition number $\kappa_2(T)$ is a natural choice, it is not commonly used due to its nondifferentiability with respect to T. Another problem associated with the minimization of the spectral condition number $\kappa_2(T)$ as an objective function is due to the fact that T defined in (1) is not unique. This prompts us to reformulate the problem and rewrite (1) as

$$AT - T\Lambda + BG = 0 \qquad (2)$$

$$G = KT \qquad (3)$$

Here, $T = T(G)$ is considered as a function of G. To ensure that the condition number is well-defined, we restrict $G \in \mathcal{G} \subset \mathbb{R}^{q \times n}$ where

$$\mathcal{G} := \left\{ G \in \mathbb{R}^{q \times n} \mid T(G) \text{ is nonsingular} \right\}$$

It can be shown that $\mathcal{G}$ is open and dense in $\mathbb{R}^{q \times n}$ (Bhattacharyya and de Souza 1982). Moreover, $T = T(G)$ via (2) is an injective function which subsequently determines a unique $K = K(G) = GT^{-1}$ via (3) for each $G \in \mathcal{G}$.

The objective function to be considered is given by

$$J(G) := \kappa_2(T(G)) = \|T(G)\|_2 \|T^{-1}(G)\|_2 \qquad (4)$$

Unfortunately, (4) has an inherent difficulty due to $J(G) = J(\alpha G)$ for any nonzero scalar α which leads to the singularity of the Hessian of $J(G)$ at a minimum point. In the following section, we will see how such difficulty can be overcome through a choice of an auxiliary objective function and how it may be minimized via a sequence of unconstrained minimizations involving $G \in \mathcal{G}$.

3 Optimization of the Spectral Condition Number

Let

$$\phi_i(T) \;:\; = \|T\|_i^2 + \|T^{-1}\|_i^2$$
$$\kappa_i(T) \;:\; = \|T\|_i \|T^{-1}\|_i$$

where $i = 2$ or F. We have

$$1 \;\le\; \kappa_2(T) \le \frac{1}{2}\phi_2(T)$$
$$2n \;\le\; \kappa_F(T) \le \frac{1}{2}\phi_F(T)$$

The following theorem summarizes the important fact that $\phi_i(T(G))$ where $T(G)$ is defined using (2) has a global minimum point in $\mathcal{G}$ and that any minimum point (global or local) of $\phi_i(T(G))$ is also a minimum point of $\kappa_i(T(G))$.

Theorem 1 *With $\phi_i(T)$ where $T = T(G)$ given by (2) for $G \in \mathcal{G}$,*

(i) $\phi_i(T(G))$ has a global minimum in $\mathcal{G}$;

(ii) if $G^* \in \mathcal{G}$ is a minimum point of $\phi_i(T(G))$, then $\|T(G^*)\|_i = \|T^{-1}(G^*)\|_i$;

(iii) if $G^* \in \mathcal{G}$ is a minimum point of $\phi_i(T(G))$, then G^* is also a minimum point of $\kappa_i(T(G))$.

Proof:

(i) Set $\phi^* = \inf_{G \in \mathcal{G}} \phi(T(G))$. Then there exists a sequence G_k in $\mathcal{G}$ such that

$$\lim_{k \to \infty} \phi(T(G_k)) = \phi^*$$

Note that the sequence $T(G_k)$ is bounded as ϕ^* is finite. Since

$$\|G_k\|_i \le \|(B^T B)^{-1} B^T\|_i \left(\|\Lambda\|_i + \|A\|_i\right) \|T(G_k)\|_i < \infty$$

there exists a convergent subsequence G_{k_n}. We have $G^* := \lim_{n \to \infty} G_{k_n}$ must be in $\mathcal{G}$ since $\phi_i(T(G_{k_n}))$ will go unbounded otherwise. Consequently, from the continuity of $\phi_i(T(G))$, it follows that

$$\phi^* = \lim_{n \to \infty} \phi_i(T(G_{k_n})) = \phi_i\left(T\left(\lim_{n \to \infty} G_{k_n}\right)\right) = \phi_i(T(G^*))$$

That is, G^* is a global minimum point of $\phi_i(T(G))$.

