A DESIGN OF ROBUST FEEDBACK CONTROL FOR FLEXIBLE ROBOTIC MECHANISMS

SIU-PANG YUNG†, GEN-QI XU‡ AND JUN-MIN WANG††

†Department of Mathematics, University of Hong Kong,
Hong Kong, P.R. CHINA, spyung@hku.hk

‡Department of Mathematics, Shanxi University,
TaiYuan, 030006, P.R. CHINA, gqxu@mail.sxu.edu.cn

††Department of Mathematics, University of Hong Kong,
Hong Kong, P.R. CHINA, wangjc@hkusua.hku.hk

Abstract. We study flexible robotic arms that are free to rotate and bend in the horizontal plane but are stiff in vertical bending and torsion. A motor connected to the hinged end drives the arm to a prescribed target position and it is the aim of this paper to design boundary feedback controller to stabilize the arm once it reaches the target position. A distinct difficulty is the non-dissipativity that arises from the requirement that the designed feedback should retain 0 as an eigenvalue in order not to change the rigid body mode shape of the arm. On preserving this zero eigenvalue, we have successfully constructed a boundary feedback that is robustly stable with respect to the target position by showing it is exponentially stable there.

Key Words. Euler-Bernoulli beam, boundary feedback control, non-dissipativity.

1. INTRODUCTION

In the past two decades, the Euler-Bernoulli beam with boundary control have been extensively studied as a distributed parameter system, see for instance Guo [1] and references therein. In most literatures on the boundary control problem of Euler-Bernoulli beam, the feedback control is put on the free end, which is also the loaded end. Recently, J. Knani in [2] studied a different model. The free end is controlled by putting the boundary feedback on the other end and this suits the practical situation more that the former. This is the Knani's model that we shall study in this paper. Without disrupting the rigid mode shape, we design a boundary feedback control such that the closed loop system is exponentially stable with respect to the rigid mode shape.

The present paper is organized as follows. In section 2, we recall our problem from [2] and the details that we shall need in this paper, and propose our control law. In section 3, we find asymptotic expressions for the spectrum of the closed loop system. Finally in section 4, we prove that the closed loop system is exponentially stable, while retaining the rigid mode shape, by showing the system is a Riesz one, and hence satisfy the spectrum determined growth condition.

2. MATHEMATICAL MODELLING OF THE FLEXIBLE ARM

2.1. Model Description

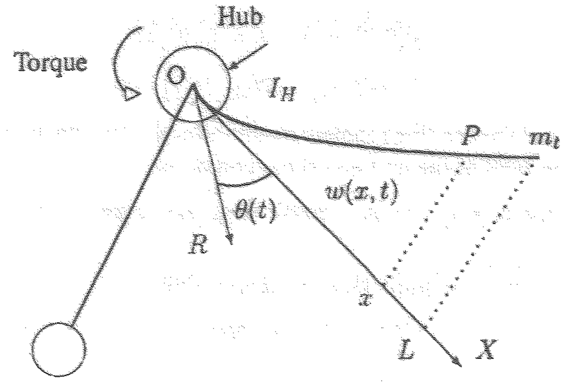

Fig. 1. The skeleton of the flexible arm.

Due to the lack of space, we only give a short description of the flexible arm and refer the readers to [2] and [4]. The arm is free to rotate and bend along the horizontal direction, but it is stiff vertically both in bending and in torsion. It is powered by a direct dc drive motor connected to the hinged end of the arm. The arm is modelled as a continuous, pinned-free beam of length ℓ whose moment of inertia about the root is I_b, with an additional lumped inertia I_h at the actuator end. A tip mass m_t (of negligible moment of inertia) is added at the other end of the arm (tip).

Line "OR" is the fixed reference line; "OX" is the tangent line to the beam's neutral axis at the hub. The displacement of any point P along the beam's neutral axis at a distance x from the hub is given by the hub angle $\theta(t)$ and the small elastic deflection $w(x,t)$ measured from line "OX", as shown in Fig. 1 (see also [2]). Axial deformations are neglected. The hub angle $\theta(t)$ can be arbitrarily large.

To set up a mathematical model for the implementation of controllers for the flexible arm, one can adopt the truncated dynamical model of an idealized one-link direct-drive elastic manipulator. Several noteworthy approximations will be made. First, deformation of the beam was ignored and second, rotational inertia moments were not considered. For slender beams, such as flexible arm with thickness only 0.6 mm and length about 1000 mm, either shear nor rotational effects are important and so are neglected.

2.2. Equations of motion – Fourth-order partial differential equation

Let us define a variable $y(x,t)$ as

$$y(x,t) := w(x,t) + x\theta(t) \tag{2.1}$$

which is the distance departed from the reference line "OR", $w(x,t)$ is the deformation of the beam from "OX", $\theta(t)$ is the angle between "OR" and "OX". Then $y(x,t)$ satisfies a fourth-order partial differential equation of motion ([3], [9]) of a one-link flexible arm given by

$$EIy_{xxxx} + \rho y_{tt} = 0, \quad 0 < x < \ell \tag{2.2}$$

together with the boundary conditions

$$y(0,t) = 0, \quad \theta = y_x(0,t),$$
$$\eta = y(\ell,t), \quad EIy_{xx}(\ell,t) = 0$$

and the bending moment condition at the original point as well as the tip mass shear condition

$$I_h\ddot{\theta} = EIy_{xx}(0,t) + T(t); \quad m_t\eta_{tt}(t) = EIy_{xxx}(\ell,t), \tag{2.3}$$

where $\theta := y_x(0,t), \eta := y(\ell,t)$. Putting

$$Y(t) := [y(\cdot,t), \theta(t), \eta(t)]^T, \tag{2.4}$$

then system (2.2) and (2.3) can be rewritten as

$$\ddot{Y}(t) = \mathbb{A}Y(t) + BT(t) \tag{2.5}$$

where

$$\begin{cases} \mathbb{A}Y := [-\frac{EI}{\rho}y_{xxxx}, \frac{EI}{I_h}y_{xx}(0), \frac{EI}{m_t}y_{xxx}(\ell)]^T, \\ B := [0,1,0]^T. \end{cases} \tag{2.6}$$

Lemma 2.1. *Operator* $-\mathbb{A}$ *with domain*

$$\begin{aligned} \mathcal{D}(\mathbb{A}) := \{Y = [y,\theta,\eta] \in L^2[0,\ell] \times \mathbb{C} \times \mathbb{C} \\ \mid y \in H^4(0,\ell), y(0) = 0, y(\ell) = \eta, \\ y_x(0) = \theta, y_{xx}(\ell) = 0\} \end{aligned} \tag{2.7}$$

is a nonnegative self-adjoint operator in $L^2[0,\ell]\times\mathbb{C}\times\mathbb{C}$ *endowed with norm*

$$\|Y\|^2 := \int_0^\ell \rho|y(x)|^2dx + I_h|\theta|^2 + m_t|\eta|^2, \tag{2.8}$$

and the corresponding inner product. The spectrum of $\mathbb{A}$ *are all eigenvalues, and* $0 \in \sigma(\mathbb{A})$.

