Memory is relevant in the symmetric phase of the minority game

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Minority game is a simple-mined econophysical model capturing the cooperative behavior among selfish players. Previous investigations, which were based on numerical simulations up to about 100 players for a certain parameter $\alpha$ in the range $0.1 \leq \alpha \leq 1$, suggested that memory is irrelevant to the cooperative behavior of the minority game in the so-called symmetric phase. Here using a large scale numerical simulation up to about 3000 players in the parameter range $0.01 \leq \alpha \leq 1$, we show that the mean variance of the attendance in the minority game actually depends on the memory in the symmetric phase. We explain such dependence in the framework of crowd-anticrowd theory. Our findings conclude that one should not overlook the feedback mechanism buried under the correlation in the history time series in the study of minority game.

I. INTRODUCTION

Minority game (MG) [1,2] is the most studied econophysical model capturing the minority seeking behavior of independent selfish players. MG is a repeated game of $N$ players, each picking one out of two alternatives independently in every time step based on the publicly posted minority choices of the previous $M$ turns. Those correctly picking the minority choice are awarded one mark while the others are deducted one. The aim of each player is to maximize his/her own mark. To help players making their choices, each of them is assigned once and for all $S$ deterministic strategies. Here, each strategy is a map from the set of all possible minority choices of the previous $M$ turns, which we call the history, to the set of the two alternatives. The performance of each strategy is evaluated according to its virtual score, which is defined as the hypothetical mark it would get if the strategy were used throughout the game. Among the $S$ assigned strategies, every player follows the suggestion of the one with the highest current virtual score to choose an alternative (in case of a tie, a player randomly uses one of his/her current best working strategies) [1,3].

The number of possible strategies in MG equals $2^M$. Nonetheless, out of these $2^M$ strategies, only $2^{M-1}$ of them are significantly different. These significantly different strategies formed the so-called reduced strategy space. In fact, for a fixed $S$ and up to first order approximation, the dynamics of MG is robust with $\alpha$, which is the ratio of the reduced strategy space size $2^{M+1}$ to the number of strategies at play $NS$ [2,3].

The attendance of an alternative $A(t)$ at turn $t$ is defined as the number of players choosing that alternative in that turn. Since there is no prior bias in choosing the two alternatives in MG, $\langle A(t) \rangle_{t=\infty} = N/2$ where the average is taken over time $t$ and initially assigned strategies $\xi$. In contrast, the variance of attendance per player averaged over initially assigned strategies, $\langle \sigma^2(A(t)) \rangle_{t=\infty} / N$, is a more instructive quantity to study.

(We write $\langle \sigma^2(A(t)) \rangle_{t=\infty}$ as $\sigma^2$ and call it simply the variance of attendance from now on when confusion is not possible.) Numerical simulation shows that, over a wide range of parameters, the $\sigma^2/N$ against $\alpha$ curve is lower than the value in which all players make their choices randomly. This implies that these selfish and independent players maximize their own marks cooperatively. More importantly, a cusp is found at $\alpha = \alpha_c$; this second order phase transition point $\alpha_c$ divides the parameter space into the so-called symmetric ($\alpha < \alpha_c$) and asymmetric ($\alpha > \alpha_c$) phases [4].

The first numerical study on the effect of history concerning the dynamics and cooperative behavior of MG was carried out by Cavagna using $N=101$, who suggested that memory is irrelevant in MG. That is to say, the $\sigma^2/N$ against $\alpha$ curve is unaltered if we replace the history in each turn with a randomly and independently generated $M$-bit string, while keeping the virtual score calculation method unchanged [5]. (From now on, we denote the original MG and the one played using random history strings by MGreal and MGrand respectively.) Later on, by exploring a wider range of parameters in their numerical simulations, Challet and Marsili pointed out that history is irrelevant if and only if $\alpha < \alpha_c$. As long as the occurrence of history is uniform, $\sigma^2/N$ should not change when we replace the history with a randomly and independently generated time series of $M$-bit string [6]. Furthermore, Lee argued that although the time series of the attendance shows a strong periodic signal, this signal does not affect the volatility of the attendance [7]. To summarize, the current understanding is that memory plays no role on the volatility in the symmetric phase of MG.

In this paper, we perform a large scale numerical simulation of MG played using real and random histories with up to 3232 players for $0.01 \leq \alpha \leq 10$. Our simulation results show that memory is in fact relevant in the symmetric phase whenever $\alpha \approx 0.2$. Specifically, we discover that in such a low value of $\alpha$, the $\sigma^2/N$ against $\alpha$ curves corresponding to the games played using real and random histories split. We explain this split by crowd-anticrowd theory developed by Hart et al. [8,9] and argue that finite size effect prevents earlier investigations from revealing this discrepancy. Finally, we...
against MG real is consistently higher than that of MG rand for 

Figure (a) compares the dependence of history for \( N=101 \) (inset) 

and 3232 (main plot). We represent the MGreal and MGrand data by 

triangles and squares, respectively. Figure (b) shows the dependence of \( \sigma^2/N \) on \( N \) for MGreal.