(ii) Since G^* is a minimum point of $\phi_i(T(G))$, we have

$$\phi_i(T(G^*)) \le \phi_i(T(G))$$

for all $G \in \mathcal{G}$ sufficiently close to G^* (in the case where G^* is a global minimum, the inequality is valid for all $G \in \mathcal{G}$). Suppose $\|T^{-1}(G^*)\|_i = k\|T(G^*)\|_i$ for $k \neq 1$, then

$$\begin{aligned} \phi_i(G^*) - \phi_i(\alpha G^*) &= \left[(1 - \alpha^2) + k^2\left(1 - \frac{1}{\alpha^2}\right)\right] \|T(G^*)\|_i^2 \\ &= \left(1 - \alpha^2\right)\left(\alpha^2 - k^2\right) \frac{\|T(G^*)\|_i^2}{\alpha^2} \end{aligned}$$

The RHS is positive if α is such that $1 < \alpha^2 < k^2$ when $k > 1$ or $k^2 < \alpha^2 < 1$ when $k < 1$. For any value of $k \neq 1$, there exists some α arbitrarily close to one such that RHS is always positive and hence contradicting that G^* is a minimum point. Thus, $k = 1$ and the result follows.

(iii) Since for any nonzero scalar α ,

$$\kappa_i(\alpha T) = \kappa_i(T)$$

with

$$\alpha_0 = \sqrt{\frac{\|T^{-1}(G)\|_i}{\|T(G)\|_i}} \neq 0$$

it is clear that $\alpha_0 G \in \mathcal{G}$ and $\alpha_0 G$ can be chosen arbitrarily close to G^* provided that G is

sufficiently close to G^* . Then

$$\begin{aligned} \kappa_i(T(G)) &= \kappa_i(\alpha_0 T(G)) \\ &= \|\alpha_0 T(G)\|_i \|(\alpha_0 T)^{-1}(G)\|_i \\ &= \frac{1}{2}\phi_i(\alpha_0 T(G)) \\ &\geq \frac{1}{2}\phi_i(T(G^*)) \\ &= \kappa_i(T(G^*)) \end{aligned}$$

where the result in (ii) is used in the last step. Hence G^* is also a minimum point of $\kappa_i(T(G))$. □

Therefore, we can minimize $\phi_i(T(G))$ and its minimum point will serve as a minimizer of $\kappa_i(T(G))$ (see also (Lam and Yan 1995)). Now, we state a well-known result in matrix theory which will eventually lead to a sequence of minimization problems that ultimately solves the minimization problem of $\kappa_2(T(G))$.

Lemma 1 (Horn and Johnson (1985, p.299)) *For any* $M \in \mathbb{R}^{n\times n}$,

$$\lim_{p\to\infty} \|M^p\|^{1/p} = \rho(M)$$

where $\rho(\cdot)$ *denotes the spectral radius of* $(\cdot)$ *and* $\|\cdot\|$ *stands for any matrix norm.*

The following application of the above lemma will give us an indication of how to minimize $\kappa_2(T)$ via a sequence of minimization problems.

Proposition 1

$$\begin{aligned} \kappa_2(T) &= \lim_{p\to\infty} \kappa_F^{1/(2p)}\left(\left[T^T T\right]^p\right) \quad (5) \\ &= \lim_{p\to\infty} \left[\left\|\left[T^T T\right]^p\right\|_F \left\|\left[T^{-T}T^{-1}\right]^p\right\|_F\right]^{1/(2p)} \\ &= \lim_{p\to\infty} \left[\mathrm{tr}\left(\left[T^T T\right]^p\right) \mathrm{tr}\left(\left[T^{-T}T^{-1}\right]^p\right)\right]^{1/p} \end{aligned}$$

where $\mathrm{tr}(\cdot)$ *denotes the trace of* $(\cdot)$.