PROOF For any $Y = [y,\theta,\eta]^T \in \mathcal{D}(\mathbb{A})$ and any $Z = [z,\xi,\zeta] \in H^4(0,\ell)\times\mathbb{C}\times\mathbb{C}$, direct computations show that

$$\begin{aligned} &< \mathbb{A}Y, Z > \\ = &-\int_0^\ell EIy(x)z_{xxxx}(x)dx - EIy_{xxx}(\ell)z(\ell) \\ &+EIy_{xxx}(0)z(0) + EIy_{xx}(\ell)z_x(\ell) \\ &-EIy_{xx}(0)z_x(0) - EIy_x(\ell)z_{xx}(\ell) \\ &+EIy_x(0)z_{xx}(0) + EIy(\ell)z_{xxx}(\ell) \\ &-EIy(0)z_{xxx}(0) + EIy_{xx}(0)\xi + EIy_{xxx}(\ell)\zeta. \end{aligned}$$

If we take $z(0) = 0, z_{xx}(\ell) = 0, \xi = z_x(0), \zeta = z(\ell)$, and use $y(0) = 0, y_{xx}(\ell) = 0$, we have

$$\begin{aligned} &< \mathbb{A}Y, Z > \\ = &-\int_0^\ell EIy(x)z_{xxxx}(x)dx + EIy_x(0)z_{xx}(0) \\ &+EIy(\ell)z_{xxx}(\ell) \\ = &-\int_0^\ell \rho\frac{EI}{\rho}y(x)z_{xxxx}(x)dx + I_h\theta\frac{EI}{I_h}z_{xx}(0) \\ &+m_t\eta\frac{EI}{m_t}z_{xxx}(\ell) \\ = &< Y, \mathbb{A}Z >. \end{aligned}$$

So $\mathbb{A}$ is self-adjoint in $L^2[0,\ell]\times\mathbb{C}\times\mathbb{C}$. In particular, we have

$$< \mathbb{A}Y, Y > = -\int_0^\ell EI|y_{xx}(x)|^2dx,$$

therefore, $-\mathbb{A} \geq 0$. Direct verification shows that $0 \in \sigma(\mathbb{A})$ with $Y_0 := [x, 1, \ell]$ being an eigenvector.

Since $I - \mathbb{A}$ is positive and $(I - \mathbb{A})^{-1}$ is bounded from $L^2[0,\ell]\times\mathbb{C}\times\mathbb{C}$ to $\mathcal{D}(\mathbb{A})$, so the Sobolev Embedding Theorem ensures that $(I - \mathbb{A})^{-1}$ is compact, and hence $\mathbb{A}$ has discrete spectrum. □

Remark 2.1. *Let $\{\lambda_n : n \in \mathbb{N}$ with $\lambda_n \leq \lambda_{n+1}\}$ be the spectrum of $-\mathbb{A}$. We see that $\lambda = 0$ is a simple eigenvalue of $\mathbb{A}$, which is usually called the* **rigid body mode shape**. *Its eigenvector $Y_0 := [x, 1, \ell]$ is called the* **total moment of initial associated this mode shape** *because*

$$\|Y_0\|^2 = \rho \int_0^\ell |x|^2 dx + I_h + m_t \ell^2 =: I_T.$$

Thus for any $Y = [y, \theta, \eta]^T \in L^2[0, \ell] \times \mathbb{C} \times \mathbb{C}$, we have

$$Y = <Y, Y_0> Y_0 + \sum_{n=1}^{\infty} <Y, Y_n> Y_n$$

where $\{Y_n : n \geq 0\}$ is the orthogonal basis in $L^2[0, \ell] \times \mathbb{C} \times \mathbb{C}$ satisfying the normalized condition

$$<Y_j, Y_k> = I_T \delta_{kj}$$

formed by the eigenfunctions of the self-adjoint $\mathbb{A}$. In particular, we have

$$\int_0^\ell EI y_j''(x) \overline{y_k''(x)} dx = I_T \delta_{jk}.$$

Goal of the problem: Let $Y(0) := [y_0(x), y_0'(0), y_0(\ell)]$ be the initial state, which denote the arm position at the resting state. The target state (final arm position) is $Y_{final} := [y_1(x), y_1'(0), y_1(\ell)]$. Our goal is "To find a control $T(t)$" under which the solution $Y(t)$ of the closed-loop controlled system makes

$$\|Y(t) - Y_{final}\|$$

goes to zero exponentially or there exists $t_0 > 0$ such that $\|Y(t) - Y_{final}\| = 0, \forall\, t \geq t_0$.

This situation arises from a computer-controlled manipulator that is required to follow a prescribed trajectory in its workspace and then stay at a target location in order to perform a required task.

The difficult part of this problem is that the feedback control should not change the rigid mode shape, which means that $\lambda_0 = 0$ will still be an eigenvalue of closed loop system but at the same time the feedback must achieve exponential stability with respect to the target state of the controlled closed loop system.

Design of the feedback control law: Let $\beta > 0$. We observe that the values of the rate of change of moment at $x = 0$ is

$$-EI y_{xxt}(0, t).$$

If we let the torque of the motor be

$$T(t) := \beta EI y_{xxt}(0, t), \tag{2.9}$$

then the closed loop system is

$$\begin{cases} EI y_{xxxx} + \rho y_{tt} = 0, \quad 0 < x < \ell, \\ EI y_{xx}(0, t) + \beta EI y_{xxt}(0, t) \\ \quad -I_h y_{xtt}(0, t) = 0, \\ y(0, t) = 0, \quad EI y_{xx}(\ell, t) = 0, \\ EI y_{xxx}(\ell, t) = m_t y_{tt}(\ell, t). \end{cases} \tag{2.10}$$

Under the state variable (2.4),

$$Y(t) = [y(x, t), \theta(t) := y_x(0, t), \eta(t) := y(\ell, t)]^T,$$

system (2.10) is just

$$\ddot{Y}(t) = \mathbb{A} Y(t) + \mathbb{B} \dot{Y}(t), \tag{2.11}$$

where

$$\begin{cases} \mathbb{B}Y := [0, \beta \frac{EI}{I_h} y_{xx}(0), 0]^T, \\ \mathcal{D}(\mathbb{B}) := H^3[0, \ell] \subset \mathcal{D}(\mathbb{A}). \end{cases} \tag{2.12}$$

Lemma 2.2. *The energy of the closed-loop system (2.11), defined by*

$$\begin{aligned} E(t) := \; & \tfrac{1}{2}\Big[\int_0^\ell (EI|y_{xx}(x, t)|^2 \\ & + \rho |y_t(x)|^2) dx + I_h |y_{xt}(0, t) \\ & - \tfrac{\beta EI}{I_h} y_{xx}(0, t)|^2 + m_t |y_t(\ell, t)|^2 \Big], \end{aligned} \tag{2.13}$$

is a non-increasing function in t.

PROOF Direct computation leads to

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_0^\ell EI y_{xx}(x, t) y_{xxt}(x, t) dx \\ & \quad - \int_0^\ell EI y_t(x, t) y_{xxxx}(x, t) dx \\ & \quad + [y_{xt}(0, t) - \frac{EI}{I_h} \beta y_{xx}(0, t)] \\ & \quad [I_h y_{xtt}(0, t) - \beta EI y_{xxt}(0, t)] \\ & \quad + m_t y_t(\ell, t) y_{tt}(\ell, t) \\ &= -\frac{(EI)^2}{I_h} \beta |y_{xx}(0, t)|^2 \\ &\leq 0. \end{aligned}$$

Lemma 2.3. *For the closed-loop system (2.10), there is no eigenvalue on the imaginary axis besides $\lambda = 0$.*

Proof Let $\lambda = i\tau$ be an eigenvalue with $\tau \in \mathbb{R}$ and let $Y = [y(x), \theta, \eta]^T \in \mathcal{D}(\mathbb{A})$ be the corresponding nontrivial eigenvector. Then $Y(t) = e^{\lambda t} Y$ is a solution to (2.11) (or (2.10)), and $y(x)$ satisfies

$$\begin{cases} EI y_{xxxx} + \rho \lambda^2 y(x) = 0, \quad 0 < x < \ell, \\ EI y_{xx}(0) + \beta \lambda EI y_{xx}(0) - I_h \lambda^2 y_x(0) = 0, \\ EI y_{xxx}(\ell) = \lambda^2 m_t y(\ell), \\ y(0) = 0, \quad EI y_{xx}(\ell) = 0. \end{cases} \tag{2.14}$$