We find that the same conclusions are reached for the case of odd 

\( N \).

Thus, the discrepancy between the two volatilities 

\[ \sigma^2_{\text{rand}} \sim f(\alpha, S) \] 

in the symmetric phase (\( \alpha < \alpha_c \)), where \( \sigma^2_{\text{rand}} \) is the variance of the attendance for MGrand and \( f \) is a scaling function depending only on \( \alpha \) and \( S \). Moreover, Fig. 2 shows that, for a 

fixed memory size \( M \), the value of \( \sigma^2/N^2 \) is independent of \( N \) in the symmetric phase. That is to say, 

\[ \frac{\sigma^2_{\text{real}}}{N^2} \sim g_{\text{real}}(M, S) > g_{\text{rand}}(M, S) \sim \frac{\sigma^2_{\text{rand}}}{N^2} \] 

provided that \( \alpha < \alpha_c \), where \( \sigma^2_{\text{real}} \) is the variance of attendance for MGreal and \( g_i \)'s are scaling functions depending on 

\( M \) and \( S \) only.

II. NUMERICAL RESULTS

All the \( \sigma^2 \)'s reported in our simulation are averaged over 

1000 independent runs. And in each run, the variance of 

attendance is computed from the attendance of 15000 

consecutive turns after the system equilibrates. Furthermore, we 

looked at the attendance time series over at least \( 8 \times 10^6 \) 

iterations in about 50 independent runs with various values of 

\( \alpha \) to verify that the system has indeed equilibrated. Our 

computation requires about 7 Gflops/yr of instructions.

In Fig. 1, we plot \( \sigma^2/N \) against \( \alpha \) for different values of \( N \) 

for both the MG played using real and random histories. By 

putting \( N=101 \), we successfully reproduce the two \( \sigma^2/N \) 

against \( \alpha \) curves for MGreal and MGrand obtained by Challet 

and Marsili in Ref. [6] in the range of \( 0.1 \leq \alpha \leq 10 \). In fact, 

these two curves coincide in the symmetric phase. Surprising- 

ly, by increasing the number of players \( N \), \( \sigma^2/N \) for 

MGreal is consistently higher than that of MGrand for \( \alpha \) 

\( \leq 0.2 \). Furthermore, Fig. 1(b) shows that for a fixed \( \alpha \) 

\( \leq 0.2 \), \( \sigma^2/N \) for MGreal increases as \( N \) increases. In contrast, 

we find that the \( \sigma^2/N \) against \( \alpha \) curve for MGrand is independent of \( N \). Thus, the discrepancy between the two volatilities increases with \( N \) when \( \alpha \leq 0.2 \). (Although the values of \( N \) 

used in all curves reported in this paper are in the form \( 2^k \times 101 \) for some integer \( k \), our numerical simulation shows 

that the same conclusions are reached for the case of odd \( N \).)

From the above discussions, MGrand follows the scaling relation 

\[ \sigma^2_{\text{real}} \sim f(\alpha, S) \]

III. THE CROWD-ANTICROWD EXPLANATION

The above numerical findings can be explained by the 

crowd-anticrowd theory proposed by Hart et al. using the 

notion of reduced strategy space [8,9]. Recall that decisions 

made by two distinct strategies in a reduced strategy space 

are either mutually anti-correlated or uncorrelated when 

averaged over all possible history strings. Besides, the 

dynamics is very close to the original MG if all strategies are picked 

from the reduced strategy space [3]. In this formalism, the 

crowd-anticrowd theory states that every pair of anti- 

correlated strategies contributes independently to the vari- 

ance of attendance \( \sigma^2 \) [8,9].