Proof: Let $\sigma_1(\cdot)$ and $\sigma_n(\cdot)$ denote the maximum and minimum singular values of $(\cdot)$. Notice that

$$\begin{aligned} \sigma_1^2(T) = \rho(T^T T) &= \lim_{p\to\infty} \left\|\left[T^T T\right]^p\right\|_F^{1/p} \\ &= \lim_{p\to\infty} \left[\mathrm{tr}\left(\left[T^T T\right]^p \left[T^T T\right]^p\right)\right]^{1/(2p)} \\ &= \lim_{p\to\infty} \left[\mathrm{tr}\left(\left[T^T T\right]^{2p}\right)\right]^{1/(2p)} \\ &= \lim_{p\to\infty} \left[\mathrm{tr}\left(\left[T^T T\right]^p\right)\right]^{1/p} \end{aligned}$$

and similarly we have

$$\begin{aligned} \sigma_n^2(T) = \sigma_1^2(T^{-1}) &= \lim_{p\to\infty} \left\|\left[T^{-T}T^{-1}\right]^p\right\|_F^{1/p} \\ &= \lim_{p\to\infty} \left[\mathrm{tr}\left(\left[T^{-T}T^{-1}\right]^p\right)\right]^{1/p} \end{aligned}$$

Hence,

$$\begin{aligned} \kappa_2(T) &= \sigma_1(T)\sigma_1(T^{-1}) \\ &= \lim_{p\to\infty} \left[\mathrm{tr}\left(\left[T^T T\right]^p\right) \mathrm{tr}\left(\left[T^{-T}T^{-1}\right]^p\right)\right]^{1/(2p)} \end{aligned}$$

and the result follows. □

Consequently, it is conceivable that the nonsmooth $\kappa_2(T(G))$ can be minimized via (5), that is,

$$\min_G \kappa_2(T(G)) = \lim_{p\to\infty} \min_G \kappa_F^{1/(2p)}\left(\left[T(G)^T T(G)\right]^p\right)$$

This view is indeed feasible based on the result established in the following theorem.

Theorem 2 *For* $G \in \mathcal{G}$,

$$\inf_G \kappa_2(T(G)) = \inf_G \lim_{p\to\infty} \kappa_F^{1/(2p)}\left(\left[T(G)^T T(G)\right]^p\right)$$

Proof: Put

$$J_p(G) := \kappa_F^{1/(2p)}\left(\left[T^T(G)T(G)\right]^p\right)$$

and let the singular values of $T(G)$ be

$$\sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq \sigma_n(G)$$

Then

$$\begin{aligned} \|[T^T(G)T(G)]^p\|_F^{1/(2p)} &= \left[\sum_{i=1}^n \sigma_i^{4p}\right]^{1/(4p)} \geq \sigma_1 \\ \|[T^{-T}(G)T^{-1}(G)]^p\|_F^{1/(2p)} &= \left[\sum_{i=1}^n \frac{1}{\sigma_i^{4p}}\right]^{1/(4p)} \geq \frac{1}{\sigma_n} \end{aligned}$$

So, there holds

$$J_p(G) \geq \frac{\sigma_1}{\sigma_n} = J(G) \quad (6)$$

Now it will be proved that

$$\lim_{p\to\infty} \bar{J}_p = \bar{J}$$

with

$$\bar{J}_p := \inf_G J_p(G), \quad \bar{J} := \inf_G J(G)$$

To this end, take an arbitrary small number $\epsilon > 0$. Then there exists G_ϵ such that

$$J(G_\epsilon) - \bar{J} \leq \epsilon$$

leading to

$$\begin{aligned}\bar{J}_p - \bar{J} &= \bar{J}_p - J_p(G_\epsilon) + J_p(G_\epsilon) - J(G_\epsilon) + J(G_\epsilon) - \bar{J} \\ &\leq \epsilon + J_p(G_\epsilon) - J(G_\epsilon)\end{aligned}$$