Multiplying $-\overline{y(x)}, \overline{y_x}(0), \overline{y(\ell)}$ respectively to the first, second and third equation in (2.14), integrating and adding them up, we get

$$\begin{aligned} & -EI y_{xxx}(\ell) \overline{y}(\ell) - EI y_{xx}(0) \overline{y_x}(0) \\ & - \int_0^\ell EI |y_{xx}(x)|^2 dx - \lambda^2 \int_0^\ell \rho |y(x)|^2 dx \\ & + EI y_{xx}(0) \overline{y_x}(0) + EI \beta \lambda y_{xx}(0) \overline{y_x}(0) \\ & - I_h \lambda^2 |y_x(0)|^2 + EI y_{xxx}(\ell) \overline{y}(\ell) - m_t \lambda^2 |y(\ell)|^2 \\ & = 0. \end{aligned}$$

Thus

$$
\begin{aligned}
&-\int_0^\ell EI|y_{xx}(x)|^2dx+\tau^2\int_0^\ell \rho|y(x)|^2dx\\
&+I_h\tau^2|y_x(0)|^2+m_t\tau^2|y(\ell)|^2=0,\\
&iEI\beta\tau y_{xx}(0)\overline{y_x}(0)=0. \qquad (2.15)
\end{aligned}
$$

Combining these with the second equation of (2.14), we get $y_x(0)=y_{xx}(0)=0$. Therefore, $y(x)$ should satisfy

$$
\begin{cases}
EIy_{xxxx}-\rho\tau^2y(x)=0, \quad 0<x<\ell,\\
EIy_{xxx}(\ell)=\lambda^2m_ty(\ell),\\
y(0)=y_x(0)=y_{xx}(0)=y_{xx}(\ell)=0.
\end{cases} \qquad (2.16)
$$

Letting $\omega^4=\frac{\rho\tau^2}{EI}$, then

$$
\begin{aligned}
y(x) &= A\sinh\omega x+B\sin\omega x\\
&\quad +C\cosh\omega x+D\cos\omega x,\\
y_x(x) &= \omega[A\cosh\omega x+B\cos\omega x\\
&\quad +C\sinh\omega x-D\sin\omega x],\\
y_{xx}(x) &= \omega^2[A\sinh\omega x-B\sin\omega x\\
&\quad +C\cosh\omega x-D\cos\omega x],\\
y_{xxx}(x) &= \omega^3[A\cosh\omega x-B\cos\omega x\\
&\quad +C\sinh\omega x+D\sin\omega x].
\end{aligned}
$$

Using the boundary condition: $y(0)=y_x(0)=y_{xx}(0)=0$, we see that $A=-B$, $C=-D=0$, and so

$$y(x)=A[\sinh\omega x-\sin\omega x].$$

From $y_{xx}(\ell)=0, EIy_{xxx}(\ell)=-m_t\tau^2y(\ell)$, we also have

$$
\begin{cases}
A[\sinh\omega\ell+\sin\omega\ell]=0,\\
A[\cosh\omega\ell+\cos\omega\ell]=\frac{m_t}{\rho}\omega A[\sinh\omega\ell-\sin\omega\ell].
\end{cases}
$$

As $A\neq 0$ since Y is a nonzero eigenvector, so

$$
\begin{cases}
\sinh\omega\ell+\sin\omega\ell=0,\\
\cosh\omega\ell+\cos\omega\ell=\frac{m_t}{\rho}\omega[\sinh\omega\ell-\sin\omega\ell],
\end{cases}
$$

which has a unique zero $\omega=0$. Hence, $\tau=0$ and so there is no eigenvalue on the imaginary axis besides $\lambda=0$. □

Remark 2.2. *The energy of the closed-loop system is non-increasing in t (Lemma 2.2) implies that there is no eigenfrequency in right open half plane of $\mathbb{C}$. Combining it with Lemma 2.3, we conclude that all the nonzero eigenfrequencies are located in the left open half plane of $\mathbb{C}$.*

3. ANALYSIS OF EIGEN-FREQUENCIES OF THE SYSTEM

We are now ready to calculate the nonzero eigenvlues of the closed loop system. For simplicity, we normalize equation (2.10) by rescaling the spatial coordinate and time as

$$x=s\ell, \quad 0\le s\le 1, \quad t=\nu\tau,$$

with $\nu^2=\frac{\rho\ell^4}{EI}$, and let $z(s,\tau)=y(x,t)$ so that

$$\ell^k\frac{\partial^ky}{\partial x^k}=\frac{\partial^kz}{\partial s^k}, \quad \nu^2\frac{\partial^2y}{\partial t^2}=\frac{\partial^2z}{\partial\tau^2}.$$

equation (2.10) becomes

$$
\begin{cases}
z_{ssss}+z_{\tau\tau}=0, \quad 0<s<1,\\
z_{ss}(0,\tau)+\beta_1z_{ss\tau}(0,\tau)\\
\qquad -\mu_1z_{s\tau\tau}(0,\tau)=0,\\
z(0,\tau)=0,\ z_{ss}(1,\tau)=0,\\
z_{sss}(1,\tau)=\mu_2z_{\tau\tau}(1,\tau),
\end{cases}
$$

where $\beta_1:=(\beta/\ell^2)(EI/\rho)^{1/2}$, $\mu_1:=I_h/(\rho\ell^3)$ and $\mu_2:=m_t/(\rho\ell)$.

For the sake of convenience, instead of z, s and τ, we will still denote the unknown function by y and the spatial and time variable respectively by x and t, with β_1 renamed to β. Under these renamings, the torque control is just

$$T(t)=\beta y_{xxt}(0,t), \qquad (3.1)$$

where $\beta>0$ can be regarded as the feedback control gain that can be tuned in practice. We rewrite the closed-loop-controlled system as:

$$
\begin{cases}
y_{tt}(x,t)+y_{xxxx}(x,t)=0,\ 0\le x\le 1,\ t>0,\\
y(0,t)=0,\\
\mu_1y_{xtt}(0,t)-y_{xx}(0,t)-\beta y_{xxt}(0,t)=0,\\
y_{xx}(1,t)=0,\\
\mu_2y_{tt}(1,t)-y_{xxx}(1,t)=0.
\end{cases} \qquad (3.2)
$$

To find all the eigenvalues of (3.2), we let λ be an eigenvalue, and $f(x)$ be the eigenfunction such that $y(x,t)=e^{\lambda t}f(x)$ is a solution to (3.2). Then we have

$$
\begin{cases}
f^{(4)}(x)+\lambda^2f(x)=0, \quad 0<x<1,\\
U_4(f,\lambda):=f(0)=0,\\
U_3(f,\lambda):=f''(1)=0,\\
U_2(f,\lambda):=\mu_1\lambda^2f'(0)-\beta\lambda f''(0)-f''(0)=0,\\
U_1(f,\lambda):=\mu_2\lambda^2f(1)-f'''(1)=0.
\end{cases} \qquad (3.3)
$$

