Classifying rational densities using two one-dimensional cellular automata

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Given a (finite but arbitrarily long) string of zeros and ones, we report a way to determine if the number of ones is less than, greater than, or equal to a prescribed number by applying two sets of cellular automaton rules in succession. Thus, we solve the general one-dimensional density classification problem using two cellular automata. [S1063-651X(98)07402-9]

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I. INTRODUCTION

Cellular automaton (CA) is an extremely simple local interaction model of natural systems [1,2]. In this paper, we restrict ourselves to considering a conventional definition of CA, namely, that both the number of possible states at each site and the number of CA rules in the rule table are *finite*. That is to say, the state of each site in the next time step depends only on a finite number of neighboring sites in the system. Although the rules of CA are simple and local, complicated spatial patterns can be formed. Moreover, from the computer science point of view, CA can be regarded as a very special kind of Turing machine with no internal memory. In fact, tailor-made CA can be used to simulate certain logical operations [3]. Therefore, it is instructive to know the power and limitations of CA in computation.

Since CA is essentially a special Turing machine with no internal memory, it is natural to ask if CA can be used to perform certain tasks that require global counters. An example is the so-called density classification problem. Consider a one-dimensional (finite but arbitrarily long) chain of sites with periodic boundary condition. Each site is either in state zero or state one. The problem is to change the state of every site to zero if the number of zeros is more than the number of ones in the chain. Otherwise, every site is set to state one. Clearly, the density classification problem is trivial if one can invoke an internal memory to count the number of zeros in the chain. Alternatively, one can solve this problem if the rule table scales with the system size N so that the CA model becomes nonlocal. However, Land and Belew showed that density classification cannot be done perfectly using a single one-dimensional CA [4].

Later on, Fukś pointed out that the density classification problem in fact can be solved if we apply *two* CA rules in succession [5]. For a one-dimensional lattice of N sites, he first applies the Wolfram elementary CA rule 184 (the so-called traffic rule [6]) $\lfloor (N-2)/2 \rfloor$ times. Then, he applies the Wolfram elementary CA rule 232 (the so-called majority rule) $\lfloor (N-1)/2 \rfloor$ times. The combined automaton solves the density classification problem succinctly [5].

With slight modifications, Fukś is able to classify density in the form $1/n$ for some integer $n \geq 2$ [5]. Then, Fukś went on to ask if it is possible to use combined CA to classify an arbitrary density ρ_c. Here, we report a simple solution to his problem using a combination of two CAs provided that ρ_c is a rational number. But before stating our CA rules, let us restate the density classification problem in the most general context.

First, let us define the problem in a more precise way. We are given a critical density ρ_c between zero and one. Then we consider a one-dimensional lattice made up of N sites with periodic boundary condition. (We can always assume that N is sufficiently large to avoid trivial cases.) The state of each site can be either zero or one. And the initial density of ones in the system is denoted by ρ. (That is, there are $N\rho$ sites in the system with states equal to one.) Our goal is to find a combination of two CA rules (with finite rule tables) such that after going through the CA, the final states of the sites on the lattice all become zeros (and all ones) if $\rho < \rho_c$ (and $\rho > \rho_c$, respectively). And in the case of $\rho = \rho_c$, the final state consists of exactly $N\rho_c$ sites with states equal to one. (From the nonlinear-dynamics point of view, all the zeros and all the ones are the two stable fixed points of the CA dynamics, whereas the a density equals to ρ_c is an unstable fix point of the CA dynamics.) In addition, the two CA rules are independent of the lattice size N for sufficiently large N. (Thus, our two CA rules are truly local.)

Since the number of sites N is finite, it suffices for us to consider the case where ρ_c is a rational number written in the form p/q for some relatively prime integers p and q. Furthermore, we can also assume that $p \neq 0$ and $p \neq q$, since classifying $\rho_c = 0$ and 1 are trivial using CA. Now we report the two CA rules we use to solve the density classification problem for rational critical density.

II. MODIFIED TRAFFIC RULE

Our generalized traffic rule goes as follows. First, we regard a site with state one as being occupied by a car. Otherwise, that site is empty. Then a car can move in the next time step to the right by one site if and only if (a) its immediate right-neighboring site is unoccupied and (b) the $q-1$ consecutive right-neighboring sites are occupied by at most $p-1$ cars. Otherwise, the car stays in its original position. For example, if $\rho_c = 2/5$, then the first cars in the sequences 10000 and 10100 can move one step forward in the next time step, while the first cars in the sequences 11000 and 10011 cannot. Readers can readily verify that the total number of cars in the system is conserved under the above set of rules.

Moreover, one can easily convert the above set of traffic rules into a *finite* CA rule table consisting of 2^{q+1} rules. In addition, when $\rho_c = 1/2$, the modified traffic rule is reduced to Wolfram's elementary CA rule 184.

We define the local car density at each site to be the total number of cars contained in that site and in its $q-1$ immediate right-neighboring sites divided by q. Thus, the local car density of a site changes when and only when a car enters that site or a car leaves the $(q-1)$th site to its right. For simplicity, a collection of sites is said to be in a low-density region if the local car density of each site in the collection is less than or equal to ρ_c. And a high-density region is defined as a collection of sites with local car densities greater than ρ_c. Now, we prove two theorems concerning the distribution of high- and low-density regions under the repeated actions of the modified traffic rules.

