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Statistical analysis of magnetic-field spectra

Jian Wang
Department of Physics, The University of Hong Kong,Pokfulam Road, Hong Kong

Hong Guo
Centre for the Physics of Materials, Department of Physics,McGill University,Montreal, Quebec, Canada H3A 2T8

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We have calculated and statistically analyzed the magnetic-field spectrum (the \( B \) spectrum) at fixed electron Fermi energy for two quantum dot systems with classically chaotic shape. This problem arises naturally in transport measurements where the incoming electron has a fixed energy while one tunes the magnetic field to obtain resonance conductance patterns. The \( B \) spectrum, defined as the collection of values \( \{ B_i \} \) at which conductance \( g(B_i) \) takes extremal values, is determined by a quadratic eigenvalue equation, in distinct difference to the usual linear eigenvalue problem satisfied by the energy levels. We found that the lower part of the \( B \) spectrum satisfies the distribution belonging to the Gaussian unitary ensemble, while the higher part obeys a Poisson-like behavior. We also found that the \( B \) spectrum fluctuations of the chaotic system are consistent with the results we obtained from random matrices. [S0163-1829(98)03543-7]

Due to recent advances in controlled crystal growth and lithographic techniques, it is now possible to fabricate various “artificial atoms” or quantum dots the sizes of which are so small such that transport is in the ballistic regime.\(^1\) Among the many interesting phenomena associated with quantum ballistic transport, it was proposed\(^2\) that these quantum dots could be used to examine the theoretical notion of transport that belongs to the Gaussian orthogonal ensemble.\(^9\) It is well established that, for a classically nonchaotic system such as a particle confined to move inside a rectangular box, the normalized nearest-neighbor energy spacings \( \{ s \} \) obey a Poisson distribution\(^10\) \( P(s) = e^{-s} \). On the other hand, for a classically chaotic system such as a spinless particle confined to move inside a stadium shaped box,\(^11,12\) the spacings follow the Wigner distribution that belongs to the Gaussian orthogonal ensemble (GOE) in the language of RMT.\(^13\) Furthermore, when time-reversal invariance is broken, say, by applying a magnetic field, the system is described\(^14\) by the Gaussian unitary ensemble (GUE), where \( P_2(s) = (32s^2/\pi^2)e^{-(4s^2/\pi)} \).

In order to study certain aspects of quantum chaos using quantum transport techniques,\(^2,15,6\) one must deal with open systems where a scattering problem of charge carriers by some peculiar boundary must be solved. Experimentally it is quite difficult to study conduction as a function of the incoming electron energy in a quantitatively accurate fashion,\(^16\) although measurements on a tunnel junctions made of Al nanoparticles have recently been made\(^17\) and its relation to quantum chaos discussed.\(^18\) Using a two-dimensional (2D) electron gas fabricated with compound semiconductors, experiments usually measure conductance as a function of external magnetic fields,\(^2,15,6,19\) \( g(B) \), at a fixed electron Fermi energy \( E_o \). When a quantum dot is weakly coupled to the external leads, the magnetoconductance \( g(B) \) may show resonancelike behavior as the magnetic field \( B \) is varied,\(^5\) if the measurement is indeed in a regime that probes the internal electronic states.\(^17\) This behavior may be understood as follows. As \( B \) is varied, the energy levels \( \{\epsilon_i\} \) of the scattering states labeled by indices \( i = 1,2,\ldots \) in the quantum dot change with it: \( \epsilon_i = \epsilon_i(B) \). These levels are well separated since the the dot region is weakly coupled to the leads. For an incoming electron with a fixed Fermi energy \( E_o \), each time when the internal state energy \( \epsilon_i(B) \) is tuned to be equal to \( E_o \) as \( B \) is varied, a resonance peak occurs in \( g(B) \) due to a junction resonance. This was indeed observed in numerical simulations\(^5\) by solving the quantum scattering problem. Clearly this behavior should be observable if the Coulomb blockade effects are small, which happens when the system has a large capacitance, thus small charging energy. We shall assume this to be the case.

Hence in this junction resonance regime, it is interesting to define a \( B \) spectrum as the collection of values of the special magnetic fields \( B = \{ B_i \} \) at which \( g(B_i) \) is peaked.\(^5\) It is important to ask the following questions: what are the statistical properties of this \( B \) spectrum? How do its statistical properties change with the system shape? These are also useful questions to answer because it is increasingly possible to directly probe the internal electronic states of an isolated quantum dot, as demonstrated by the experiment reported in Ref. 17.