Setting $\lambda=\rho^2$, and divide the complex plane into eight equal sectors S_m $(m=0,1,2,\dots,7)$ with sector $S_0:=\left\{z\in\mathbb{C}:0\le\arg z\le\frac{\pi}{4}\right\}$. Arrange the fourth roots of -1 into

$$
\begin{aligned}
\omega_1&=e^{i\frac{3}{4}\pi}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i,\\
\omega_2&=e^{i\frac{5}{4}\pi}=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i
\end{aligned} \qquad (3.4)
$$

and $\omega_3=-\omega_2$, $\omega_4=-\omega_1$, so that they satisfy

$$\mathrm{Re}(\rho\omega_1)\le\mathrm{Re}(\rho\omega_2)\le\mathrm{Re}(\rho\omega_3)\le\mathrm{Re}(\rho\omega_4), \forall\,\rho\in S_0.$$

Then in sector S_0, the equation

$$f^{(4)}(x)+\rho^4f(x)=0 \qquad (3.5)$$

has 4 fundamental solutions $f_1(x), f_2(x), f_3(x), f_4(x)$ that are functionally independent, and they possess the following expressions, for $s = 1, 2, 3, 4$

$$\begin{aligned} f_s(x) &= e^{\rho\omega_s x}, \\ f_s'(x) &= \rho\omega_s e^{\rho\omega_s x}, \\ f_s''(x) &= (\rho\omega_s)'' e^{\rho\omega_s x}, \\ f_s'''(x) &= (\rho\omega_s)''' e^{\rho\omega_s x}. \end{aligned} \tag{3.6}$$

Using (3.6), we can obtain asymptotic expansions for the boundary condition U_1, U_2, U_3, U_4 of system (3.3). While those for boundary conditions $U_3(f, \lambda)$ and $U_4(f, \lambda)$ are clear, the rest are

$$\begin{aligned} U_1(f_m, \rho) &= \left(\mu_2\rho^4 - (\rho\omega_m)^3\right) e^{\rho\omega_m} \quad (3.7) \\ &= \rho^4 e^{\rho\omega_m}(\mu_2 - \omega_m^3(\rho^{-1})), \end{aligned}$$

$$\begin{aligned} &U_2(f_m, \rho) \\ = \; & \mu_1\rho^4(\rho\omega_m) - \beta\rho^2(\rho\omega_m)^2 - (\rho\omega_m)^2 \quad (3.8) \\ = \; & \rho^5\left(\mu_1\omega_m - \beta\omega_m^2\rho^{-1} + O(\rho^{-2})\right) \\ := \; & \rho^5[\mu_1\omega_m - \beta\omega_m^2\rho^{-1}]_2, \end{aligned}$$

where $[a]_2 := a + O(\rho^{-2})$. Since the eigenvalues of system (3.3) are exactly the zeros of the characteristic determinant $\Delta(\rho)$ (cf. [11, pp.13-15]), where

$$\Delta(\rho) := (U_i(y_j, \rho))_{\substack{i=4,3,2,1, \\ j=1,2,3,4}}, \tag{3.9}$$

we can substitute the asymptotic expressions of $U_j(f_m, \rho), j, m = 1, 2, 3, 4$, into (3.9) and obtain the following asymptotic expansion for the characteristic determinant.

Theorem 3.1. *In each sector S_m, the characteristic determinant $\Delta(\rho)$ of the characteristic equation (3.3) has either one of the following asymptotic expansions*

$$e^{\rho\omega_1}\Delta(\rho) = 2\sqrt{2}i\mu_1\mu_2\rho^{11}$$

$$\begin{aligned} &\left\{\left[\pm i + i\left(\tfrac{\sqrt{2}}{2\mu_2} \pm \tfrac{\sqrt{2}\beta}{\mu_1}\right)\tfrac{1}{\rho} + O(\rho^{-2})\right] e^{\rho\omega_3} \right. \\ &\left. + \left[\pm 1 + i\left(\tfrac{\sqrt{2}}{2\mu_2} \mp \tfrac{\sqrt{2}\beta}{\mu_1}\right)\tfrac{1}{\rho} + O(\rho^{-2})\right] e^{\rho\omega_2}\right\}. \end{aligned} \tag{3.10}$$

Here, the values of ω_k will be taken to be the fourth root of -1 but differently in each sector S_m so that

$$\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2) \le \mathrm{Re}(\rho\omega_3) \le \mathrm{Re}(\rho\omega_4)$$

for all ρ in that sector.

PROOF. For $\rho \in S_0$, substitute the asymptotic expansions of $U_j(f_m, \rho), j, m = 1, 2, 3, 4$, into the characteristic determinant (3.9) and we have

$$\Delta(\rho) = \left|\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & (\rho\omega_2)^2 e^{\rho\omega_2} & (\rho\omega_3)^2 e^{\rho\omega_3} & (\rho\omega_4)^2 e^{\rho\omega_4} \\ \rho^5\left[\mu_1\omega_1 - \beta\frac{\omega_1^2}{\rho}\right]_2 & \rho^5\left[\mu_1\omega_2 - \beta\frac{\omega_2^2}{\rho}\right]_2 & \rho^5\left[\mu_1\omega_3 - \beta\frac{\omega_3^2}{\rho}\right]_2 & 0 \\ 0 & \rho^4 e^{\rho\omega_2}\left[\mu_2 - \frac{\omega_2^3}{\rho}\right] & \rho^4 e^{\rho\omega_3}\left[\mu_2 - \frac{\omega_3^3}{\rho}\right] & \rho^4 e^{\rho\omega_4}\left[\mu_2 - \frac{\omega_4^3}{\rho}\right] \end{matrix}\right|$$

$$\begin{aligned} = \; & \rho^{11} e^{\rho\omega_4}\left\{\left[2i\mu_1\mu_2(\omega_1 - \omega_2)\right.\right. \\ & + \left(\mu_1(\omega_1 - \omega_2)(\omega_3 - \omega_4) - 4\beta\mu_2\right)(\rho^{-1}) \\ & \left. + O(\rho^{-2})\right] e^{\rho\omega_3} + \left[2i\mu_1\mu_2(\omega_3 - \omega_1)\right. \\ & + \left(\mu_1(\omega_3 - \omega_1)(\omega_2 - \omega_4) + 4\beta\mu_2\right)(\rho^{-1}) \\ & \left.\left. + O(\rho^{-2})\right] e^{\rho\omega_2}\right\}, \end{aligned}$$

because in sector S_0, $\omega_1^2 = -i, \omega_2^2 = i, \omega_3^2 = i, \omega_4^2 = -i$ and

$$\begin{aligned} \omega_1 - \omega_2 &= \sqrt{2}i, & \omega_3 - \omega_4 &= \sqrt{2}i, \\ \omega_3 - \omega_1 &= \sqrt{2}, & \omega_2 - \omega_4 &= -\sqrt{2} \end{aligned}$$

Thus, we arrive to the asymptotic expansion of $\Delta(\rho)$ in (3.10).

