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ERROR PROBABILITIES OF SYNCHRONOUS DS/CDMA COMMUNICATIONS OVER MULTIPATH RAYLEIGH FADING CHANNELS

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Abstract — This paper derives error probabilities for BPSK synchronous DS/CDMA communications over multipath Rayleigh fading WSSUS channels using a RAKE receiver. Both cases of perfectly and approximately synchronized signal transmission are considered. Numerical results are presented.

I. INTRODUCTION

Recently, synchronous DS/CDMA techniques on multipath fading channels have received considerable interest because in addition to exploiting the multipath diversity it is possible to increase the system capacity by reducing the multiple-access interference through employing optimized signature sequences having small crosscorrelations within a small region around the origin [1]. Various sets of such sequences have been proposed, e.g., [2]-[4]. In evaluating the performance and estimating the system capacity, however, analytical techniques in previous published literature have been available only for asynchronous DS/CDMA over multipath fading channels, e.g., [5], and synchronous DS/CDMA on AWGN channels [6], [7]. In this paper, we derive the error probability of BPSK synchronous DS/CDMA communications on multipath Rayleigh fading WSSUS channels using a RAKE receiver for multipath diversity reception. Both cases of perfectly and approximately synchronized transmission are considered. The present work thus enables computation of the performance and estimation of the system capacity for synchronous DS/CDMA using optimized signature sequences.

II. SYSTEM MODEL

The system under consideration consists of K users. Suppose that the receiver is intended to detect the user-L signal. For the user \( L \) (\( 1 \leq L \leq K \)), the transmitted signal is

\[
s_r(t) = \text{Re}[u_r(t) \exp(j2\pi f_c t)]
\]

where \( f_c \) is the carrier frequency and

\[
u_r(t) = e^{j\varphi_r} \sqrt{2P_c} \sum_{i=-\infty}^{\infty} b_i^{(t)} \Psi(t - T_i - iT_b).
\]

In (1), \( \varphi_r \) is the carrier phase, \( P_r \) is the signal power, \( b_i^{(t)} \in \{+1,-1\} \) is the \( i \) th bit, \( T_i \) is the bit duration, \( T_i \in [0,T_b] \) is the time delay between bit boundaries of the user-\( L \) and the user-\( L \) signals, and \( \Psi(t) \) is the spectral-spread waveform defined as follows. Let \( (a_0^{(t)}, a_1^{(t)}, \ldots, a_{N-1}^{(t)}) \) be the user-\( L \) signature sequence of length \( N \) satisfying \( a_n^{(t)} \in \mathbb{R} \) and

\[\sum_{n=0}^{N-1} a_n^{(t)} = N.\]

Then

\[
\Psi(t) = \sum_{n=0}^{N-1} a_n^{(t)} \psi(t - nT_c)
\]

where \( T_c = T_b/N \) is the chip duration, and \( \psi(t) \), time-limited within \([0,T_c]\) and satisfied \( \int_0^{T_c} \psi^2(t)dt = T_c \), is the (real) chip waveform. Since synchronized transmission is considered, it is assumed that bit boundaries of all \( K \) signals fall within a time window of \( LT_c \) seconds where \( L \) is a nonnegative integer. In perfectly synchronous DS/CDMA, one gets \( L = 0 \) so that \( T_o = 0 \forall t \). In approximately synchronized transmission, \( L \) is nonzero, so that (i) \( T_L = 0 \), and (ii) if \( \ell \neq L \), \( T_o \) follows triangular distribution over \([-LT_o,LT_o]\). For the sake of later use, let \( k_o \equiv \lfloor T_o/T_c \rfloor \) and \( T_o^{(t)} \equiv T_o \mod T_c \). It can be shown that, for \( L \neq 0 \) and \( \ell \neq L \),

\[
\Pr\{k_o = n\} = \begin{cases} (2L)^{-1}(2L-2n-1) & \text{ne} \{0,1,\ldots,L-1\} \\ (2L)^{-1}(2L+2n+1) & \text{ne} \{-L,-L+1,\ldots,-1\} \\ 0 & \text{otherwise} \end{cases}
\]

and \( T_o^{(t)} \) is uniformly distributed over \([0,T_o] \).