Motivated by these questions, in this paper we report our studies on closed quantum dot systems where these questions can be answered clearly. Simons, Szafer, and Altshuler\(^4\) have investigated the correlations of slopes of the energy levels as a function of an external parameter such as the magnetic field \( B \). We, however, emphasize that in their studies, the focus is...
on how a particular energy level $E_i$ varies with the external parameter $B$. Our investigation, on the other hand, is completely different as it does not focus on any energy level: the energy $E_o$ is a fixed parameter in the Schrödinger equation by the equilibrium Fermi energy of the system. Rather, we investigate the statistical properties of the set $\{B_i\}$, which makes $E_i$ an eigenvalue of the Hamiltonian. Indeed, we note that fixing $B$ and studying energy levels $\{E_i\}$, or fixing $E_o$ and studying the $B$ spectrum $\{B_i\}$, are two different problems. The former is about the energy eigenvalue spectra and its relation to an external parameter $B$, while the latter is relevant for transport situations where the incoming electron has energy $E_o$, which is fixed by the electron reservoir and cannot change, while one tunes $B$ to special values $\{B_i\}$ where conductance $g(B_i)$ is peaked. To our knowledge the statistical ensemble that is satisfied by the $B$ spectrum has not previously been determined. As we shall see below, this $B$ spectrum is determined by a quadratic eigenvalue problem, in distinct contrast to the usual energy spectrum at a fixed $B$, which is a linear eigenvalue problem. When a quantum dot is weakly coupled with the external leads, our answers are valid since in the weak coupling regime the levels are well separated and statistical properties should not change.\textsuperscript{20,21}

To make the problem at hand clearer, the inset of Fig. 3 shows the two-dimensional quantum dot systems we have studied: a Sinai-like billiard and a stadium-shaped quantum dot. These systems are classically chaotic systems. Experimentally transport measurements have been reported through these quantum dots, which were fabricated using the splitgate technology.\textsuperscript{15,6,2} Figure 1 shows the energy levels of a Sinai-like billiard as a function of magnetic field $B$. For a given $B=B_o$, the levels satisfy GUE as mentioned above. These are the intersections of the line $B=B_o$ (the vertical solid line) with the spectrum. However, for a fixed energy $E=E_o$, we are interested in the intersections of the horizontal solid line with the spectrum, which defines the $B$ spectrum. From the curves of Fig. 1 it is clearly not obvious what statistics the $B$ spectrum will satisfy.

In the presence of a magnetic field, the single-particle Schrödinger equation can be written as

$$\{-\nabla^2 - e_o + b^2 A_o^2 + 2b i A_o \cdot \nabla\} \psi = 0,$$  \hfill (1)

where $\vec{A} = B \vec{A}_0 = B \vec{x}$ is the vector potential, $B$ is the magnetic field, $E = \hbar^2 e_o / 2m$ is the energy, $b = eB / \hbar c$, and $i^2 = -1$. Discretizing the spatial derivative, Eq. (1) can be cast into a matrix form,

$$(M_1 + ib M_2 + b^2 M_3) \psi = e_0 \psi,$$  \hfill (2)

where $M_1, M_2, M_3$ are real matrices and $M_3$ is also diagonal. From Eq. (2) it is clear that for a fixed magnetic field $E$, the solution of all the allowed energies forms the usual linear eigenvalue problem, which has been studied intensively. However, for a fixed energy $e_0$, the solution of all the allowed magnetic fields $B$, the $B$ spectrum, forms a quadratic eigenvalue problem.

There are two methods to find the $B$ spectrum. The first method is by brute force: one directly calculates the energy eigenvalues $E$ for a given magnetic field $B$ and traces out the curves of $E$ versus $B$. These curves may cross over each other. One then calculates, from this $E$ versus $B$ curve, the set of magnetic fields $B$ for a fixed energy $E_o$. Figure 1 shows the $E$ versus $B$ curves for the first 40 eigenstates obtained using the Lanczos eigenvalue technique\textsuperscript{22} for the Sinai-like billiard system. However, this method gets time consuming very quickly if higher and higher states are needed.