To understand the difference in volatility between MGreal 

and MGrand, we first review the periodic dynamics observed 

in the symmetric phase of MGreal [6,8–12]. First, a prominent 

period \( 2^{M+1} \) peak in the Fourier transform of the minority 

choice time series for MGreal is found. We call this phenom- 

enon “period \( 2^{M+1} \) dynamics.” In addition, a conspicuous 

period two peak in the Fourier transform of the time series of 

the minority choice conditioned on an arbitrary but fixed 

history string in MGreal is also observed. We refer this as the 

“period two dynamics” in our subsequent discussions. Note 

that the above two dynamics are also observed by replacing 

the minority choice with attendance.
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For MG_real in the symmetric phase, the number of strategies at play is much larger than the reduced strategy space size. So, it is likely that each strategy in a reduced strategy space is assigned to more than one player in this phase. Initially, for a given history string \( \mu \), every alternative has equal chance to win. And the virtual score of a strategy \( \beta \) that has correctly predicted the winning alternative is increased by one while that of its anti-correlated strategy \( \bar{\beta} \) is decreased by one. Since the \( M \)-bit history string in MG_real provides complete information of the winning choices in the previous \( M \) turns, players will prefer to use strategy \( \beta \) to \( \bar{\beta} \) at the time when the same history string appears next. As more players are using the strategy \( \beta \) in the symmetric phase, these players are less likely to guess the minority choice correctly. In MG_real, the history strings of two consecutive turns are highly correlated. Actually, one can convert the history string in turn \( t \) to that of turn \( (t+1) \) by deleting the \( (t-1-M) \)th turn minority choice from one end of the former string and then appending the minority choice in turn \( t \) to the other end. Using these observations, Challet and Marsili studied the dynamics of MG_real by means of a de Bruijn graph [6]. A consequence of their analysis is that the history strings in the symmetric phase from the \( (2^Mk+1) \)th to the \( (2^M(k+1)) \)th turn is likely to form a de Bruijn sequence for any natural number \( k \). In other words, for any natural number \( k \), an arbitrarily given history string \( \mu \) is likely to appear exactly once between \( (2^Mk+1) \)th and \( (2^M(k+1)) \)th turns. Besides, the history \( \mu \) is likely to appear exactly twice between \( (2^M(k+1)+1) \)th and \( (2^M(k+1)+1) \)th turns—one of the turn at which a particular alternative wins and the other turn at which the same alternative loses. As a result, it is highly probable that the virtual scores of all strategies in the \( (2^M+1) \)th turn agree. Since the decision of a player depends on the difference between the virtual scores of his/her strategies, the dynamics of the game MG_real has a strong tendency to “reset” itself once every \( 2^M+1 \) turns. This is the origin of the period two and period \( 2^M+1 \) dynamics [6,8–12].

Because of the above two constraints on the history time series, the time series of the minority choice as well as that of the virtual score of a strategy in MG_real are not random walks.

From the above discussions, we know that the existence of period \( 2^M+1 \) dynamics is closely related to the following three facts that are unique for MG_real: the \( M \)-bit history string gives complete information on the winning alternatives in the previous \( M \) turns, the virtual score of a strategy is updated according to the current history string, and history strings of two consecutive turns are highly correlated. In contrast, for MG_rand, the history string does not correlate with the winning alternatives, the virtual score is updated according to the historical winning alternatives rather than the randomly generated history string, and the randomly generated history strings in two distinct turns are uncorrelated. Consequently, although MG_rand has a uniformly distributed history, it does not have a mechanism to ensure the game to “reset” itself once every \( 2^M+1 \) turns. Thus there is no reason for its history time series to follow a period \( 2^M+1 \) dynamics. Indeed, the autocorrelation on the minority choice time series in Fig. 3 confirms the absence of period \( 2^M+1 \) dynamics in MG_rand although Fig. 4 shows that MG_rand still exhibits period two dynamics. Our finding is consistent with Lee’s observation that the history occurrence probability density function for MG_real and MG_rand are different [7]. To summarize, for a

![Image](https://via.placeholder.com/150)

FIG. 3. The autocorrelation function \( C_n \) of the minority choice time series averaged over 50 runs for (a) MG_real and (b) MG_rand. The parameters used are the same as that in Fig. 3.

![Image](https://via.placeholder.com/150)

FIG. 4. The autocorrelation function \( C_n^\mu \) of the minority choice time series conditioned on a given history string \( \mu \) averaged over 50 runs for (a) MG_real and (b) MG_rand. The parameters used are the same as that in Fig. 3.
sufficiently small $\alpha$, the entropy of the minority choice time series for MG$_\text{rand}$ is higher than that of MG$_\text{real}$. We have gathered enough information to explain the discrepancy in the $\sigma^2/N$ against $\alpha$ curves between MG$_\text{real}$ and MG$_\text{rand}$. Recall that period two dynamics is observed in the symmetric phase of both games. This is because a significant number of players are using a particular strategy. And according to the crowd-anticrowd theory, this results in a high volatility in the symmetric phase in both games [8,9]. Further recall that the game MG$_\text{real}$ is very likely to “reset” itself once every $2^{M+1}$ turns leading to the period $2^{M+1}$ dynamics in the symmetric phase. In contrast, there is no mechanism to “reset” the game MG$_\text{rand}$ once a while for $\alpha \ll \alpha_c$. Therefore, the absolute deviation of the virtual score difference between two distinct strategies for MG$_\text{real}$ is less than that for MG$_\text{rand}$ whenever $\alpha \ll \alpha_c$. Consequently, a player in MG$_\text{rand}$ is more likely to stick to a strategy. Hence, players in MG$_\text{rand}$ cooperate slightly better than those in MG$_\text{real}$ in the symmetric phase. This explains why for any given sufficiently small $\alpha$, the variance of attendance per player $\sigma^2/N$ for MG$_\text{real}$ is higher than that of MG$_\text{rand}$.

