MAPPINGS PRESERVING SPECTRA OF PRODUCTS OF MATRICES

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(Dedicated to Professor Ahmed Sourour on the occasion of his sixtieth birthday.

Abstract. Let $M_n$ be the set of $n \times n$ complex matrices, and for every $A \in M_n$, let $\text{Sp}(A)$ denote the spectrum of $A$. For various types of products $A_1 \ast \cdots \ast A_k$ on $M_n$, it is shown that a mapping $\phi : M_n \rightarrow M_n$ satisfying $\text{Sp}(A_1 \ast \cdots \ast A_k) = \text{Sp}(\phi(A_1) \ast \cdots \ast \phi(A_k))$ for all $A_1, \ldots, A_k \in M_n$ has the form

$$X \mapsto \xi S^{-1}XS \quad \text{or} \quad A \mapsto \xi S^{-1}A^tS,$$

for some invertible $S \in M_n$ and scalar $\xi$. The result covers the special cases of the usual product $A_1 \ast \cdots \ast A_k = A_1 \cdots A_k$, the Jordan triple product $A_1 \ast A_2 = A_1 \ast A_2 \ast A_1$, and the Jordan product $A_1 \ast A_2 = (A_1 A_2 + A_2 A_1)/2$. Similar results are obtained for Hermitian matrices.

1. Introduction

Let $M_n$ be the set of all $n \times n$ complex matrices. In [5], Marcus and Moyls proved that if a linear mapping $\phi : M_n \rightarrow M_n$ preserves the eigenvalues (counting multiplicities) of each matrix in $M_n$, then there exists an invertible matrix $S$ such that $\phi$ has the form

$$A \mapsto S^{-1}AS \quad \text{or} \quad A \mapsto S^{-1}A^tS,$$

where $A^t$ denotes the transpose of $A$. The assumption on multiplicity is not really necessary. Let $\text{Sp}(A)$ denote the spectrum of $A$, i.e., the set of all eigenvalues of $A$ without counting multiplicities. Then by a result of Jafarian and Sourour [3], the above conclusion holds if $\text{Sp}(\phi(A)) = \text{Sp}(A)$.

The result has been generalized in different directions. For example, in [8], Omladič and P. Šemrl considered spectrum-preserving mappings that are just additive. In [6] Molnár studied surjective maps $\phi$ on bounded linear operators such that

$$\text{Sp}(\phi(A)\phi(B)) = \text{Sp}(AB) \quad \text{for all linear operators } A, B.$$
In particular, such a map on $M_n$ has the form
\begin{equation}
A \mapsto \xi S^{-1}AS \quad \text{or} \quad A \mapsto \xi S^{-1}A^tS
\end{equation}
for some invertible matrix $S$ and $\xi \in \{1, -1\}$. Continuous differentiable maps on $M_n$ preserving the spectrum were characterized in [1].

In this paper, we consider different types of products $A * B$ on $M_n$ including the usual product $A * B = AB$, the Jordan triple product $A * B = ABA$, and the Jordan product $A * B = (AB + BA)/2$. We obtain a general result, which implies that a mapping $\phi : M_n \to M_n$ satisfying
\begin{equation}
\text{Sp}(A * B) = \text{Sp}(\phi(A) * \phi(B)) \quad \text{for all } A, B \in M_n
\end{equation}
has the form [12] for some invertible $S \in M_n$ and scalar $\xi$. As we do not require the surjective assumption on $\phi$, our result refines that of Molnár in the finite-dimensional case.

Note that a characterization of those $\phi : M_n \to M_n$ such that $AB$ and $\phi(A)\phi(B)$ have the same eigenvalues counting multiplicities is given in [7]. A crucial observation is the following proposition. We include the proof for the sake of completeness.

**Proposition 1.1.** Suppose $\phi : M_n \to M_n$ satisfies
\begin{equation}
\text{tr}(AB) = \text{tr}(\phi(A)\phi(B)) \quad \text{for all } A, B \in M_n.
\end{equation}
Then $\phi$ is an invertible linear map.