On the other hand, it follows from (6) that

$$\bar{J}_p - \bar{J} \geq 0$$

Consequently, there results

$$0 \leq \bar{J}_p - \bar{J} \leq \epsilon + J_p(G_\epsilon) - J(G_\epsilon)$$

This implies that

$$\limsup_{p\to\infty}(\bar{J}_p - \bar{J}) - \liminf_{p\to\infty}(\bar{J}_p - \bar{J}) \leq \epsilon$$

due to

$$\lim_{p\to\infty}[J_p(G_\epsilon) - J(G_\epsilon)] = 0$$

But, ϵ is arbitrary; hence, we obtain

$$\lim_{p\to\infty}(\bar{J}_p - \bar{J}) = 0$$

as required. □

As a by-product of the above proof, it is valid that if G_p is such that

$$J_p(G_p) = \bar{J}_p$$

then

$$\begin{aligned}&\lim_{p\to\infty} \kappa_F^{1/(2p)}\left(\left[T^T(G_p)T(G_p)\right]^p\right) \equiv \lim_{p\to\infty} J_p(G_p) \\ &= \lim_{p\to\infty} J(G_p) = \lim_{p\to\infty} \bar{J}_p = \bar{J} \equiv \inf_{G\in\mathcal{G}} \kappa_2(T(G))\end{aligned}$$

From Theorem 1, we conclude that

$$\lim_{p\to\infty}\left\{\frac{1}{2}\phi_F\left(\left[T(G_p)^T T(G_p)\right]^p\right)\right\}^{1/(2p)} = \inf_{G\in\mathcal{G}} \kappa_2(T(G))$$

In other words, if G_p globally minimizes

$$\left\{\frac{1}{2}\phi_F\left(\left[T(G_p)^T T(G_p)\right]^p\right)\right\}^{1/(2p)}$$

for each given positive integer p , the sequence $\{G_p\}$ converges to a global minimizer of $\kappa_2(T(G))$. For this reason, we refer

$$\psi_p(G) := \left\{\frac{1}{2}\phi_F\left(\left[T(G)^T T(G)\right]^p\right)\right\}^{1/(2p)}$$

to as the *auxiliary objective function* to be minimized. In practice, the solution G_p is approximated by, say, $\widehat{G}_p$ which will then be used as the initial guess of the minimization problem with auxiliary objective function $\psi_{p+\nu}(G)$ for some positive integer ν (itself is in general a function of p). The minimizations can be carried out easily apart from the possiblity of local minima.

3.1 Analytic Formulas for Optimization

Another advantage of the present formulation is that analytic formulas of the gradient of the auxiliary objective function for different p is available. This allows efficient implementation of the minimization process. Let $G = [g_{ij}]_{q\times n}$ and we have

$$\frac{\partial G}{\partial g_{ij}} = E_{ij}$$

where $E_{ij} \in \mathbb{R}^{q\times n}$ is a matrix with zero elements except a value of one at the (i, j) position. The gradient of $\psi_p(G)$ is given by

$$\nabla_G \psi_p(G) = \left[\frac{\partial}{\partial g_{ij}}\psi_p(G)\right]_{q\times n}$$

The following theorem gives an explicit representation of the gradient function.

Theorem 3 *Let*

$$\mathcal{S}(T) := \left(T^T T\right)^{2p} - \left(T^T T\right)^{-2p} = \mathcal{S}(T)^T$$