Why do the $\sigma^2/N$ against $\alpha$ curves for MG$_\text{real}$ and MG$_\text{rand}$ coincide in the symmetric phase in the simulations of Cavagna [5], Challet and Marsili [6] together with Lee [7]? We believe this is due to the finite size effect as they picked $N = 101$ and $S = 2$ in their simulations. Due to the small number of players and the small value of $M$ involved, the fluctuation between different runs overwhelms the discrepancy between MG$_\text{real}$ and MG$_\text{rand}$.

After discussing the reason why $\sigma_\text{real}^2$ and $\sigma_\text{rand}^2$ differ, we move on to explain the two scaling relations (1) and (2). Researchers have argued that $\sigma^2$ is well approximated by a function of $\alpha$ only [3,10]. Indeed, when $\alpha \geq 0.1$ and hence the number of strategies at play is at most ten times the reduced strategy space size, the above approximation is reasonably good for MG$_\text{real}$ and MG$_\text{rand}$. Nevertheless, our simulation shows that this approximation breaks down in MG$_\text{real}$ when $\alpha \approx 0.1$. On the other hand, using mean field approximation, Manuca et al. showed that

$$\sigma^2 \approx \frac{N}{4} + \frac{N^2}{3(2^M)} \chi^2(S)$$

where $\chi$ is a slow varying function of $S$ provided that $\alpha \ll \alpha_c$ [11]. Since the correlation between history strings in successive turns is ignored in this derivation, the scaling relation (1) is applicable to MG$_\text{rand}$ in the regime of $\alpha \ll 0.1$. In contrast, our simulation shows that this relation is not applicable to MG$_\text{real}$ in the same regime.

Since we have already argued that $\sigma^2_{\text{real}}(M, S)/N \approx \sigma^2_{\text{rand}}(M, S)/N$ in the symmetric phase, in order to prove the validity of relation (2), it suffices to show that $\sigma^2_{\text{real}}(M, S)$ and $\sigma^2_{\text{rand}}(M, S)$ both scale as $N^2$. According to the crowd-anticrowd theory, for a sufficiently small $\alpha$, all players in MG$_\text{real}$ hold strategies whose virtual scores are similar. Therefore, most players will not stick to a particular strategy when making their choices. Besides, the existence of period two dynamics in the symmetric phase implies that both $\sigma^2_{\text{real}}$ and $\sigma^2_{\text{rand}}$ scales as $N^2$ [8–11]. Thus relation (2) holds.

Finally, we briefly explain why $\sigma^2$ decreases as $M$ increases. Since the average time between successive appearances of a given history increases exponentially with $M$, the absolute deviation of virtual score difference between two distinct strategies increases. Therefore, the period two dynamics is weakened. So, $\sigma^2$ for MG$_\text{real}$ and MG$_\text{rand}$ decreases as $M$ increases.

IV. OUTLOOK

In this paper, we report that history is relevant in determining the mean variance of attendance per player for MG in the symmetric phase when $\alpha \approx 0.2$ by an extensive numerical simulation using real and random histories. We explain our finding using crowd-anticrowd theory. Although all graphs shown in this paper are drawn by fixing the number of strategies per player $S$ to 2 and by letting the strategies to be drawn from the full strategy space, our conclusions apply equally well to the case of $S > 2$ as well as the case of drawing strategies from the reduced strategy space. Our findings show that the feedback mechanism buried under the correlation in the history time series is important in the study of volatility in the minority game.

It is instructive to study the relevance of history in the parameter region of $0.2 \leq \alpha \leq \alpha_c$. Our numerical simulation suggests that memory is irrelevant in this regime and hence the symmetric phase of MG can be subdivided into two phases. Further investigation is needed to test our hypothesis.

Finally, it would be nice if one could analytically solve the dynamics of the game in the symmetric phase. Since the difference in $\sigma^2/N$ for MG$_\text{real}$ and MG$_\text{rand}$ in the symmetric phase originates from the period $2^{M+1}$ dynamics, any such attempt must take the periodic dynamics of the minority choice time series into account—something that all attempts using replica trick [13,14] and generating functional method [15,16] so far have not been very successful to incorporate.

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