**Proof.** For every $X = (x_{ij}) \in M_n$, let $R_X$ be the $n^2$ row vector
\[ R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}), \]
and $C_X$ the $n^2$ column vector
\[ C_X = (x_{11}, x_{21}, \ldots, x_{1n}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t. \]
Then for any $X, Y \in M_n$,
\begin{equation}
R_{\phi(X)}C_{\phi(Y)} = \text{tr}(\phi(X)\phi(Y)) = \text{tr}(XY) = R_XC_Y.
\end{equation}
Let $\{Y_1, \ldots, Y_{n^2}\}$ be a basis for $M_n$. Let $\mathcal{Y}$ have columns $C_{Y_1}, \ldots, C_{Y_{n^2}}$, and $Z \in M_{n^2}$ have columns $C_{\phi(Y_1)}, \ldots, C_{\phi(Y_{n^2})}$. Then by (1.4), for any $X \in M_n$ we have
\[ R_{\phi(X)}Z = R_X\mathcal{Y}. \]
Next, we show that $Z$ is invertible. To this end, let $\{X_1, \ldots, X_{n^2}\}$ be a basis for $M_n$, $\mathcal{X} \in M_{n^2}$ with rows $R_{X_1}, \ldots, R_{X_{n^2}}$, and $W \in M_{n^2}$ with rows $R_{\phi(X_1)}, \ldots, R_{\phi(X_{n^2})}$. Then $WZ = \mathcal{X}\mathcal{Y}$ for the invertible matrices $\mathcal{X}$ and $\mathcal{Y}$. So, $Z$ is invertible, and for any $X \in M_n$,
\[ R_{\phi(X)} = R_X\mathcal{Y}Z^{-1}. \]
Hence, $\phi$ is an invertible linear map. \qed

The problem of characterizing mappings that preserve the spectra of the product of matrices is more challenging. Our results will give a characterization of mappings preserving the spectrum of various products of $k$ matrices $X_1 \cdots X_k$ defined as follows.

Let $k \geq 2$, and let a sequence $(j_1, \ldots, j_m)$ be given so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$. We consider products of the form
\[ X_1 \cdots X_k = X_{j_1} \cdots X_{j_m}. \]
which cover the usual product $A \ast B = AB$ and the Jordan triple product $A \ast B = ABA$. We also consider products of the form

$$X_1 \ast \cdots \ast X_k = (X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1}) / 2,$$

which cover the Jordan product $A \ast B = (AB + BA)/2$.

In Section 2, we obtain the results on the set $M_n$ of $n \times n$ complex matrices. Using a transfer principle in model-theoretic algebra (see [2]), one sees that the results also hold for square matrices over an algebraically closed field. In Section 3, similar results are proved for the set $H_n$ of $n \times n$ complex Hermitian matrices.

The same results and proofs are valid for $n \times n$ real symmetric matrices as well.

2. RESULTS ON COMPLEX MATRICES

**Theorem 2.1.** Suppose $k \geq 2$, and let a sequence $(j_1, \ldots, j_m)$ be given so that

$$\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$$

and there is $j_r$ not equal to $j_s$ for all $s \neq r$. Consider

$$X_1 \ast \cdots \ast X_k = X_{j_1} \cdots X_{j_m},$$

Then a mapping $\phi : M_n \rightarrow M_n$ satisfies

$$(2.1) \quad \text{Sp}(\phi(X_1) \ast \cdots \ast \phi(X_k)) = \text{Sp}(X_1 \ast \cdots \ast X_k) \quad \text{for all } X_1, \ldots, X_k \in M_n$$

if and only if there exist an invertible matrix $S \in M_n$ and a scalar $\xi$ satisfying $\xi^m = 1$ such that

(a) $\phi$ has the form $A \mapsto \xi S^{-1}AS$, or

(b) $(j_{r+1}, j_m, j_1, \ldots, j_{r-1}) = (j_{r-1}, j_1, \ldots, j_{r+1})$ and $\phi$ has the form

$$A \mapsto \xi^{-1}A\xi S.$$

Note that the assumption that there is $j_r \notin \{j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_m\}$ is necessary. For example, if $A \ast B = ABBA$, then mappings $\phi$ satisfying $\text{Sp}(\phi(A) \ast \phi(B)) = \text{Sp}(A \ast B)$ may not have a nice structure. For instance, $\phi$ can send all involutions, i.e., those matrices $X \in M_n$ such that $X^2 = I_n$, to a fixed involution, and $\phi(X) = X$ for other $X$.

**Proof of Theorem 2.1.** It is clear that if (a) or (b) holds, then $\phi$ satisfies (2.1).

We need only prove the necessity part. We divide the proof into several assertions. Since $\text{Sp}(X_{j_1} \cdots X_{j_m}) = \text{Sp}(X_{j_r} \cdots X_{j_m}X_{j_1} \cdots X_{j_{r-1}})$, we may assume that $j_1 \notin \{j_2, \ldots, j_m\}$. Define

$$S = \{X \in M_n : X \text{ has } n \text{ distinct eigenvalues}\}.$$

 Assertion 1. For every $A \in S$, there is a neighborhood of $\mathcal{N}_A$ such that the restriction of $\phi$ on $\mathcal{N}_A$ equals an invertible linear map $L_A$.