Also, from [11, pp.56-74] we know by repeating the same arguments in the other sector $S_m, m = 1, 2, \ldots, 7$, we will end up with one of the two expansions in (3.10). □

Theorem 3.2. *If $\mu_1\mu_2 \ne 0$, then an asymptotic expansion for the eigenvalues λ_n of system (3.3) is, for $k := \frac{3}{4} - n$ and $n = 1, 2, \ldots$,*

$$\lambda_n = -\frac{2\beta}{\mu_1} \pm \frac{i}{\mu_2} \pm (k\pi)^2 i + O(\frac{1}{n}). \tag{3.11}$$

Moreover, each eigenvalue λ_n with sufficiently large modulus is algebraically simple and satisfies

$$\mathrm{Re}\lambda_n \to -\frac{2\beta}{\mu_1}, \quad n \to \infty. \tag{3.12}$$

PROOF. In sector S_0, since $\omega_3 = -\omega_2$ and from $\Delta(\rho) = 0$, equation (3.10) is equivalent to

$$\begin{aligned} & e^{\rho\omega_2} + ie^{-\rho\omega_2} + i\left(\tfrac{\sqrt{2}}{2\mu_2} - \tfrac{\sqrt{2}\beta}{\mu_1}\right)\tfrac{1}{\rho}e^{\rho\omega_2} \\ & + i\left(\tfrac{\sqrt{2}}{2\mu_2} + \tfrac{\sqrt{2}\beta}{\mu_1}\right)\tfrac{1}{\rho}e^{-\rho\omega_2} + O(\rho^{-2}) = 0, \end{aligned} \tag{3.13}$$

which can be rewritten as

$$e^{\rho\omega_2} + ie^{-\rho\omega_2} + O(\rho^{-1}) = 0. \tag{3.14}$$

Since the equation $e^{\rho\omega_2} + ie^{-\rho\omega_2} = 0$ possess the solution

$$\rho_n = \frac{1}{\omega_2}k\pi i, \quad n = 1, 2, \ldots,$$

with $k = \frac{3}{4} - n$, so by Rouche's theorem, equation (3.14) has solutions

$$\tilde{\rho}_n = \frac{1}{\omega_2} k\pi i + O(\frac{1}{n}), \quad n = 1, 2, \ldots. \tag{3.15}$$

From

$$\begin{aligned}
& e^{\rho\omega_2} + ie^{-\rho\omega_2} \\
= \; & e^{k\pi i + \omega_2 O(n^{-1})} + ie^{-(k\pi i + \omega_2 O(n^{-1}))} \\
= \; & \Big[\cos(k\pi) + i\sin(k\pi) + i\cos(k\pi) \\
& + \sin(k\pi)\Big] \cos(\tfrac{\omega_2}{i} O(n^{-1})) \\
& + \Big[-\sin(k\pi) + i\cos(k\pi) - i\sin(k\pi) \\
& + \cos(k\pi)\Big] \sin(\tfrac{\omega_2}{i} O(n^{-1})),
\end{aligned}$$

using $ie^{k\pi i} + e^{-k\pi i} = i\cos(k\pi) - \sin(k\pi) + \cos(k\pi) - i\sin(k\pi)$, $e^{k\pi i} + ie^{-k\pi i} = \cos(k\pi) + i\sin(k\pi) + i(\cos(k\pi) - i\sin(k\pi)) = 0$ and

$$\sin(\frac{\omega_2}{i} O(n^{-1})) = \frac{\omega_2}{i} O(n^{-1}) + O(n^{-2}),$$

we obtain

$$e^{\rho\omega_2} + ie^{-\rho\omega_2} = (ie^{k\pi i} + e^{-k\pi i}) \frac{\omega_2}{i} O(n^{-1}) + O(n^{-2}). \tag{3.16}$$

Furthermore,

$$e^{\pm k\pi i + O(\frac{1}{n})} = e^{\pm k\pi i} + O(\frac{1}{n}),$$

hence (3.13) changes into

$$\begin{aligned}
& (ie^{k\pi i} + e^{-k\pi i}) \tfrac{\omega_2}{i} O(n^{-1}) + \left(\tfrac{\sqrt{2}}{2\mu_2} - \tfrac{\sqrt{2}\beta}{\mu_1}\right) \tfrac{\omega_2}{k\pi} e^{k\pi i} \\
& + \left(\tfrac{\sqrt{2}}{2\mu_2} + \tfrac{\sqrt{2}\beta}{\mu_1}\right) \tfrac{\omega_2}{k\pi} e^{-k\pi i} + O(n^{-2}) = 0,
\end{aligned}$$

$$\begin{aligned}
-\frac{\omega_2}{i} O(n^{-1}) = \; & \tfrac{\sqrt{2}i}{2\mu_2} \tfrac{\omega_2}{k\pi i} \tfrac{e^{k\pi i} + e^{-k\pi i}}{ie^{k\pi i} + e^{-k\pi i}} \\
& - \tfrac{\sqrt{2}\beta i}{\mu_1} \tfrac{\omega_2}{k\pi i} \tfrac{e^{k\pi i} - e^{-k\pi i}}{ie^{k\pi i} + e^{-k\pi i}} \\
& + O(n^{-2}).
\end{aligned} \tag{3.17}$$

Note that $k = \frac{3}{4} - n$, so we have

$$\begin{aligned}
& \frac{e^{k\pi i} + e^{-k\pi i}}{ie^{k\pi i} + e^{-k\pi i}} \\
= \; & \frac{2\cos(k\pi)}{(1+i)(\cos(k\pi) - \sin(k\pi))} \\
= \; & \frac{2}{(1+i)(1 + \tan(\frac{\pi}{4}))} \\
= \; & \frac{1}{1+i}
\end{aligned}$$

and

$$\frac{e^{k\pi i} - e^{-k\pi i}}{ie^{k\pi i} + e^{-k\pi i}} = \frac{-i}{1+i}.$$

Substituting those identities into (3.17), we have

$$-\frac{\omega_2}{i} O(n^{-1}) = \frac{\sqrt{2}}{1+i} \left(\frac{1}{2\mu_2} - \frac{\beta}{i\mu_1}\right) \frac{\omega_2}{k\pi} + O(n^{-2}). \tag{3.18}$$

From (3.15) and (3.18), we obtain a more accurate asymptotic expansion for $\tilde{\rho}_n$: for $k = \frac{3}{4} - n$ and $n = 1, 2, \ldots$,

$$\tilde{\rho}_n = \frac{k\pi i}{\omega_2} - \frac{\sqrt{2}}{1+i} \left(\frac{i}{2\mu_2} - \frac{\beta}{\mu_1}\right) \frac{1}{k\pi} + O(n^{-2}).$$

Since in sector $\mathcal{S}_0$, $\lambda_k = \tilde{\rho}_k^2$, $\omega_2 = e^{i5/4\pi}$ and $\omega_2^2 = i$, so

$$\lambda_n = -\frac{2\beta}{\mu_1} + \frac{i}{\mu_2} + (k\pi)^2 i + O(n^{-1}). \tag{3.19}$$

In sector $\mathcal{S}_2$, the eigenvalues of problem (2.6) can be obtained by similar arguments provided that we choose

$$\omega_1 = e^{i\frac{1}{4}\pi} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i,$$
$$\omega_2 = e^{i\frac{3}{4}\pi} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

and $\omega_3 = -\omega_2$, $\omega_4 = \omega_1$, so that the following inequality holds:

$$\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2) \le \mathrm{Re}(\rho\omega_3) \le \mathrm{Re}(\rho\omega_4), \forall \rho \in \mathcal{S}_2.$$

Hence, in sector $\mathcal{S}_2$, the asymptotic expression of $\Delta(\rho)$ is

$$\begin{aligned}
& e^{\rho\omega_1} \Delta(\rho) = (2\sqrt{2}\mu_1\mu_2)\rho^{11} \\
= \; & \Bigg\{ \left[-i + \left(\frac{\sqrt{2}}{2\mu_2} - \frac{\sqrt{2}\beta}{\mu_1}\right) \frac{1}{\rho} + O(\rho^{-2})\right] e^{\rho\omega_3} \\
& + \left[-1 + \left(\frac{\sqrt{2}}{2\mu_2} + \frac{\sqrt{2}\beta}{\mu_1}\right) \frac{1}{\rho} + O(\rho^{-2})\right] e^{\rho\omega_2} \Bigg\}.
\end{aligned}$$