The signal \( s_r(t) \) is transmitted via a multipath Rayleigh fading WSSUS channel with

\[
\frac{1}{2}E[|h_r(\tau)|^2] = Q_r(\tau) \delta(\tau - \tau)
\]

where \( h_r(\tau) \) is the low-pass equivalent channel impulse response and \( Q_r(\tau) \) is the delay power spectrum. It is assumed that (i) all \( K \) channels are statistically independent, (ii) all channels are slowly varying relative to \( T_o \), (iii) \( Q_r(t) = 0 \) for \( t < 0 \) and \( t > T_M^{(t)} \) where \( T_M^{(t)} \) is the...
maximum delay spread, and (iv) \( T^{(f)}_M = \ell = 1, \ldots, K \), are small compared to \( T_c \). The receiver input is given by 
\[
r(t) = \text{Re}(v(t) \exp(j2\pi f_c t))
\]
where 
\[
v(t) = n(t) + \sum_{\ell=1}^{K} u_\ell(t-\tau) h_\ell(t) dt.
\]

The complex AWGN \( n(t) \) has a one-sided power spectral density \( \eta \), i.e., 
\[
\frac{1}{2} E\{n(t) n^*(t+\tau)\} = \eta |\delta(\tau)|.
\]

The RAKE receiver under consideration (Fig. 1) consists of multiple branches followed by a combiner. A set of time-delayed uniformly spaced user-\( L \) signature sequences is used in the correlation, so that each branch output represents signal energy arrived from multipaths within a particular \( T \)-width time segment of \( h_L(t) \) [8]. Let \( M \) be the number of branches. In the \( q \) th branch (\( 0 \leq q \leq M-1 \)), a matched filter matched to \( \Psi_L(t) \) delayed by \( qT_c \) seconds is used to process \( s(t) \). Coherent detection followed by maximal ratio combining (MRC) on all branch outputs is assumed. Without loss of generality, we analyze the case of detecting \( b^{(L)}_m \).

After some algebraic manipulations we get (Appendix 1) 
\[
X_q = T_c^{-1} \psi_L^* (t-qT_c) dt.
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It is easy to show that $G_r$ is the average power gain due to transmitting a signal over $h_r(t)$. Let $s_m^{(t)} = \sqrt{G_r}(x_m^{(t)} + jy_m^{(t)})$, so that (i) $x_m^{(t)}$ and $y_m^{(t)}$ are independent zero-mean Gaussian random variables conditioned on $T_r^{(t)}$, and (ii) $x_m^{(t)} (y_m^{(t)})$, $m = 0, \ldots, M_r - 1$, are correlated with $E(x_m^{(t)}y_m^{(t)}) = (2G_r)^{-1}E(s_m^{(t)}s_m^{(t)*})$. In the derivation of BER, we analyze

$$Z = (G_L \sqrt{P_L N})^{-1} Z$$

(14)

and consider the case of, say, $b_0^{(t)} = +1$ so that the BER, $P_e$, is given by $P_e = \int_{0}^{\infty} p(Z) dZ$. The p.d.f. $p(Z)$ is obtained by taking the inverse Fourier transform on the characteristic function of $Z$.

Expanding $Z$ gives

$$Z = \sum_{q=0}^{M_r-1} \sum_{k=0}^{M_r-1} x_k^{(t)} y_q^{(t)} + \sum_{q=0}^{M_r-1} \sum_{m=0}^{N_q^{(t)}} (x_m^{(t)} + y_m^{(t)}) C_{q,m} \frac{H_{q,m,t}}{N}$$

(15)

where

$$\Gamma = G_L P_L T_r / \eta$$

(16)

is the received bit energy to noise ratio, and $N_q = \sqrt{\eta / T_r}(N_q^{(Re)} + jN_q^{(Im)})$. Let $x_k^{(t)} \triangleq N^{-1/2} \sum_{m=0}^{M_r-1} a_k^{(t)} H_{q,m,t}$ and $y_q^{(t)} \triangleq N^{-1/2} \sum_{m=0}^{M_r-1} a_q^{(t)} H_{q,m,t}$. Let $x_L^T \triangleq [x_0^{(L)}, \ldots, x_{M_L-1}^{(L)}]$, $y_L^T \triangleq [y_0^{(L)}, \ldots, y_{M_L-1}^{(L)}]$, $\tilde{x}_L^T \triangleq [\tilde{x}_0^{(L)}, \ldots, \tilde{x}_{M_L-1}^{(L)}]$, $\tilde{y}_L^T \triangleq [\tilde{y}_0^{(L)}, \ldots, \tilde{y}_{M_L-1}^{(L)}]$, $\tilde{\mathbf{R}}_{Re} \triangleq [N_0^{(Re)}, \ldots, N_{M_L-1}^{(Re)}]$ and $\tilde{\mathbf{R}}_{Im} \triangleq [N_0^{(Im)}, \ldots, N_{M_L-1}^{(Im)}]$. Let $\mathbf{C}_{L,L}$ be an $M_r \times M_r$ matrix with the $(q,m)$th element $N^{-1/2} C_{q,m,t}(q-m)$. Consequently, $Z$ can be expressed as the sum of bilinear forms in correlated Gaussian random variables:

$$Z = \frac{1}{\sqrt{2T}} \mathbb{E}[\mathbf{R}_{Re}\tilde{x}_L^T] + \mathbb{E}[\tilde{\mathbf{R}}_{Im}\tilde{y}_L^T] + x_L C_{L,L} y_L^T + \tilde{x}_L C_{L,L} \tilde{y}_L^T$$

$$+ \sum_{t=1}^{K} \sum_{t=1}^{T} \sqrt{G_L P_L} (\tilde{x}_L^t \tilde{y}_L^t + \tilde{y}_L^t \tilde{x}_L^t).$$

(17)

Although the conditional joint distribution of $x_q^{(t)} \forall q$ (or $y_q^{(t)} \forall q$) are jointly zero-mean Gaussian, the unconditional one may not. However, for mathematical tractability in the derivation of $P_e$, it is assumed that $x_q^{(t)} \forall q$ and $y_q^{(t)} \forall q$ are approximately jointly zero-mean Gaussian with $E(x_q^{(t)}x_q^{(t)})$ and $E(y_q^{(t)}y_q^{(t)})$ averaged over random variables involved in them ($T_r^{(t)}$, $x_t^{(t)}$, $b_0^{(t)}$, $b_1^{(t)}$, $b_0^{(t)}$ and $b_1^{(t)}$). Expressions of $E(x_q^{(t)}x_q^{(t)})$ and $E(y_q^{(t)}y_q^{(t)})$ are listed in Appendix 2. It is easy to show that $N_q^{(Re)}(N_q^{(Im)})'$s are jointly zero-mean Gaussian with $E(x_q^{(t)}x_q^{(t)})$ and $E(y_q^{(t)}y_q^{(t)})$ independently. Let $S_r, N$ and $S_L$ be $M_L \times M_L$ matrices with the $(q,q')$th elements given by $E(x_q^{(t)}x_q^{(t)})$, $E(y_q^{(t)}y_q^{(t)})$ and $(2G_r)^{-1}E(s_q^{(t)}s_q^{(t)*})$, respectively. Since these matrices are covariance matrices, they are positive semidefinite [9, ch. 22.3]. Although these matrices can be singular with zero determinants, in practice we are only interested in the case that $S_r, N$ and $S_L$ are non-singular, so that they are positive definite.

Consequently, we have $p(\tilde{x}_L) = (2\pi)^{-M_r/2} |S_r|^{-1/2} \exp(-1/2 \tilde{x}_L S_r^{-1} \tilde{x}_L^T)$. Similar expressions exist for $p(y_L)$, $p(x_L)$, $p(\tilde{y}_L)$, $p(\tilde{\mathbf{R}}_{Re})$ and $p(\tilde{\mathbf{R}}_{Im})$.

The characteristic function of $Z$ is given by

$$\mathbb{E}[\exp(-j\omega \tilde{Z})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-j\omega \tilde{x}_L) p(x_L) p(y_L) p(\tilde{\mathbf{R}}_{Re}) p(\tilde{\mathbf{R}}_{Im})$$

$$dx_L dy_L d\tilde{\mathbf{R}}_{Re} d\tilde{\mathbf{R}}_{Im} \left[ \prod_{t=1}^{T} \prod_{t=1}^{T} \mathbb{E}[\mathbf{R}_{Re}\tilde{x}_L^t] p(x_L^t) p(y_L^t) d\tilde{x}_L^t d\tilde{y}_L^t \right].$$

(18)

Consider

$$\mathbb{E}[\exp(-j\omega \sqrt{G_L P_L} x_L^t)] x_L^t$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-j\omega \sqrt{G_L P_L} \tilde{x}_L^t) p(\tilde{x}_L^t) d\tilde{x}_L^t.$$ 