**Proof.** For every $A \in S$, $\text{Sp}(AI_n^{m-1})$ has $n$ distinct elements. By the continuity of the eigenvalues, there are neighborhoods $\mathcal{N}_I$ of $I_n$ and $\mathcal{N}_A$ of $A$ such that $XY^{m-1}$ has $n$ distinct eigenvalues for every $X \in \mathcal{N}_A$ and $Y \in \mathcal{N}_I$. By (2.1), $\phi(X)\phi(Y)^{m-1}$ has $n$ distinct eigenvalues equal to those of $XY^{m-1}$. Hence

$$\text{(2.2) } \text{tr} \phi(X)\phi(Y)^{m-1} = \text{tr} (XY^{m-1}) \quad \text{for every } X \in \mathcal{N}_A \text{ and } Y \in \mathcal{N}_I.$$

As in the proof of Proposition [2.1] for every $X = (x_{ij}) \in M_n$, let $R_X$ be the $n^2$ row vector

$$R_X = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}).$$
and $C_X$ the $n^2$ column vector
\[ C_X = (x_{11}, x_{21}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t. \]

Then
\[ R_{\phi(X)} C_{\phi(Y)}^{m-1} = \text{tr}(\phi(X)\phi(Y)^{m-1}) = \text{tr}(XY^{m-1}) = R_X C_{Y^{m-1}} \]
for every $X \in \mathcal{N}_A$ and $Y \in \mathcal{N}_I$.

Now, suppose $\text{tr}(XZ) = 0$ for each $Z \in \{Y^{m-1} : Y \in \mathcal{N}_I\}$. Then for any $R \in M_n$,
\[ \text{tr}(X(I + tR)^{m-1}) = \sum_{j=0}^{m} t^j \binom{m-1}{j} \text{tr}(XR^j) = 0 \]
for sufficiently small $t > 0$. We have $\text{tr}(XR) = 0$. It follows that $X = 0$. So, \{\{Y^{m-1} : Y \in \mathcal{N}_I\}\} is a spanning set of $M_n$, and contains a basis \{\{Y^{m-1} : 1 \leq j \leq n^2\}\} for $M_n$ with $Y_j \in \mathcal{N}_I$ for each $j = 1, \ldots, n^2$. Let $\mathcal{Y}$ and $\mathcal{Z}$ be the $n^2 \times n^2$ matrices with columns $C_{Y^{m-1}_1}, \ldots, C_{Y^{m-1}_{n^2}}$ and $C_{\phi(Y_1)^{m-1}}, \ldots, C_{\phi(Y_{n^2})^{m-1}}$ respectively. By (2.3) and (2.2),
\[ R_{\phi(X)} \mathcal{Z} = R_X \mathcal{Y} \quad \text{for every } X \in \mathcal{N}_A. \]

We claim that the matrix $\mathcal{Z}$ is invertible. To this end take a basis \{\{X_1, \ldots, X_{n^2}\}\} of $M_n$ in $\mathcal{N}_A$ and let $\mathcal{X}$ and $\mathcal{W}$ be the $n^2 \times n^2$ matrices with rows $R_{X_1}, \ldots, R_{X_{n^2}}$ and $R_{\phi(X_1)}, \ldots, R_{\phi(X_{n^2})}$ respectively. Then $\mathcal{WZ} = \mathcal{X} \mathcal{Y}$ for invertible matrices $\mathcal{X}$ and $\mathcal{Y}$. It follows that $\mathcal{Z}$ is invertible, and
\[ R_{\phi(X)} = R_X \mathcal{XZ}^{-1} \quad \text{for every } X \in \mathcal{N}_A. \]

Hence the restriction of $\phi$ to $\mathcal{N}_A$ is some invertible linear mapping $L_A$. The proof of Assertion 1 is complete.

**Assertion 2.** All the linear maps $L_A$ in Assertion 1 are the same, i.e., $\phi$ is equal to an invertible linear mapping $L$ on the dense subset $\mathcal{S}$.