Since $\omega_3 = -\omega_2$, so the root ρ_n in $\mathcal{S}_2$ is given by

$$\rho_n = \frac{1}{\omega_2} k\pi i + \frac{\sqrt{2}}{1+i} \left(\frac{1}{2\mu_2} - \frac{\beta i}{\mu_1}\right) \frac{1}{k\pi} + O(n^{-2}),$$

where $k = \frac{3}{4} - n$, and $n = 1, 2, \ldots$. In sector $\mathcal{S}_2$, $\lambda_k = \tilde{\rho}_k^2$, $\omega_2 = e^{i3/4\pi}$ and $\omega_2^2 = -i$, so we have the conjugate of the eigenvalues of problem (3.3) namely,

$$\lambda_n = -\frac{2\beta}{\mu_1} - \frac{i}{\mu_2} - (k\pi)^2 i + O(n^{-1}), \tag{3.20}$$

for $k = \frac{3}{4} - n$ and $n = 1, 2, \ldots$. The exact same procedures can be carried out for the remaining sectors and end up either (3.19) or (3.20) and so the theorem follows. □

Remark 3.1. *Let $\{\lambda_n : n \ge 1\}$ be an enumeration of all nonzero eigenfrequencies of the closed loop system. As a consequence of Theorem 3.2, we have*

$$\inf_{n \ge 1} \mathrm{dist}(\lambda_n, i\mathbb{R}) = \delta > 0,$$

and there are positive constants N and M such that

$$\inf_{n \ne m, n, m \ge N} |\lambda_n - \lambda_m| > 0, \quad |Re\lambda_n| \le M, \quad \forall n \in \mathrm{N}.$$

4. GENERATION OF SEMIGROUP AND RIESZ BASIS PROPERTY OF GENERALIZED EIGENFUNCTIONS

To discuss the exponential stability of the target state of system (2.11), we put it into a "double-sized" state space and show that the eigenfunctions of the system form a Riesz basis. For convenience, we denote by X the Hilbert space

$$X := L^2[0,\ell]\times \mathbb{C}\times\mathbb{C},$$

and define a subset of X by $V_0 := \{Y = [y(x), y_x(0), y(\ell)] \in X \mid y \in H^2[0,\ell], y(0) = 0\}$, and endow it with a norm

$$\|Y\|_1^2 := \int_0^\ell EI|y_{xx}(x)|^2dx + |y_x(0)|^2,$$

and the corresponding inner product. It is easy to see that $(V_0, \|\cdot\|_1)$ is also a Hilbert space. We now define the state Hilbert space as

$$\mathcal{H} := V_0\times X = V_0 \times L^2[0,1]\times\mathbb{C}\times\mathbb{C},$$

with an inner product defined by:

$$\begin{aligned}&< [Y_1, Z_1], [Y_2, Z_2] >_{\mathcal{H}}\\ =& \int_0^\ell EIy_1''(x)\overline{y_2''(x)} + y_1'(0)\overline{y_2'(0)}\\ &+\int_0^\ell \rho z_1(x)\overline{z_2(x)}dx + I_h\theta_1\overline{\theta_2} + m_t\eta_1\overline{\eta_2},\end{aligned}$$

where, for $j = 1, 2$,

$$Y_j = [y_j(x), y_j(0), y_j(\ell)] \in V_0,$$

and

$$Z_j = [z_j(x), \theta_j, \eta_j] \in X.$$

In $\mathcal{H}$, we define operator $\mathcal{A}$ by

$$\mathcal{A}[Y, Z] := [Z + \mathbb{B}Y, \mathbb{A}Y], \tag{4.1}$$

$$\begin{aligned}\mathcal{D}(\mathcal{A}) := \Big\{&[Y, Z] \in \mathcal{H} \,\Big|\, Y \in V_0, y \in H^4[0,\ell],\\ &y''(\ell) = 0, Z = [z(x), \theta, \eta], z \in H^2[0,\ell],\\ &z(0) = 0,\quad z''(\ell) = 0,\\ &\theta = z'(0) - \beta\tfrac{EI}{I_h}y''(0),\quad \eta = z(\ell)\Big\}.\end{aligned} \tag{4.2}$$

If we put

$$Z(t) := \dot{Y}(t) - \mathbb{B}Y(t),\quad \Psi(t) := [Y(t), Z(t)] \in \mathcal{H}, \tag{4.3}$$

then equation (2.11) can be rewritten into an evolutionary equation in $\mathcal{H}$:

$$\frac{d\Psi(t)}{dt} = \mathcal{A}\Psi(t),\quad t > 0, \tag{4.4}$$

with initial condition

$$\Psi(0) := [Y(0), 0] \in \mathcal{H}.$$

Lemma 4.1. *Operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by (4.2) and (4.1) is a closed densely defined linear operator with compact resolvents and generates a C_0 semigroup on $\mathcal{H}$.*

Proof. First, for any $[Y.Z] \in D(\mathcal{A})$, we have

$$\begin{aligned}&Re < \mathcal{A}[Y, Z], [Y, Z] >_{\mathcal{H}}\\ =& < Z + \mathbb{B}Y, Y >_1 + < Y, Z + \mathbb{B}Y >_1\\ &+ < \mathbb{A}Y, Z >_X + < Z, \mathbb{A}Y >_X\\ =& Re z_x(0)\overline{y_x(0)} - \frac{(EI)^2\beta}{I_h}|y_{xx}(0)|^2\\ \le& \epsilon|z_x(0)|^2 + \frac{1}{\epsilon}|y_x(0)|^2 - \frac{(EI)^2\beta}{I_h}|y_{xx}(0)|^2\\ \le& 2\epsilon\Big[\Big|z_x(0) - \frac{(EI)\beta}{I_h}y_{xx}(0)\Big|^2 + \Big|\frac{(EI)\beta}{I_h}y_{xx}(0)\Big|^2\Big]\\ &+\frac{1}{\epsilon}|y_x(0)|^2 - \frac{(EI)^2\beta}{I_h}|y_{xx}(0)|^2\\ \le& 2\epsilon\|Z\|^2 + \frac{1}{\epsilon}\|Y\|_1^2\\ &+\Big[2\epsilon\frac{(EI\beta)^2}{I_h^2} - \frac{(EI)^2\beta}{I_h}\Big]|y_{xx}(0)|^2.\end{aligned}$$

By choosing ϵ sufficiently small such that

$$\Big[2\epsilon\frac{(EI\beta)^2}{I_h^2} - \frac{(EI)^2\beta}{I_h}\Big] < 0$$

and taking $M = \max\{2\epsilon, \frac{1}{\epsilon}\}$, we have

$$\begin{aligned}&Re < \mathcal{A}[Y, Z], [Y, Z] >_{\mathcal{H}}\\ \le& M\|[Y, Z]\|_{\mathcal{H}}^2 + \Big[2\epsilon\frac{(EI\beta)^2}{I_h^2}\\ &-\frac{(EI)^2\beta}{I_h}\Big]|y_{xx}(0)|^2.\end{aligned}$$

So, $(\mathcal{A} - MI)$ is a dissipative operator on $\mathcal{H}$.

