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NUMERICAL RADIUS PERSERVING OPERATORS ON $B(H)$

JOR-TING CHAN

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $H$ be a Hilbert space over $\mathbb{C}$ and let $B(H)$ denote the vector space of all bounded linear operators on $H$. We prove that a linear isomorphism $T : B(H) \rightarrow B(H)$ is numerical radius-preserving if and only if it is a multiply of a $C^*$-isomorphism by a scalar of modulus one.

1. INTRODUCTION

Let $H$ be a Hilbert space over $\mathbb{C}$ and let $B(H)$ denote the vector space of all bounded linear operators on $H$. For every $A$ in $B(H)$, the numerical range and the numerical radius of $T$ are defined respectively by

$$W(A) = \{(Ax, x) : x \in H, \|x\| = 1\},$$
$$w(A) = \sup \{|\lambda| : \lambda \in W(A)\}.$$

It is well known that $w(\cdot)$ is a norm on $B(H)$ and that this norm is equivalent to the usual operator norm. (See [4, p. 117].) A classical theorem of Kadison [4, Theorem 7] asserts that every linear isomorphism on $B(H)$ which is isometric with respect to the operator norm is a $C^*$-isomorphism followed by left multiplication by a fixed unitary operator. A $C^*$-isomorphism is a linear isomorphism of $B(H)$ such that $T(A^*) = T(A)^*$ for all $A$ in $B(H)$ and $T(A^n) = T(A)^n$ for all selfadjoint $A$ in $B(H)$ and all natural number $n$. A description of $C^*$-isomorphisms on $B(H)$ can be obtained. First of all we have from [6, Corollary 11] that a $C^*$-isomorphism on $B(H)$ is either a $^*$-isomorphism or a $^*$-anti-isomorphism. Suppose that $T$ is an algebra isomorphism on $B(H)$. Then by [3, Theorem 2], there is an invertible operator $V$ on $H$ such that $T(A) = VAV^{-1}$ for all $A$ in $B(H)$. If we also assume that $T(A^*) = T(A)^*$ for all $A$ in $B(H)$, then $VA^*V^{-1} = (V^{-1})^*A^*V*$ and hence $(V^*V)A^* = A^*(V^*V)$ for all $A$ in $B(H)$. It follows that $V^*V$ is a scalar multiple of the identity operator $I$. Say $V^*V = kI$. As $V^*V$ is always a positive operator and $k$ cannot be zero, $k > 0$. Let $U = \frac{1}{\sqrt{k}}V$. Then $U$ is unitary and $T(A) = UAU^*$ for all $A$ in $B(H)$. For a $^*$-anti-isomorphism $T$, it can be shown (e.g., see [5, Remark 2]) that there is a unitary operator $U$ in $B(H)$ such that $T(A) = UA^*U^*$ for all $A$ in $B(H)$, where $A^t$ denotes the transpose.
of $A$ relative to a fixed orthonormal basis of $H$. Clearly operators of these two types are $C^*$-isomorphisms.

Let us turn to numerical range and numerical radius. Pellegrini [9, Theorem 3.1] proved that an operator $T$ on $B(H)$ is a $C^*$-isomorphism exactly when $T$ preserves the "numerical range" of each element in $B(H)$. It should be noted that Pellegrini obtained his result in a general Banach algebra, and his definition of numerical range is different from ours. In fact, for each $A$ in $B(H)$, the "numerical range" of $A$ defined by Pellegrini reduces to the closure of $W(A)$. When the underlying space $H$ is finite-dimensional, $W(A)$ is compact and hence the two sets are identical. Despite the discrepancy we still have that $T$ is a $C^*$-isomorphism if and only if $W(T(A)) = W(A)$ for every $A$ in $B(H)$.

For simplicity we shall call an operator $T$ with the latter property numerical range-preserving. Likewise we say that $T$ is numerical radius-preserving if $w(T(A)) = w(A)$ for all $A$ in $B(H)$.

In the finite-dimensional situation, the above result was extended by Li. In [1, Theorem 1] he proved that $T$ is numerical radius-preserving if and only if $T$ is a scalar multiple of a $C^*$-isomorphism by a complex number of modulus one. It is immediate that if $T$ is numerical range-preserving, then $T$ is numerical radius-preserving and hence the scalar in question is one. In this note we prove that the conclusion of Li remains valid without the dimension constraint.

**2. Results**

In what follows $T$ denotes a linear isomorphism on $B(H)$ which is numerical radius-preserving on $B(H)$. We shall prove that $T$ maps the identity mapping $I$ to a scalar multiple of $I$. The scalar is necessarily of modulus one. Multiplying by the complex conjugate of the scalar, we get a numerical radius-preserving operator $T_1$ with an additional property that $T_1(I) = I$. The result is concluded by showing that $T_1$ is a $C^*$-isomorphism.