Then

$$\nabla\psi_p(G) = \frac{1}{\psi_p^{2p-1}(G)} B^T P \tag{7}$$

where P is the solution to the Sylvester equation

$$A^T P - P\Lambda^T + T^{-T}\mathcal{S}(T) = 0 \tag{8}$$

Moreover, the jth column of P is given by the jth column of

$$\begin{cases} \left((\alpha_{\lceil \frac{j}{2}\rceil} I - A^T)^2 + \beta^2_{\lceil \frac{j}{2}\rceil} I\right)^{-1}\left(T^{-T}\mathcal{S}(T)\Lambda - A^T T^{-T}\mathcal{S}(T)\right) & \text{if} \quad j = 1, 2, \ldots, 2n_1 \\ \left(\gamma_{j-2n_1} I - A^T\right)^{-1} T^{-T}\mathcal{S}(T) & \text{if} \quad j = 2n_1 + 1, \ldots, n \end{cases} \tag{9}$$

where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

Proof: See Appendix A. □

Now we are ready to summarize the following schematic algorithm for computing the robust pole assignment feedback gain.

Robust Pole Assignment Algorithm:

1. Choose $\widehat{G}_0 \in \mathcal{G}$ as an initial guess and a small positive number ϵ . Set $p = 1$.

2. Find $\widehat{G}_p$ as an approximate solution to the unconstrained minimization problem

$$\min_{G\in\mathbb{R}^{q\times n}} \psi_p(G)$$

based on some globally convergent descent minimization algorithm which makes use of the gradient given by (7).

3. While

$$\psi_p(\widehat{G}_p) - \psi_{p-1}(\widehat{G}_{p-1}) \geq \epsilon$$

then update $p := p+1$ and use $\widehat{G}_{p-1}$ as the next initial guess, repeat STEP 2.

4. Construct $\widehat{K} = \widehat{G}_{p-1}T\left(\widehat{G}_{p-1}\right)^{-1}$, where $T = T(G)$ is given by (2), as the required feedback gain.

□

Observe that the elements of P in (8) can be represented via $(A^T \otimes I - I \otimes \Lambda)\text{vec}(P) = -\text{vec}(T^{-T}S(T))$ where $\otimes$ denotes the Kronecker product and vec$(\cdot)$ denotes the column vector formed by lexicographical ordering the elements in matrix $(\cdot)$. Since it is only the right-hand-side of this linear matrix equation that is varying in the optimization process, P can be computed efficiently using Bartels-Stewart's algorithm (Golub and van Loan 1989). Alternatively, the explicit column representation in (9) may be used. In this case, the LU decomposition of $\left((\alpha_{\lceil\frac{j}{2}\rceil}I - A^T)^2 + \beta^2_{\lceil\frac{j}{2}\rceil}I\right)$ and $\left(\gamma_{j-2n_1}I - A^T\right)$ are computed and stored only once which will be used in STEP 2. Although it is difficult to guarantee that $\widehat{G}_p$ in STEP 2 to be close to the global minimizer, the algorithm works well in all numerical experiments carried out. In fact, this may be modified in such a way only a fixed number of minimization iterations are computed. Such modification will prevent the iterations from staying in each step for excessively long as we want p to be eventually sufficient large. In practice, it seldom requires p to be greater than 5 to give a settled solution.

4 Numerical Example

Consider a distillation column model (Kautsky, Nichols, and van Dooren 1985; Lam and Yan 1995) given by

$$A = \begin{pmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.00227 & 0 & 0.1636 & -0.1625 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & 0.0638 & 0.0838 & 0.1004 & 0.0063 \\ 0 & 0 & -0.1396 & -0.2060 & -0.0128 \end{pmatrix}$$

with eigenvalues at -0.077324, -0.014232, -0.89531, -2.8408 and -5.9822. The desired closed-loop poles are -0.2, -0.5, -1 and $-1 \pm j$ and we have

$$\Lambda = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We start with an initial guess given by

$$G(0) = \begin{pmatrix} -2.8143 & 3.0453 & 7.2018 & -10.7410 & 6.9278 \\ 4.6115 & 12.2449 & 10.1953 & 0.2029 & 6.5302 \end{pmatrix}$$