Next we show that if $\lambda > 0$, then $\lambda \in \rho(\mathcal{A})$. For $\lambda > 0$ and for each $[W, U] \in \mathcal{H}$, we consider the resolvent equation

$$(\lambda I - \mathcal{A})[Y, Z] = [W, U],\quad [Y, Z] \in \mathcal{D}(\mathcal{A}),$$

i.e.,

$$\lambda Y - Z - \mathbb{B}Y = W \in V_0,\quad \text{and}\quad \lambda Z - \mathbb{A}Y = U \in X.$$

Then,

$$\lambda^2 Y - \mathbb{A}Y - \lambda\mathbb{B}Y = \lambda W + U\quad\text{in}\quad X,$$

which is just, for $0 < x < \ell$,

$$\begin{cases} EIy_{xxxx} + \rho\lambda^2 y(x) = \rho[\lambda w(x) + u(x)],\\ I_h\lambda^2 y_x(0) - EIy_{xx}(0)\\ \quad -\beta\lambda EIy_{xx}(0) = I_h[\lambda w_\theta + u_\theta],\\ \lambda^2 m_t y(\ell) - EIy_{xxx}(\ell) = m_t[\lambda w_\eta + u_\eta],\\ y(0) = 0,\quad EIy_{xx}(\ell) = 0.\end{cases} \tag{4.5}$$

Since $\Delta(\lambda) \neq 0$ when $\lambda > 0$, so from the theory of ordinary differential equations, for $\lambda > 0$, there is a unique function $y(\cdot,\lambda) \in H^4[0,\ell] \times \mathbb{C} \times \mathbb{C}$ such that

$$y(x,\lambda) = \rho \int_0^\ell G(x,\lambda,s)[\lambda w(s) + u(s) + \lambda^2 h(s)]ds \tag{4.6}$$

that satisfies equation (4.5), where $G(x,\lambda,s)$ is the Green's function (see [11]) and

$$h(s) := c_1 s + c_2 s^2 + c_3 s^3$$

with

$$c_3 := \frac{m_t(\lambda w_\theta + u_\theta - \lambda w_\eta - u_\eta)}{6m_t I_h{}^{-1} l(1+\beta\lambda)EI + 2\lambda^2 m_t l^3 + 6EI},$$

$$c_2 := -3lc_3,$$

$$c_1 := \frac{I_h(\lambda w_\theta + u_\theta) - 6c_3 l(1+\beta\lambda)EI}{\lambda^2 I_h}.$$

Thus

$$Z = \lambda Y - \mathbb{B}Y - W \in H^2[0,\ell] \times \mathbb{C} \times \mathbb{C} \tag{4.7}$$

and so $[Y,Z] \in \mathcal{D}(\mathcal{A})$. Since the Sobolev Imbedding Theorem ensures that $R(\lambda,\mathcal{A})[W,U] = [Y,Z]$ is a compact operator, so the Lumer-Phillips theorem says that $\mathcal{A}-MI$ generates a C_0−semigroup of contraction $e^{\mathcal{A}t}$ on Hilbert space $\mathcal{H}$, and so does $\mathcal{A}$. □

Corollary 4.1. *The spectrum of $\mathcal{A}$ is $\sigma(\mathcal{A}) = \{\lambda_0 = 0, \lambda_1, \lambda_2, \ldots\}$, where each λ_n ($n \in \mathbb{N}$) is an eigenvalue of the closed loop system that can be determined by Theorem 3.2.*

Direct calculations show that $\lambda = 0$ is an eigenvalue of $\mathcal{A}$ of degree two with generalized eigenfunctions $[Y_0, 0]$ and $[0, Y_0]$.

As a direct application of Theorem 2.1 in [12, pp.1326] on (4.6), we may conclude the following lemma about the completeness of the generalized eigenfunctions of $\mathcal{A}$ in $\mathcal{H}$.

Lemma 4.2. *Let $\mathcal{A}$ be defined by (4.2) and (4.1). Then the system of generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

The Riesz basis property can be deduced from Theorem 1.1 of [13], which is recalled as follows.

Lemma 4.3. *Let X be a separable Hilbert space, and A be the generator of C_0 semigroup $S(t)$. Suppose that*

1) $\sigma(A) = \sigma_1(A) \cup \sigma_2(A)$ and $\sigma_2(A) = \{\lambda_k,\}_{k=1}^\infty$ is consisted of isolated eigenvalues of finite multiplicity;

2) $\sup_{k\geq 1} m_a(\lambda_k) < \infty$, where $m_a(\lambda_k) = dim E(\lambda; A)X$ and $E(\lambda_k; A)$ is the Riesz projector associated with λ_k;

3) there is a constant C such that

$$\sup\{Re\lambda : \lambda \in \sigma_1(A)\} \leq C \leq \inf\{Re\lambda : \lambda \in \sigma_2(A)\}$$

and

$$\inf_{k\neq m,} |\lambda_k - \lambda_m| > 0.$$

Then the following assertions are true:

i) There exist two $S(t)$-invariant closed subspaces X_1 and X_2 with property that $\sigma(A|_{X_1}) = \sigma_1(A)$ and $\sigma(A|_{X_2}) = \sigma_2(A)$, and $\{E(\lambda_k; A)X_2\}_{k=1}^\infty$ forms a Riesz basis of subspaces for X_2. Furthermore,

$$X = \overline{X_1 \oplus X_2}.$$

ii) If $\sup_{k\geq 1} \|E(\lambda_k; A)\| < \infty$, then

$$D(A) \subset X_1 \oplus X_2 \subset X.$$

iii) X has the topological direct sum decomposition

$$X = X_1 \oplus X_2$$

if and only if

$$\sup_{n\geq 1} \|\sum_{k=1}^n E(\lambda_k; A)\| < \infty.$$

Combining Lemma 4.1, 4.2 and 4.3 with Theorem 3.2, we arrive to the following conclusion.

Theorem 4.1. *There is a sequence of generalized eigenvector of $\mathcal{A}$ that forms a Riesz basis for $\mathcal{H}$.*

Proof Take $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$ and $\sigma_1(\mathcal{A}) = \{\infty\}$. It is easy to see from Theorem 3.2 that $\sigma_2(\mathcal{A})$ satisfies all conditions of Lemma 4.3. Thus the first assertion of the lemma says that $\{E(\lambda_k; \mathcal{A})\mathcal{H}\}_{k=0}^\infty$ forms a subspace Riesz basis for $\overline{span\{E(\lambda_k; \mathcal{A}), k \in \mathbb{N}\}}$. Lemma 4.2 in turns implies that

$$\overline{span\{E(\lambda_k; \mathcal{A})\mathcal{H} : k \in \mathbb{N}\}} = \mathcal{H}.$$

Therefore, $\{E(\lambda_k; \mathcal{A})\mathcal{H}\}_{k=0}^\infty$ forms a subspace Riesz basis for $\mathcal{H}$. From Theorem 3.2, we know that the eigenvalue λ_k for k large enough is simple, so we can choose a sequence $\{\Psi_{k,j} : k \in \mathbb{N}, 0 \leq j \leq m_a(\lambda_k) - 1\}$ of generalized eigenvectors of $\mathcal{A}$ to form a Riesz basis for $\mathcal{H}$. Denote by $\{\Psi^*_{k,j} : k \in \mathbb{N}, 0 \leq j \leq m_a(\lambda_k)\}$ its biorthogonal system, and assume without loss of generality that λ_k for $k \geq 1$ are simple. Then for any $\Psi := [Y,Z] \in \mathcal{H}$, we have

$$\begin{aligned}\Psi &= [Y,Z] \\ &= <\Psi, \Psi^*_{0,0}>_{\mathcal{H}} [Y_0, 0] + <\Psi, \Psi^*_{0,1}>_{\mathcal{H}} [0, Y_0] \\ &\quad + \sum_{k=1}^\infty <\Psi, \Psi^*_k>_{\mathcal{H}} \Psi_k,\end{aligned}$$

where Y_0 is given in Remark 2.1. Let $S(t)$ be the semigroup generated by $\mathcal{A}$. Then

$$\begin{aligned}S(t)\Psi &= S(t)[Y,Z] \\ &= [<\Psi, \Psi^*_{0,0}>_{\mathcal{H}} + t <\Psi, \Psi^*_{0,1}>_{\mathcal{H}}][Y_0, 0] \\ &\quad + <\Psi, \Psi^*_{0,1}>_{\mathcal{H}} [0, Y_0] \\ &\quad + \sum_{k=1}^\infty e^{\lambda_k t} <\Psi, \Psi^*_k>_{\mathcal{H}} \Psi_k.\end{aligned}$$