Applying (11.12a) of [9] gives

$$E[\exp(-j\omega \sqrt{G_L P_L} \tilde{x}_L^t)] x_L^t = \exp(-1/2 \omega^2 G_L P_L x_L S_L x_L^T x_L).$$

Similar relationships exist for $E[\exp(-j\omega \sqrt{G_L P_L} \tilde{y}_L^t)] y_L^t$, etc. Consequently, (18) is reduced to

$$E[\exp(-j\omega \tilde{Z})]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(x_L U x_L^T + y_L U y_L^T) p(x_L) p(y_L) dx_L dy_L,$$

where

1 Throughout this paper, the upper left element of a matrix is indexed by (0,0).
Applying (11.12.2) of [9] yields
\[
E\{\exp(-j\omega Z)\} = I + \omega^2 A + 2j\omega B \quad (19)
\]
where \( I \) is the \( M \times M \) identity matrix, \( A = \sum_{l=1}^{K} \frac{G_{PL}}{G_{PL}} S_l S_l^t + \frac{1}{2} S_l^t N_l N_l^t \), and \( B = S L C_{L^L} \).

Let \( \mu_0, \mu_1, \ldots, \mu_{2M-1} \) be the \( 2M \) latent roots of \( I - s^2 A + 2sB \), that is, \( I - \mu_i^2 A + 2\mu_i B = 0 \). It is known that [10, ch. 14.11] these latent roots coincide with \( \{ \pm \sqrt{\tau_{m\omega}}, \pm \sqrt{\tau_{m\omega}} \} \) where \( \tau_{m\omega} = 0.5T_c \) (r.m.s. delay spread) was assumed. The multipath spread was truncated at \( T_M = 2T_c \). Fig. 2 plots the \( P_e \) against \( \Gamma \) for systems with \( L = 3 \) (approximately synchronized transmission) using both sets of signature sequences under consideration. The result demonstrates that synchronous DS/CDMA communications using optimized cyclic codes provide significant suppression of the multiple-access interference in the multipath Rayleigh fading environment.

**APPENDIX I**

Substituting (2) into (5) yields

\[
X_q = \sum_{k=0}^{N-1} a_k^{(L)} \xi_{k+q}
\]

where \( \xi_k = T_c^{-1} \int_{t-kT_c}^{t+(k+1)T_c} \nu(t) \psi(t-kT_c)dt \). Let

\[
\xi_k^{(t)} = T_c^{-1} \int_{t-kT_c}^{t+(k+1)T_c} \int_0^T u(t-\tau)\psi(t-kT_c)\psi(t-kT_c)dt \quad (A1)
\]

so that \( \xi_k = N_k + \sum_{q=1}^{K} \xi_k^{(t)} \) where \( N_k = T_c^{-1} \int_{kT_c}^{(k+1)T_c} \int_0^T u(t-\tau)\psi(t-kT_c)\psi(t-kT_c)dt \). It follows that \( N_k \)'s are i.i.d. zero-mean complex Gaussian random variables with \( E(N_k N_k^*) = 2\pi T_c \). Substituting (2) into (1) gives

\[
u(t) = u(t) = \frac{1}{2\pi T_c} \int_{t-kT_c}^{t+(k+1)T_c} \int_0^T u(t-\tau)\psi(t-kT_c)\psi(t-kT_c)dt dx
\]
$$e^{j\phi_k \sum_{m=0}^{\infty} c_{m-k}^{(t)} a_m^{(t)} \psi(t - T_k^{(t)} - mT_c)} \text{ where } a_m^{(t)} \equiv a_{m \mod N}^{(t)} \text{ and } c_{m \mod N}^{(t)} \equiv b_{m \mod N}^{(t)} \text{ for } 0 \leq n \leq N - 1.$$

Rewriting $u(t)$ as $u(t) = e^{j\phi_k \sqrt{2P_T} \sum_{m=0}^{\infty} c_{m-k}^{(t)} a_m^{(t)} mW(t - T_k^{(t)} - (k-m)T_c)$ and evaluating $\xi^{(t)}_k$ using (A1), we get

$$\xi^{(t)}_k = \sqrt{2P_T} \sum_{m=0}^{\infty} c_{k-k_m}^{(t)} a_{k-k_m}^{(t)}$$

where

$$g_m^{(t)} = e^{j\phi_k \int_0^{T_m^{(t)}} h_t^{(t)} W(t - T_k^{(t)} - (k-m)T_c) dt} \quad (A2)$$