**Proof.** Note that for any $A, B \in \mathcal{S}$, there is a continuous curve $f : [0, 1] \to \mathcal{S}$ such that $f(0) = A$ and $f(1) = B$. Consider the set
\[ \mathcal{C} = \{t \in [0, 1] : \phi = L_A \text{ on an open neighborhood of } f(t)\}. \]

Then clearly $\mathcal{C}$ is an open subset of $[0, 1]$. But $\mathcal{C}$ is also closed in $[0, 1]$. Let $t_0 \in \mathcal{C}^c$. There is an open neighborhood $\mathcal{N}_{f(t_0)}$ of $f(t_0)$ on which $\phi$ is equal to the linear mapping $L_{f(t_0)}$. Take $t \in f^{-1}(\mathcal{N}_{f(t_0)}) \cap \mathcal{C}$. Then on some open neighborhood $\mathcal{N}_{f(t)}$ of $f(t)$, $\phi = L_A$. On the nonempty open set $\mathcal{N}_{f(t_0)} \cap \mathcal{N}_{f(t)}$, $L_{f(t_0)} = \phi = L_A$. Hence $L_{f(t_0)} = L_A$, and $t_0 \in \mathcal{C}$. We conclude that $\mathcal{C} = [0, 1]$, and $L_A = L_B$. The proof of Assertion 2 is complete.

**Assertion 3.** The mapping $L$ in Assertion 2 has the form $A \mapsto \xi S^{-1} AS$ or $A \mapsto \xi S^{-1} A^t S$ for some invertible $S \in M_n$ and $\xi \in \mathbb{C}$ with $\xi^n = 1$. Moreover, if the latter case holds, then $(j_2, \ldots, j_m) = (j_m, \ldots, j_2)$.

**Proof.** By the continuity of $L$ and the spectrum, we have that
\[ \text{Sp}(L(X_1) \cdots L(X_k)) = \text{Sp}(X_1 \cdots X_k) \]
for all $X_1, \ldots, X_k \in M_n$. If $A$ is invertible, then
\[ 0 \notin \text{Sp}(A \cdots A) = \text{Sp}(L(A) \cdots L(A)), \]
and hence $L(A)$ is also invertible. It follows that $L$ is nonsingular and preserves invertible matrices. By [5], there are invertible matrices $M, N$ such that $L$ has the form

$$(2.4) \quad A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN.$$  

We claim that $NM$ is a scalar matrix. Otherwise, there exists an invertible $R \in M_n$ such that $RNMR^{-1}$ is a direct sum of companion matrices so that its second row has the form $(1, 0, \ldots, \ast)$. Let $A = R^{-1}E_{12}R$ or $A^t = R^{-1}E_{12}R$ depending on $L$ have the first or the second form in (2.4) where $E_{12}$ is the $n \times n$ matrix with $1$ at the $(1, 2)$ position and $0$ everywhere else. Then $\text{Sp}(A^m) = \text{Sp}(A) = \{0\}$. Now

$$\text{Sp}(L(A)) = \text{Sp}(M(R^{-1}E_{12}R)N) = \text{Sp}(E_{12}RNMR^{-1}).$$

It follows that $1 \in \text{Sp}(L(A))$ and hence $1 \in \text{Sp}(L(A)^m)$ whereas $\text{Sp}(A^m) = \{0\}$, which contradicts (2.1).

We have proved that $L$ has the form $A \mapsto \xi S^{-1}AS$ or $A \mapsto \xi S^{-1}A^tS$ for some $\xi$. Since $\{\xi_m\} = \text{Sp}(L(I_m)^m) = \text{Sp}(I_n^m) = \{1\}$, $\xi_m = 1$.

Now, suppose $L$ has the form $A \mapsto \xi S^{-1}A^tS$. Replacing $L$ by the mapping $A \mapsto \xi SL(A)S^{-1}$, we may assume that $L(A) = A^t$ for all $A \in M_n$. Then

$$\text{Sp}(X_{j_1} \cdots X_{j_m}) = \text{Sp}(X_1 \ast \cdots \ast X_k) = \text{Sp}(L(X_1) \ast \cdots \ast L(X_k))$$

$$= \text{Sp}(X_{j_1} \cdots X_{j_m}) = \text{Sp}(X_{j_1}, X_{j_2}, \ldots, X_{j_m})$$

for any $X_1, \ldots, X_k \in M_n$. We have to show that $(j_2, \ldots, j_m) = (j_m, \ldots, j_2)$. Suppose it is not true. Let $l \geq 2$ be the smallest integer such that $j_l \neq j_{m+2-l}$. Then $l \leq (m+1)/2$. Let $A_{j_l} = A = \text{diag}(\lambda, 1, \ldots, 1)$, and for every $k \notin \{1, j_l\}$, let $A_k = B = B_1 \oplus I_{n-2}$, where $B_1 \in M_2$ is a symmetric invertible matrix with positive entries. Then