chosen randomly. The minimization process in STEP 2 is performed based on the gradient descent algorithm with line search and Hessian (secant) update due to Dennis and Schnabel (Dennis Jr. and Schnabel 1983). We choose a fixed number of iterations equals 53 for every p ($p = 1, 2, 3$). This gives a total of 159 iterations in the 3-stage minimization process. The final values of G and T are given by respectively

$$G_{final} = \begin{pmatrix} -53.4132 & -23.2664 & 22.8727 & 51.7407 & 57.7332 \\ -12.9532 & -0.5655 & 5.7369 & 47.8198 & 53.8995 \end{pmatrix}$$

and

$$T_{final} = \begin{pmatrix} 0.1401 & -0.0442 & -0.5453 & -0.2424 & -0.1710 \\ -1.2835 & 2.8574 & 0.7867 & 1.5077 & 2.4244 \\ -1.1069 & 3.5619 & 0.7880 & -0.5341 & -0.7298 \\ -0.7923 & 2.4956 & -0.0302 & -0.9567 & -1.7769 \\ 0.3112 & 0.0426 & -1.8002 & 1.3014 & 0.7300 \end{pmatrix}$$

These give the feedback gain as

$$K_{final} = \begin{pmatrix} -69.6961 & 91.7345 & -194.8262 & 163.1511 & -39.5294 \\ -39.3138 & 22.9888 & -33.1815 & 20.0475 & 3.9069 \end{pmatrix}$$

Notice that

$$\kappa_2(T(G)) \leq \kappa_F^{1/(2p)}\left(\left[T^T(G)T(G)\right]^p\right) \leq \psi_p(G)$$

with $\lim_{p\to\infty}\psi_p(G_p) = \kappa_2(T(G_p))$. The monotonically decreasing behavior of $\psi_p(G)$ is ensured by the minimization algorithm chosen. The gap between $\psi_p(G)$ and $\kappa_2(T(G))$ is closing during the minimization process and reaches a value equal to 0.0404 at the last iterate. The end results are summarized and the present asymptotic approximation method is compared with other methods (Kautsky, Nichols, and van Dooren 1985; Lam and Yan 1995; Byers and Nash 1989; Yang and Tits 1993) in Table 1.

	$\kappa_2(T)$	$\kappa_F(T)$	$\|K\|_2$	$\|K\|_F$
Asymptotic method	31.6	47.7	286.5	288.1
Lam-Yan's method	33.6	39.3	337.0	337.4
Kautsky's method 1	39.4	47.3	311.5	312.1
Kautsky's method 2	66.1	98.8	283.1	292.6
Byers-Nash method	33.1	39.1	354.9	-
Yang-Tits method	39.7	42.5	228.7	230.1

Table 1. Comparison of condition numbers and gains

It can be seen the our method gives the smallest $\kappa_2(T)$. During the 159 iterations, it turned out that the choice of feedback gain at the 88th iterate may be more acceptable in terms of size of the feedback gain ($\|K\|_2 = 250.2$ and $\|K\|_F = 251.3$) as well as the conditioning ($\kappa_2(T) = 36.8$ and $\kappa_F(T) = 48.7$).

5 Conclusion

We have presented an approach towards achieving robust pole assignment under the robust measure using spectral condition number of the closed-loop eigenvector matrix. This is done via a sequence of unconstrained minimizations of some auxiliary objective functions. It was shown that the sequence of minimum points obtained from these minimization problems converges to a minimum point of spectral condition number of the closed-loop eigenvector matrix. Explicit formulas, including the gradient of the auxiliary objective functions, are given for implementation. The method is simple, efficient, and can be easily implemented. A numerical example was employed to clearly demonstrate the superiority of our method when compared with other well-established algorithms.