The second method is to transform the quadratic eigenvalue problem Eq. (2) into a usual linear eigenvalue problem.\textsuperscript{23} Let $\Psi(t) = e^{i \Omega t} \psi$ and define $N_1 = M_1^{-1}(M_1 - e_0)$, $N_2 = M_3^{-1} M_2$, so that Eq. (2) becomes $-i \Psi'' + N_2 \Psi' + N_1 \Psi = 0$ where $\Psi' = iB \Psi$ is the first derivative with respect to the parameter $t$, and $I$ is the unit matrix. With $\Psi' = \Phi$ ($\Phi' = iB \Phi$), we have $\Phi = \Psi' = iB \Psi$ and $N_2 \Phi + N_1 \Psi = iB \Psi$, or

$$\begin{pmatrix} 0 & I \\ N_1 & N_2 \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = iB \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}.$$  \hfill (3)

This is a linear eigenvalue problem for $b$. The matrix in Eq. (3) is, however, not Hermitian, hence its eigenvalues may or may not be real. To obtain the physical solution, we look for all imaginary eigenvalues so that $b$ is real. We verified that the results coincide with those obtained with the conventional Lanczos technique described above. In this way we have calculated the $B$ spectrum from Eq. (3) for the two chaotic systems (inset of Fig. 3), for different energies $e_o$. The confining potential is assumed to be hard wall.

For the Sinai-like billiard, we considered a 4.7 $\times$ 4.3 $\mu$m rectangular quantum dot with three hard disks inside the dot. The radius of the disks were fixed at 0.9, 0.8, and 0.5 $\mu$m with their centers randomly chosen but without the disks overlapping each other. Randomly changing positions of the disks allows us to generate different configurations for ensemble averaging to obtain reasonable statistics. The number of physical solutions $N$ obtained from Eq. (3) depends on the fixed energy $e_o$, with $N$ increasing with the value of the energy $e_o$. For instance, when $e_o$ is fixed at 30 meV, we have $N \sim 1433$ physical solutions for the $B$ spectrum out of a total of about 3000 eigenvalues. We found that the statistics of the $B$ spectrum behaves differently for the lower and
higher part of the spectrum, respectively. The lowest $24 \sim 300$ $B$ levels give GUE statistics and the highest $\sim 800$ $B$ levels give a Poisson-like behavior. Figure 2 plots the distribution function obtained from our numerical data of the nearest-neighbor $B$ level spacings for the Sinai-like billiards. With the ensemble average of 20 different configurations, the distribution determined from the lower part of the $B$ spectrum agrees well with GUE statistics (solid line) $P_B(s)$ discussed above. On the other hand, the higher part of the spectrum, shown in the inset of Fig. 2, has a Poisson-like behavior.

Another often used measure in studying level statistics is the spectral rigidity $\Delta_3$, defined as $\langle s^2 \rangle$ the mean-square deviation of the best local fit straight line to the staircase cumulative spectral density over a normalized energy scale. This quantity measures longer range correlations of the level spectrum and often provides a more critical test of the level statistics. To compute $\Delta_3$ we followed a scheme presented in Ref. 26, and the numerical data are compared with the analytical formula from random matrix theory. Figure 3 shows the $\Delta_3$ analysis of the lower part of the $B$ spectrum. It is clear that the data are in very good agreement with the GUE statistics.

To test the statistical properties of the $B$ spectrum further, we studied another chaotic system, namely a stadium-shaped quantum dot (inset of Fig. 3). The distribution function and $\Delta_3$ for its $B$ spectrum are included in Figs. 2 and 3. Here we did not use an ensemble average and the data is for one system only. The general trend is the same as for the Sinai-like billiards, and still clearly shows the two distinct behaviors for different parts of the $B$ spectrum, namely a GUE behavior of the lower part and a Poisson-like behavior for the higher part. Finally, we have verified that the same statistical behavior is observed for both the Sinai-like billiard and the stadium-shape billiard with many different values of the fixed energy $e_0$, and conclude that the lower part of the $B$ spectrum of both systems satisfies GUE statistics.

It is not difficult to understand that the higher part of the $B$ spectrum should behave differently. The higher part corresponds to larger values of the magnetic fields, which are known to destroy chaos. For the particular sizes of the Sinai-like billiard and the stadium-shaped dot, the magnetic-field $B_c$ that roughly separates the low-lying and high-lying part of the $B$ spectrum is about $B_c \sim 3$ to 4 $T$. The classical cyclotron radius of the electron at and above this field strength is quite small compared with the system size. Hence the electron "skips" along the wall of the confining potential or makes circular motion inside the quantum dot, thereby reducing the effect of chaotic scattering by the geometry. Nevertheless, the field range up to $B_c$ is quite wide and it should be possible to investigate the $B$ spectrum experimentally using resonant magneto-conductance measurements for systems having weak coupling to the leads.