$$A_{j_2} \cdots A_{j_m} = RA^{r_1}B^{s_1}A^{r_2}B^{s_2} \cdots A^{r_t}B^{s_t}R^t$$

for positive integers $r_i, s_i$, where $R = A_{j_2} \cdots A_{j_{l-1}}$. Note that

$$A^{r_i}B^{s_i} = \begin{pmatrix} \lambda^{r_i}b_{11}^{(s_i)} & \lambda^{r_i}b_{12}^{(s_i)} \\ b_{21}^{(s_i)} & b_{22}^{(s_i)} \end{pmatrix} \oplus I_{n-2},$$

for positive numbers $b_{11}^{(s_i)}, b_{12}^{(s_i)}, b_{21}^{(s_i)}$ and $b_{22}^{(s_i)}$. An induction argument shows that the $(1,2)$ entry of $A_{j_2} \cdots A_{j_{m-1}}$ is a polynomial of degree $r_1 + \cdots + r_t$ in $\lambda$. Similarly, the $(1,2)$ entry of $A_{j_{m-1}} \cdots A_{j_2}$, is a polynomial of degree $r_2 + \cdots + r_t$. So, there is $\lambda > 0$ such that

$$A_{j_1} \cdots A_{j_{m-1}} \neq A_{j_{m-1}} \cdots A_{j_1}.$$  

It follows that

$$A_{j_2} \cdots A_{j_m} = RA_{j_1} \cdots A_{j_{m-1}}R^t \neq RA_{j_1} \cdots A_{j_{m-1}}R^t = A_{j_1} \cdots A_{j_2}.$$  

Note that if $X \in M_n$ is a rank one idempotent matrix, and $\text{Sp}(A) = \text{Sp}(BX)$, then $\text{tr}(AX) = \text{tr}(BX)$. Moreover, if $\text{tr}(AX) = \text{tr}(BX)$ for all rank one idempotent $X \in M_n$, then $A = B$. By these facts, we see that there exists a rank one idempotent $A_1$ such that

$$\text{Sp}(A_{j_1} \cdots A_{j_2}A_{j_1}) \neq \text{Sp}(A_{j_1} \cdots A_{j_m}A_{j_1}) = \text{Sp}(A_{j_1}A_{j_2} \cdots A_{j_m}),$$

which is a contradiction. Hence, $(j_2, \ldots, j_m) = (j_m, \ldots, j_2)$ as asserted. \hfill $\square$
Proof. From \((2.3)\), and the continuity of \(L\) and the spectrum, we have
\[
\text{Sp} (L(A)L(B)^{m-1}) = \text{Sp} (AB^{m-1}) = \text{Sp} (\phi(A)L(B)^{m-1}) \quad \text{for every } A, B \in M_n.
\]
Since \(L\) is surjective,
\[
(2.5) \quad \text{Sp} (\phi(A)C^{m-1}) = \text{Sp} (L(A)C^{m-1}) \quad \text{for every } A, C \in M_n.
\]
Let \(A \in M_n\). If \(C \in M_n\) is a rank one idempotent, \(\phi(A)C^{m-1}\) has at most one nonzero eigenvalue, which is given by \(\text{tr} (\phi(A)C)\). The same is true for \(L(A)C\). By \((2.4)\),
\[
\text{tr} (\phi(A)C) = \text{tr} (L(A)C) \quad \text{for every rank one idempotent matrix } C \in M_n.
\]
It follows that \(\phi(A) = L(A)\) for all \(A \in M_n\). The proof of Assertion 4 is complete.

By Assertions 1–4, the theorem follows. \(\square\)

**Theorem 2.2.** Suppose \(k \geq 2\), and \(X_1 \cdots X_k = X_{j_1} \cdots X_{j_m} + X_{j_m} \cdots X_{j_1}\) for a given sequence \((j_1, \ldots, j_m)\) so that \(\{j_1, \ldots, j_m\} = \{1, \ldots, k\}\) and there exists \(j_r\) not equal to \(j_s\) for all \(s \neq r\). Then a mapping \(\phi : M_n \to M_n\) satisfies
\[
(2.6) \quad \text{Sp} (\phi(X_1) \cdots \phi(X_k)) = \text{Sp} (X_1 \cdots X_k) \quad \text{for all } X_1, \ldots, X_k \in M_n
\]
if and only if there exist an invertible matrix \(S \in M_n\) and a scalar \(\xi\) satisfying \(\xi^m = 1\) such that \(\phi\) has the form
\[
A \mapsto \xi S^{-1} AS \quad \text{or} \quad A \mapsto \xi S^{-1} A'S.
\]