□

Corollary 4.2. *For initial state $\Psi := [Y, 0]$, system (4.4) is exponentially stable to the state $< \Psi, \Psi^*_{00} >_{\mathcal{H}} Y_0$.*

Proof If we let $[Y(t), Z(t)] := S(t)\Psi = S(t)[Y, 0]$, we have

$$Y(t) = [< \Psi, \Psi^*_{00} >_{\mathcal{H}} + t < \Psi, \Psi^*_{0,1} >_{\mathcal{H}}]Y_0 + Y_1(t)$$

and

$$Y_1(t) := \mathcal{P}\sum_{k=1}^{\infty} e^{\lambda_k t} < \Psi, \Psi^*_k >_{\mathcal{H}} \Psi_k,$$

where $\mathcal{P}$ is the projection from $\mathcal{H}$ to V_0, and

$$Z(t) = \dot{Y}(t) = < \Psi, \Psi^*_{01} >_{\mathcal{H}} Y_0 + Z_1(t),$$

$$Z_1(t) = (I - \mathcal{P})\sum_{k=1}^{\infty} e^{\lambda_k t} < \Psi, \Psi^*_k >_{\mathcal{H}} \Psi_k.$$

Since $Re\lambda < -\delta < 0$ (see Remark 3.1), we have

$$||Y_1(t)||_1^2 + ||Z_1(t)||^2 = O(e^{-\delta t}) \to 0, \quad t \to \infty.$$

It is not hard to check that $\Psi^*_{0,1} = \frac{1}{I_T}[0, Y_0]$ because $[0, Y_0]$ is the corresponding eigenvector of $\mathcal{A}^*$ for the eigenvalue $\lambda = 0$. Therefore, for any initial state $\Psi(0) = \Psi = [Y, \mathbb{A}Z]$, with $Z \in D(\mathbb{A})$, we have $< \Psi, \Psi^*_{0,1} >_{\mathcal{H}} = 0$. Thus,

$$Y(t) = < \Psi, \Psi^*_{00} >_{\mathcal{H}} Y_0 + Y_1(t),$$

and when $t \to \infty$,

$$||Y(t) - < \Psi, \Psi^*_{00} >_{\mathcal{H}} Y_0|| = O(e^{-\delta t}) \to 0.$$

□

We are now in the position to give an explicit expression for $< \Psi, \Psi^*_{00} >_{\mathcal{H}} Y_0$. For this aim, we need only to calculate Ψ^*_{00}. From the theory of operator, we know that $\mathcal{A}^{*2}\Psi^*_{00} = 0$. We can assume without loss of generality that $\mathcal{A}^*\Psi^*_{00} = \Psi^*_{01}$. Let $\Psi^*_{00} = [W, U] \in \mathcal{H}$, for any $[Y, Z] \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}
& < \mathcal{A}[Y, Z], [W, U] >_{\mathcal{H}} \\
= & < Z + \mathbb{B}Y, W >_1 + < \mathbb{A}Y, U >_X \\
= & \int_0^\ell EIz_{xx}(x)\overline{w_{xx}(x)}dx + z_x(0)\overline{w_x(0)} \\
& - EIy_{xxx}(\ell)\overline{u(\ell)} - EIy_{xx}(0)\overline{u_x(0)} \\
& - \int_0^\ell EIy_{xx}\overline{u_{xx}(x)}dx + EIy_{xx}(0)\overline{u_\theta} \\
& + EIy_{xxx}(\ell)\overline{u_\eta} \\
= & EIz_x(x)\overline{w_{xx}(x)}\Big|_0^\ell - EIz(x)\overline{w_{xxx}(x)}\Big|_0^\ell \\
& + \int_0^\ell EIz(x)\overline{w_{xxxx}(x)}dx + z_x(0)\overline{w_x(0)} \\
& - EIy_{xxx}(\ell)\overline{u(\ell)} - EIy_{xx}(0)\overline{u_x(0)} \\
& - \int_0^\ell EIy_{xx}\overline{u_{xx}(x)}dx + EIy_{xx}(0)\overline{u_\theta} \\
& + EIy_{xxx}(\ell)\overline{u_\eta}.
\end{aligned}$$

Equate it, after choosing

$$u(x) = 0, u_\eta = u(\ell) = 0, u_\theta = -\beta, w_{xx}(\ell) = 0, \tag{4.8}$$

with

$$\begin{aligned}
& < [Y, Z], \mathcal{A}^*[W, U] >_{\mathcal{H}} \\
= & < [Y, Z], [0, Y_0] >_{\mathcal{H}} \\
= & \int_0^\ell \rho z(x)\, x dx \\
& + I_h[z_x(0) - \frac{EI}{I_h}\beta y_{xx}(0)] + m_t z(\ell)\ell,
\end{aligned}$$

then function $w(x)$ will satisfy the following equation

$$\begin{cases} EIw_{xxxx}(x) = \rho x, \\ w(0) = 0, w_{xx}(\ell) = 0, \\ w_x(0) - EIw_{xx}(0) = I_h \\ -EIw_{xxx}(\ell) = m_t\ell, \end{cases} \tag{4.9}$$

together with normalized condition

$$< [Y_0, 0], [W, U] >_{\mathcal{H}} = w_x(0) = 1. \tag{4.10}$$

Solving equation (4.9) we get

$$w(x) = \frac{\rho}{EI}\frac{x^5}{5!} + a\frac{x^4}{4!} + b\frac{x^3}{3!} + c\frac{x^2}{2!} + dx \tag{4.11}$$

where $a = -\frac{2(I_h-1)}{EI\ell^2} - \frac{2m_t}{EI} - \frac{2\rho\ell}{EI}$, $b = \frac{3\rho\ell^2}{2EI} + \frac{m_t\ell}{EI} + \frac{2(I_h-1)}{EI\ell}$, $c = -\frac{(I_h-1)}{EI}$ and $d = 1$. Thus, for any $\Psi = [Y, Z] \in \mathcal{H}$, we have

$$\begin{aligned}
& < \Psi, \Psi^*_{00} >_{\mathcal{H}} \\
= & < [Y, Z], [W, U] >_{\mathcal{H}} \\
= & \int_0^\ell EIy_{xx}(x)\overline{w_{xx}(x)}dx + y_x(0)\overline{w_x(0)} + I_h z_\theta \overline{u_\theta} \\
= & \int_0^\ell EIy_{xx}(x)\overline{w_{xx}(x)}dx + y_x(0) - \beta I_h z_\theta,
\end{aligned}$$

and we are free to choose the parameter β to alter this quantity to achieve our task with w given by (4.11).

Conclusion 4.1. *We show in this paper that a flexible arm robot can be controlled by a feedback control without disrupting the rigid mode shape. We also see from the proof of Corollary 4.2 that the C_0 semigroup $S(t)$ determined by the system is not uniformly bounded even if $\beta = 0$. Other stabilizing feedback control such as*

$$T(t) = EI\beta y_{xxt}(0, t) - \alpha y_x(0, t),$$

is not suitable in practice because it will drive the state of the closed loop system back to the reference line rather than the target position.

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