While the quadratic eigenvalue problems reported above were for billiard systems where the continuum Schrödinger equation is discretized to obtain the matrix equation (3), we also studied a similar quadratic eigenvalue problem using random matrices to replace the matrices $M_1$, $M_2$, and $M_3$ in Eq. (2). Two requirements must be satisfied: first, the matrices $M_1$, $M_2$, and $M_3$ must be real; second, $M_1$ must be symmetric, and $M_2$ antisymmetric so that the Hamiltonian be Hermitian. The random matrices were set up in the standard fashion: for a $N \times N$ matrix, $M_1$ has $N(N+1)/2$ independent matrix elements and $M_2$ has $N(N-1)/2$, where the matrix elements are Gaussian random numbers. We then diagonalized Eq. (3) using the same numerical methods discussed above and found all the physical solutions. We chose $N = 1500$ so that the matrix that must be diagonalized is $3000 \times 3000$. Typically we obtained about 700 physical $B$ levels for various energies $e_0$. The results are included in Figs. 2 and 3. It is clear that the $B$-level statistics from the random matrices is completely consistent with the GUE statistics. We may conclude that $B$ spectra coming from the quantum billiard systems and from random matrices are still
consistent with each other and a universality can still be established for this quadratic eigenvalue problem using the random matrices.

In summary, we have numerically investigated the statistical properties of the magnetic-field spectra (the $B$ spectrum), which is determined by a quadratic eigenvalue problem. This spectrum is defined by the allowed magnetic fields for an electron moving in a quantum dot with its fixed Fermi energy. This problem arises for systems with weak coupling to the leads in which the scattering states are well separated in energy. For two different chaotic billiards, e.g., two-dimensional quantum dots in the shape of a Sinai-like billiard and a stadium billiard, the $B$ spectra have distinctly different statistical behavior at the lower and higher parts of the spectra. In particular, the lower part is well described by the GUE statistics while the higher part is Poisson like. We found that the same quadratic eigenvalue problem can be studied using random matrices as well, and the eigenvalues from the random matrices have precisely the same behavior as those of the billiards. Thus the notion of universality classes using the random matrix theory can be carried over to this new problem. While our numerical data provided clear evidence of the statistical properties of the present quadratic eigenvalue problem, it is not at all obvious a priori that such statistical properties are controlled by the GUE universality.

Further work is needed to provide an analytical understanding.

Experimentally the statistical properties of the $B$ spectra could be examined for quantum dot systems which couple weakly to the external leads and thus the transmission is controlled by junction resonances. The weak coupling could be provided by adding constrictions at the connections of the leads with the quantum dot for which only the lowest few subbands of the leads can propagate. In typical experimental situation on submicron structures the single-particle level spacing is around 0.05 meV, thus they can be measured if the temperature is kept to less than 500 mK. We thus conclude that for a magnetotransport measurement in the junction resonance regime, the special magnetic-field strengths $\{B_i\}$ at which $g(B_i)$ takes extremal values, should satisfy GUE statistics.

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7For a short review of the fractal resistance fluctuations observed in Ref. 6, see Mark Fromhold, Nature (London) 386, 123 (1997).
24 For $B=0$ the energy levels $E(B)$ satisfy GOE statistics, and it changes to GUE when $B$ is such that approximately one flux quanta is enclosed by the quantum dot area. In our case here, below this transition field there are about $16–17$ $B$ levels. Thus in all our data analysis, we have not included the lowest twenty $B$ levels.


28 A departure from random matrix prediction was observed in Fig. 3 when $L$ becomes larger than 20. This is similar to that happens in energy level analysis.


30 For the high lying part of the $B$ spectrum, the precise distribution form is difficult to determine from our numerical analysis. However a Poisson-like behavior, namely, a $B$-level attraction, is clearly observed here. The numerical fit to a Poisson distribution is nevertheless not good, due in part to two reasons. First, there is a gradual crossover from the GUE distribution of the low-lying part of the spectrum. Second, for the very high part of the $B$ spectrum, the electron wave function forms edge states with nearly equal level spacings (the Landau levels) where the energy vs $B$ is nearly linear. These lead to a deviation from a perfect Poisson distribution.