**Proof.** The necessity of the result is clear. We consider the sufficiency part. Using similar arguments as in the proof of Theorem \(2.1\) (cf. Assertions 1 and 2), we can prove that \(\phi\) is equal to a bijective linear mapping \(L\) on the dense subset
\[
\mathcal{S} = \{X \in M_n : X \text{ has } n \text{ distinct eigenvalues}\}.
\]

By continuity of \(L\) and the spectrum, we see that
\[
\text{Sp} (L(X_1) \cdots L(X_k)) = \text{Sp} (X_1 \cdots X_k)
\]
for all \(X_1, \ldots, X_k \in M_n\). Thus, \(\text{Sp} (L(A)^m) = \text{Sp} (A^m)\) for all \(A \in M_n\). Using the argument in Assertion 3 in the proof of Theorem \(2.1\) we see that \(L\) has the form \(A \mapsto \xi S^{-1} AS\) or \(A \mapsto \xi S^{-1} A'S\). Now, replace \(\phi\) and \(L\) by the mappings \(A \mapsto \xi S \phi(A)S^{-1}\) and \(A \mapsto \xi SL(A)S^{-1}\), respectively; we may assume that \(L(A) = A\) for all \(A \in M_n\).

We will show that \(\phi = L\) on \(M_n\). From \((2.2)\), we have
\[
\text{Sp} (X^{r-1}L(Y)X^{m-r} + X^{m-r}L(Y)X^{r-1})
\]
\[
= \text{Sp} (L(X)^{r-1}L(Y)L(X)^{m-r} + L(X)^{m-r}L(Y)L(X)^{r-1})
\]
\[
= \text{Sp} (X^{r-1}YX^{m-r} + X^{m-r}YX^{r-1})
\]
\[
= \text{Sp} (\phi(X)^{r-1}\phi(Y)\phi(X)^{m-r} + \phi(X)^{m-r}\phi(Y)\phi(X)^{r-1})
\]
\[
= \text{Sp} (L(X)^{r-1}\phi(Y)L(X)^{m-r} + L(X)^{m-r}\phi(Y)L(X)^{r-1})
\]
\[
= \text{Sp} (X^{r-1}\phi(Y)X^{m-r} + X^{m-r}\phi(Y)X^{r-1})
\]
for every \(X \in \mathcal{S}\) and \(Y \in M_n\). Since the set of such matrices \(X\) is dense in \(M_n\), by continuity of the spectrum, we see that
\[
(2.7) \quad \text{Sp} (X^{r-1}\phi(Y)X^{m-r} + X^{m-r}\phi(Y)X^{r-1}) = \text{Sp} (X^{r-1}L(Y)X^{m-r} + X^{m-r}L(Y)X^{r-1})
\]
for any $X, Y \in M_n$. It remains to prove the following.

**Assertion.** Let $A, B \in M_n$. Then $A = B$ if, for every $X \in M_n$,

$$\text{Sp}(X^{-1}AX^m + X^m - AX^{r-1}) = \text{Sp}(X^{-1}BX^m + X^m - BX^{r-1})$$

**Proof.** If both $r - 1$ and $m - r$ are positive, then for any rank one idempotent $X \in M_n$ we have

$$\text{Sp}(2AX) = \text{Sp}(XAX + XAX) = \text{Sp}(X^{-1}AX^m + X^m - AX^{r-1}) = \text{Sp}(X^{-1}BX^m + X^m - BX^{r-1}) = \text{Sp}(XBX + XBX) = \text{Sp}(2BX).$$

Since $AX$ and $BX$ have the same spectrum and have rank at most one, we see that

$$\text{tr}(AX) = \text{tr}(BX).$$

It follows that $A = B$.

Suppose $r - 1$ or $m - r$ is zero. Then

$$\text{Sp}(AX + XA) = \text{Sp}(BX + XB)$$

for all $X \in \{Z^{m-1} : Z \in M_n\}$, which is a dense set in $M_n$. By continuity of the spectrum, we may assume that the above equality is true for all $X \in M_n$. We shall assume without loss of generality that $A$ is upper triangular. We claim that $B$ is also upper triangular. Suppose $A = (a_{ij})$, $B = (b_{ij})$, and for every $t \in \mathbb{C}$, let $X_t = E_{11} + tE_{12} + \cdots + t^{n-1}E_{1n}$. Then only the first row of $AX_t + X_tA$ is nonzero and equals $(2a_{11} \ast \cdots \ast)$.