We begin with a lemma which describes scalar multiples of $I$ in terms of numerical radius. Let $\Lambda = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.

**Lemma 1.** An operator $A \in B(H)$ is a scalar multiple of $I$ if and only if for every $B \in B(H)$, there is a $\lambda \in \Lambda$ such that $w(A + \lambda B) = w(A) + w(B)$.

*Proof.* It is clear that if $A$ is a scalar multiple of $I$, then $A$ satisfies the condition. For the converse we borrow the idea from Li and Tsing [2, p. 40]. We first show that elements in $W(A)$ are of constant modulus; it follows then from the convexity of $W(A)$ ([4, p. 113]) that the set is a singleton. Hence $A$ is a scalar multiple of the identity $I$. Now assume that there is an $x$ in $H$, $\|x\| = 1$, and $|\langle Ax, x \rangle| < w(A)$. Let $B$ be the orthogonal projection onto the linear span of $x$. Then $w(B) = 1$. Fix any $r$ such that $|\langle Ax, x \rangle| < r < w(A)$. We can find an $\epsilon > 0$ such that $|\langle Ay, y \rangle| < r$ whenever $\|y - x\| < \epsilon$. In fact $|\langle Ay, y \rangle| < r$ if there is a $\lambda \in \Lambda$ such that $\|y - \lambda x\| < \epsilon$. Suppose that $y \in H$, $\|y\| = 1$, and $\|y - \lambda x\| \geq \epsilon$ for every $\lambda \in \Lambda$. Then

$$e^2 \leq \langle y - \lambda x, y - \lambda x \rangle = 2 - 2\text{Re}(\langle y, \lambda x \rangle)$$

for every $\lambda \in \Lambda$.

It follows that $\langle y, x \rangle \leq 1 - \frac{1}{2}e^2$. Let $k = \min\{r + 1, w(A) + 1 - \frac{1}{2}e^2\}$. Then for every $\lambda \in \Lambda$ and $y \in H$ with $\|y\| = 1$, we have

$$\|\langle A + \lambda B \rangle y, y \| \leq |\langle Ay, y \rangle| + |\langle y, x \rangle| \leq k .$$

Hence $w(A + \lambda B) < w(A) + w(B)$. □
By the above lemma \( T(I) = \lambda I \). Clearly we have \( \lambda \in \Lambda \). Let \( T_1 = \bar{\lambda}T \). Then \( T_1(I) = I \). We need the following definitions. By a state on \( B(H) \) we mean as usual a bounded linear functional \( \rho \) on \( B(H) \) such that \( \rho(I) = \|\rho\| = 1 \). The set \( S \) of all states is called the state space of \( B(H) \). A bounded linear operator \( T : B(H) \to B(H) \) is said to be state-preserving if its adjoint \( T' \) satisfies \( T'(S) \subseteq S \). By [9, Theorem 2.3 and Theorem 3.1], \( T \) is a \( C^* \)-isomorphism if and only if it is state-preserving. Let \( x \) be a unit vector in \( H \). The linear functional \( \rho_x \) given by
\[
\rho_x(A) = \langle Ax, x \rangle \quad \text{for every} \quad A \in B(H)
\]
is a state of \( B(H) \). States of this form are called vector states.

**Lemma 2.** The operator \( T_1 \) is state-preserving.

**Proof.** Let \( w' \) denote the norm in \( B(H)' \) dual to the numerical radius. Then \( w'(\rho) \geq \|\rho\| \) for every \( \rho \) in \( B(H)' \). As \( T_1 \) is numerical radius-preserving, \( w'(T_1'(\rho)) = w'(\rho) \) for every \( \rho \) in \( B(H)' \). If \( \rho_x \) is a vector state, then \( w'(\rho_x) = 1 \) and hence \( \|T_1'(\rho_x)\| \leq w'(T_1'(\rho_x)) = 1 \). But \( T_1'(\rho_x)(I) = \rho_x(T_1(I)) = \rho_x(I) = 1 \). It follows that \( T_1'(\rho_x) \) is a state of \( B(H) \). By [4, Corollary 4.3.10] the state space is the closed convex hull of the vector states in the weak *-topology. This together with the fact that \( T_1' \) is continuous in the weak *-topology entail that \( T_1 \) is state-preserving.

By Lemma 1 and Lemma 2, we have proved

**Theorem.** A linear isomorphism \( T \) on \( B(H) \) is numerical radius-preserving if and only if \( T \) is a multiple of a \( C^* \)-isomorphism by a scalar of modulus one.

In [1] Li also studied a numerical radius-preserving real-linear operator on the selfadjoint elements in \( B(H) \). He proved ([1, Theorem 2]) that such an operator is the restriction of a \( C^* \)-isomorphism on \( B(H) \) multiplied by \( \pm 1 \). Let us remark that as the numerical radius and the operator norm coincide on selfadjoint operators, this result can alternatively be deduced from [7, Theorem 2].

**References**

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