and $W(t)$ is given by (9). We should mention that since $h_t^{(t)}$ is slowly varying relative to $T_c$, otherwise $h_t^{(t)}$ would also be a function of $t$ in (A1) and the expression of $\xi^{(t)}_k$ would be very complicated. Since $h_t^{(t)}$ and $W(t)$ are time-limited within $[0, T_M^{(t)}]$ and $[-T_c, T_c]$, respectively, $g_m^{(t)}$ is zero if $m < 0$ or $m > M$, where $M = 1 + \left\lfloor T_M^{(t)}/T_c \right\rfloor$. It is also apparent that $g_m^{(t)}$, $n = 0, 1, \ldots, M - 1$, are jointly zero-mean complex Gaussian random variables conditioned on $T^{(t)}_c$, and it is easy to show that $E\{g_m^{(t)} g_n^{(t)}\}$ is given by (7). From (7), it is apparent that $E\{g_m^{(t)} g_n^{(t)}\}$ is in general nonzero and $g_m^{(t)}$'s are correlated. Since all $K$ channels are independent, $g_m^{(t)}$ and $g_n^{(t)}$ are statistically independent if $\ell \neq \ell'$. We now arrive at

$$\xi^{(t)}_k = \sqrt{2P_T} \sum_{m=0}^{\infty} c_{k-k_m}^{(t)} a_{k-k_m}^{(t)}$$

Substituting this expression into $\xi^{(t)}_k$ and hence $X_q$ gives (6).

**APPENDIX 2**

It is found that

$$E\{\tilde{X}_q^{(t)} \tilde{X}_q^{(t)}\} = \left\{ \begin{array}{ll}
F_0(q, q', \ell) + F_1(q, q', \ell), & M \neq 1 \\
F_0(q, q, \ell), & M = 1 
\end{array} \right. \quad (A3)$$

where $F_0(\cdot)$ and $F_1(\cdot)$ are given as follows. Let

$$f_0(k_1, m) = \sum_{i=1}^{K \cdot L \cdot \ell_1} C_{L, \ell} (q - k_1 - m - iN) C_{L, \ell} (q' - k_1 - m - iN) \quad \text{ and }$$

$$f_1(k_1, m) = \sum_{i=1}^{K \cdot L \cdot \ell_1} C_{L, \ell} (q - k_1 - m - iN) C_{L, \ell} (q' - k_1 - (m+1) - iN) + C_{L, \ell} (q - k_1 - (m+1) - iN) C_{L, \ell} (q' - k_1 - m - iN).$$

Then (i) $\ell = 0$:

$$F_0(q, q', \ell) = N^{-1} \sum_{m=0}^{M-1} x_m^{(t)} \left[ \sum_{k_1=0}^{L-1} \frac{2k_1 - 2k_1}{2L} f_0(k_1, m) \right]$$

and

$$F_1(q, q', \ell) = N^{-1} \sum_{m=0}^{M-2} x_m^{(t)} \left[ \sum_{k_1=0}^{L-1} \frac{2k_1 - 2k_1}{2L} f_1(k_1, m) \right]$$

where, for $n \in \{m, m+1\}$,

$$x_m^{(t)} x_n^{(t)} = (G_T^{(t)})^{-1} \int_0^T \int_0^T Q(t) W(mT_c - t - \tau) W(mT_c - \tau - \tau) d\tau d\tau'.$$

(ii) $\ell = 0$:

$$F_0(q, q', \ell) = N^{-1} \sum_{m=0}^{M-1} x_m^{(t)} f_0(0, m) \quad \text{ and }$$

$$F_1(q, q', \ell) = N^{-1} \sum_{m=0}^{M-2} x_m^{(t)} x_{m+1}^{(t)} f_1(0, m)$$

where, for $n \in \{m, m+1\}$,

$$x_m^{(t)} x_n^{(t)} = G_T^{-1} \int_0^T Q(t) W(mT_c - t) W(mT_c - \tau) d\tau d\tau'.$$

It is easy to show that $E\{\tilde{X}_q^{(t)} \tilde{X}_q^{(t)}\} = E\{\tilde{X}_0^{(t)} \tilde{X}_0^{(t)}\}$ as $x_m^{(t)} x_n^{(t)} = \gamma_{m}^{(t)} \gamma_{n}^{(t)}$, $n \in \{m, m+1\}$.

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