Hence $\text{Sp}(AX_t + X_tA) = \{2a_{11}, 0\}$. As $\text{Sp}(BX_t + X_tB) = \text{Sp}(AX_t + X_tA) = \{2a_{11}, 0\}$, $BX_t + X_tB$ has eigenvalues $2a_{11}$ and 0 with certain multiplicities. So,

$$\text{tr}(BX_t + X_tB) \in \{2a_{11}, \ldots, 2(n-1)a_{11}\}.$$ Now

$$BX_t + X_tB = \begin{pmatrix} b_{11} & b_{11}t & \cdots & b_{11}t^{n-1} \\ b_{21} & b_{21}t & \cdots & b_{21}t^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n1}t & \cdots & b_{n1}t^{n-1} \end{pmatrix} + \begin{pmatrix} b_n + b_{21}t + \cdots + b_{n1}t^{n-1} & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

has diagonal entries

$$2b_{11} + b_{21}t + \cdots + b_{n1}t^{n-1}, b_{21}t, \ldots, b_{n1}t^{n-1},$$

and hence

$$\text{tr}(BX_t + X_tB) = 2b_{11} + 2b_{21}t + \cdots + 2b_{n1}t^{n-1}$$

is a polynomial in $t$. It cannot take on a finite number of values only unless it is a constant. The coefficients, except the constant term, are all zero. Hence $b_{21} = \cdots = b_{n1} = 0$. Similarly, by considering $X_t = E_{ii} + tE_{i,i+1} + \cdots + t^{n-1}E_{in}$, we get $b_{i+1,i} = \cdots = b_{ni} = 0$ for $i = 2, \ldots, n - 1$. The matrix $B$ is upper triangular.
To show that $A = B$, we first obtain, by putting $X = E_{i i}$, $a_{i i} = b_{i i}$ for every $i$. For $i < j$, we have

$$AE_{ji} + E_{ji}A = \begin{pmatrix}
    a_{1 j} & \cdots & \cdots & a_{i j} & \cdots & a_{i n} \\
    \vdots & \ddots & \cdots & \vdots & \ddots & \vdots \\
    \vdots & \cdots & 0 & a_{i i} + a_{j j} & \cdots & a_{i n} \\
    0 & \cdots & \cdots & 0 & \ddots & \ddots & \ddots \\
    \vdots & \cdots & \cdots & \cdots & \cdots & 0 & \ddots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}.$$ 

Expanding along the $j$th column, say, we get

$$\det (AE_{ji} + E_{ji}A - \lambda I_n) = (-\lambda)^{n-2}(a_{ij} - \lambda)^2.$$ 

Hence $Sp (AE_{ji} + E_{ji}A) = \{a_{ij}, 0\}$. Note that $Sp (BE_{ji} + E_{ji}B) = \{b_{ij}, 0\}$. So, $a_{ij} = b_{ij}$. □

3. Results on Hermitian matrices

In this section, we study mappings on $H_n$ that have similar preserving properties as in Section 2. First, we consider products of the form $X_1 \cdots X_k = X_{j_1} \cdots X_{j_m}$ such that one of the $j_r$ appears only once in $(j_1, \ldots, j_m)$. Even though $H_n$ may not be closed under this product, mappings that preserve the spectrum of the product are in nice form. If we insist that $X_1 \cdots X_k \in H_n$, then $m$ is odd, and $r = (m + 1)/2$ is the only possible value for $j_r$ to appear once; in particular, $A * B = A^kB^A$ is the only product we can define on two matrices.

**Theorem 3.1.** Suppose $k \geq 2$, and $X_1 \cdots X_k = X_{j_1} \cdots X_{j_m}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists $j_r$ not equal to $j_s$ for all $s \neq r$. Then a mapping $\phi : H_n \to H_n$ satisfies

$$Sp (\phi(X_1) \cdots \phi(X_k)) = Sp (X_1 \cdots X_k) \quad \text{for all } X_1, \ldots, X_k \in H_n$$

if and only if there exists a unitary matrix $S \in M_n$ and a scalar $\xi$ satisfying $\xi^m = 1$ such that

(a) $\phi$ has the form $A \mapsto \xi S^* AS$, or

(b) $(j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_m) = (j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1})$ and $\phi$ has the form $A \mapsto \xi S^* A^t S$.