Appendix A

Since

$$\phi_F\left(\left[T^TT\right]^p\right) = \mathrm{tr}\left((T^TT)^{2p} + (T^TT)^{-2p}\right) = \mathrm{tr}\left(\mathcal{S}(T)\right)$$

the Fréchet derivative of $\phi_F\left(\left[T^TT\right]^p\right)$ with respect to G is

$$\begin{aligned} D\phi_F|_GX &= \mathrm{tr}\left(4p[(T^TT)^{2p-1}T^T - (T^TT)^{-2p-1}T^T]DT|_GX\right) \\ &= 4p\mathrm{tr}\left(\mathcal{S}(T)T^{-1}DT|_GX\right) \end{aligned}$$

where $DT|_GX$ denotes the Fréchet derivative of T with respect to G and $DT|_GX$ satisfies

$$A(DT|_GX) - (DT|_GX)\Lambda + BX = 0$$

Suppose P is the solution to (8), then we have

$$\begin{aligned} \mathrm{tr}\left(X^TB^TP\right) &= \mathrm{tr}\left(\left[\Lambda^T(DT|_GX)^T - (DT|_GX)^TA^T\right]P\right) \\ &= \mathrm{tr}\left((DT|_GX)^TT^{-T}\mathcal{S}(T)\right) \\ &= \mathrm{tr}\left(\mathcal{S}(T)T^{-1}DT|_GX\right) \end{aligned}$$

and hence

$$D\phi_F|_GX = 4p\mathrm{tr}\left(X^TB^TP\right)$$

That is, there holds

$$\nabla\phi_F(G) = 4pB^TP$$

This together with

$$\nabla\psi_p(G) = \frac{1}{2^{1+1/(2p)}p}\phi_F^{\frac{1}{2p}-1}\nabla\phi_F(G) = \frac{1}{4p\psi_p^{2p-1}(G)}\nabla\phi_F(G)$$

yields (7).

Let

$$P = \left[\begin{array}{cccc} p_1 & p_2 & \cdots & p_n \end{array}\right]$$

To construct the columns of P, we multiply e_j, the jth standard basis of $\mathbb{R}^n$, to (8) from the right. For $j = 2n_1+1, \ldots, n$, we have

$$A^Tp_j - \gamma_{r-2n_1}p_j + T^{-T}\mathcal{S}(T)e_j = 0$$

which gives

$$p_j = \left(\gamma_{r-2n_1}I - A^T\right)^{-1}T^{-T}\mathcal{S}(T)e_j$$

For $j = 1, 3, \ldots, 2n_1-1$, we have

$$A^Tp_j - P\left(\alpha_{\lceil\frac{j}{2}\rceil}e_j + \beta_{\lceil\frac{j}{2}\rceil}e_{j+1}\right) + T^{-T}\mathcal{S}(T)e_j = 0$$

$$A^Tp_{j+1} - P\left(-\beta_{\lceil\frac{j}{2}\rceil}e_j + \alpha_{\lceil\frac{j}{2}\rceil}e_{j+1}\right) + T^{-T}\mathcal{S}(T)e_{j+1} = 0$$

These give

$$\begin{aligned} p_j &= \left(\left(\alpha_{\lceil\frac{j}{2}\rceil}I - A^T\right)^2 + \beta^2_{\lceil\frac{j}{2}\rceil}I\right)^{-1} \\ &\qquad \left(T^{-T}\mathcal{S}(T)\Lambda - A^TT^{-T}\mathcal{S}(T)\right)e_j \quad (10) \\ p_{j+1} &= \left(\left(\alpha_{\lceil\frac{j}{2}\rceil}I - A^T\right)^2 + \beta^2_{\lceil\frac{j}{2}\rceil}I\right)^{-1} \\ &\qquad \left(T^{-T}\mathcal{S}(T)\Lambda - A^TT^{-T}\mathcal{S}(T)\right)e_{j+1} (11) \end{aligned}$$

Notice that for $j = 2, 4, \ldots, 2n_1$, we have $\lceil\frac{j}{2}\rceil = \lceil\frac{j-1}{2}\rceil$. Consequently, the expression of p_j given by (10) is valid for $j = 1, 2, 3, \ldots, 2n_1$. Hence the result follows.

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