**Proof.** Again, we only need to consider the sufficiency part. Using similar arguments as in the proof of Theorem 2.1 (cf. Assertions 1 and 2), we can prove that $\phi$ is equal to a bijective (real) linear mapping $L$ on the dense subset

$$S = \{ X \in H_n : X \text{ has } n \text{ distinct eigenvalues} \},$$

and that $L$ preserves the invertible matrices in $H_n$. By [3] Theorem 6], there is an invertible matrix $S \in M_n$ such that $L$ is of the form

$$A \mapsto \pm S^* AS \quad \text{or} \quad A \mapsto \pm S^* A^t S.$$ 

From the observations

$$\{1\} = Sp (I_n^m) = Sp (L(I_n)^m) = Sp ((\pm S^* S)^m)$$
and that $S^*S$ is positive definite, we conclude that $S^*S = I_n$, i.e., $S$ is unitary. Hence $L$ has the asserted forms.

Also, we can show that $(j_{r+1}, \ldots, j_m, j_1, \ldots, j_{r-1}) = (j_m, \ldots, j_{r+1}, j_{r-1}, \ldots, j_1)$ if $L$ has the form $A \mapsto \xi S^*A^tS$ with the help of the following fact.

Two matrices $A, B \in M_n$ are equal if $\text{Sp}(XA) = \text{Sp}(XB)$ for every rank one $X \in H_n$.

[Note that we use real symmetric matrices in the proof of Assertion 3 in the proof of Theorem 2.1] Using the above fact again, we can adapt the proof of Assertion 3 in the proof of Theorem 2.1.

□

**Theorem 3.2.** Suppose $k \geq 2$, and $X_1 \ast \ast \ast X_k = X_{j_1} \ast \ast \ast X_{j_m} + X_{j_m} \ast \ast \ast X_{j_1}$ for a given sequence $(j_1, \ldots, j_m)$ so that $\{j_1, \ldots, j_m\} = \{1, \ldots, k\}$ and there exists $j_r$ not equal to $j_s$ for all $s \neq r$. Then a mapping $\phi : H_n \rightarrow H_n$ satisfies

\[
(3.2) \quad \text{Sp}(\phi(X_1)) \ast \ast \ast \phi(X_k)) = \text{Sp}(X_1 \ast \ast \ast X_k)
\]

for all $X_1, \ldots, X_k \in H_n$ if and only if there exist a unitary matrix $S \in M_n$ and a scalar $\xi$ satisfying $\xi^m = 1$ such that $\phi$ has the form

\[
A \mapsto \xi S^*AS \quad \text{or} \quad A \mapsto \xi S^*A^tS.
\]

**Proof.** We use arguments similar to those in the proof of Theorem 2.2. We need only replace the Assertion in the proof by the following.

**Assertion.** Let $A, B \in H_n$. Then $A = B$ if

\[
\text{Sp}(X^{r-1}AX^{m-r} + X^{m-r}AX^{r-1}) = \text{Sp}(X^{r-1}BX^{m-r} + X^{m-r}BX^{r-1})
\]

for every rank one idempotent $X \in H_n$.

**Proof.** If both $r - 1$ and $m - r$ are positive, we can prove the result using a similar argument as in the proof of Theorem 2.2.

If $r - 1$ or $m - r$ is zero, then we have

\[
\text{Sp}(XA + AX) = \text{Sp}(XB + BX)
\]

for every rank one idempotent $X \in H_n$.

We shall assume without loss of generality that $A$ is the diagonal matrix $\text{diag}(a_1, \ldots, a_n)$. Putting $X = E_{11}$, we have $AE_{11} + E_{11}A = \text{diag}(2a_1, 0, \ldots, 0)$, and hence $\text{Sp}(AE_{11} + E_{11}A) = \{2a_1, 0\}$. Let $B = (b_{ij})$. Then

\[
BE_{11} + E_{11}B = \begin{pmatrix}
2b_{11} & b_{12} & \cdots & a_1 \\
b_{12} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_{1n} & 0 & \cdots & 0
\end{pmatrix}.
\]

The characteristic polynomial of $BE_{11} + E_{11}B$ is

\[
(-\lambda)^{n-2}(\lambda^2 - 2b_{11}\lambda - (|b_{12}|^2 + \cdots + |b_{1n}|^2)).
\]

The zeros of the polynomial are $2a_1$ and 0. Now it is easy to see that the polynomial cannot have a nonzero double zero. Hence if $a_1 \neq 0$, $2a_1$ is a simple zero. We have $b_{11} = a_1$ and $b_{12} = \cdots = b_{1n} = 0$. It is obvious that if $a_1 = 0$, then $b_{11} = b_{12} = \cdots = b_{1n} = 0$. Similarly, by putting $X = E_{jj}$, we conclude that $A = B$. □
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