Lemma 1. If the initial density ρ of the system is less than or equal to ρ_c, then after at most $\lceil N(\max(q,2p)-1)/q \rceil + q - 2$ time steps, the local car density for every site is less than or equal to ρ_c.

Proof. Consider a collection of sites with local car density greater than ρ_c in the initial system configuration. Using the modified traffic rule, in the co-moving frame of a car, no site will have a car density exceeding $(p+1)/q$ if that car is originally located at a site with a local car density less than or equal to ρ_c. In addition, such an aggregation of cars above the density threshold ρ_c will be achieved within the first $q-1$ time steps. Since the overall car density of system is less than or equal to ρ_c, upon repeated applications of the modified traffic rules, cars will gradually move out of local regions with a car density exceeding ρ_c (if any). Note that the local car densities of those cars moving out of these "high-density regions" are less than or equal to ρ_c. Moreover, once these cars are "dissolved" from a local high-density region, the local car density of the sites containing these cars will never exceed ρ_c unless these cars merge into a high-car-density region in front.

Suppose none of the cars moving out of a high-car-density region will be stopped by another high-car-density region in front. Then, once all the high-car-density regions dissolve completely, no further high-density region will be formed thereafter. Thus, our assertion that all sites will have a local car density not exceeding ρ_c asymptotically is true in this case. So, it remains to consider the case in which some of the cars moving out of a high-car-density region move into (and hence are temporarily stopped by) a high-car-density region in front. Since the overall car density of the entire system does not exceed ρ_c, at least one of the high-car-density regions will start to dissolve. Note that N is finite, and those cars merging into a high-density region will readily redissolve once they are allowed to move. After a finite number of time steps, all the high-density regions will disappear. After that, no more high-density regions can form. Thus, our assertion that all sites will have local car density not exceeding ρ_c is also true is this case.

Finally, we estimate the number of time steps required to reach this state. Since the updating is taken in parallel, the worst case occurs when some of the local high-density regions are formed during the first $q-2$ time steps. Then all the cars merge into a single high-car-density region before they finally dissolve. According to the modified traffic rules,

if $p \leq q/2$, then all cars dissolved from a high-density region will not be blocked. In this case, it takes at most $q-1$ time steps to dissolve p cars from a high-density region. On the other hand, if $p > q/2$, then some of the dissolved cars will still be blocked occasionally. And it takes $q-1+(2p-q) = 2p-1$ time steps to dissolve p cars from a high-density region. Therefore, in any case, it requires at most $\lceil N\rho(\max(q,2p)-1)/p \rceil \leq \lceil N(\max(q,2p)-1)/q \rceil$ time steps to completely dissolve a high-density region made up of $N\rho$ cars. Hence, our assertion is proved.

Lemma 2. If the initial density ρ of the system is greater than ρ_c, then after at most $\lceil N(q-p)(\max(q,2p)-1)/pq \rceil$ time steps, the local car density for every site is greater than or equal to ρ_c.

Proof. We may assume that the local car density at each site initially is less than or equal to ρ_c. Otherwise, no car can move to the right at the beginning and our assertion is trivially true. Since the overall car density ρ is greater than ρ_c, all the low-car-density regions (sites with local car density less than ρ_c) are surrounded by high-car-density ones (sites with car density greater than or equal to ρ_c). Thus, cars can gradually move from the high-density to the low-density regions. And using similar arguments as in Lemma 1, after a finite number of time steps, the entire system is contained in a single high-density region. And from that time on, no car in the system can move.

We move on to estimate the number of time steps required to reach this "frozen" state. Similar to the argument in Lemma 1, the worst case occurs when there is only one low-density region initially. And it is easy to verify that the number of time steps required to reach a frozen state for such an initial system configuration equals $\lceil N(1-\rho)(\max(q,2p)-1)/p \rceil \leq \lceil N(q-p)(\max(q,2p)-1)/pq \rceil$.

Combining Lemmas 1 and 2, we conclude the following.

Theorem 1. By applying the modified traffic rules for $\lceil N(\max(q,2p)-1)\max(q-p,p)/pq \rceil + q - 2$ times, an initial system configuration will be segregated into one of the following three cases.

(a) If $\rho < \rho_c$, then the local car density at every site is less than or equal to ρ_c. In addition, at least one of the sites will have local car density strictly less than ρ_c.

(b) If $\rho = \rho_c$, then the local car density at every site equals ρ_c.

(c) If $\rho > \rho_c$, then the local car density at every site is greater than or equal to ρ_c. In addition, at least one of the sites will have a local car density strictly greater than ρ_c.

III. MODIFIED MAJORITY RULE

After segregating the system configuration according to its initial density ρ, the density classification problem becomes straightforward. We consider the following modified majority rule for a given critical density $\rho_c \equiv p/q$. The state of a site in the next time step is one if there are at least $2p+1$ ones in the $2q+1$ sites consisting of itself and the q consecutive left- and right-neighboring sites. Otherwise, the state of this site in the next time step is zero. For example, if $\rho_c = 1/2$, then the states of the middle sites in the next time step for 10101 and 01010 are one and zero, respectively. Clearly, the modified majority rule is local.

Now, we present the results of applying the modified ma-

jority rule to certain system configurations which are of interest.

Lemma 3. Any system configuration where the local car density at every site equals ρ_c is a fixed point of the modified majority rule dynamics.

Proof. Consider an arbitrary site α in the system. Since the local car density at every site equals ρ_c, there are precisely $2p$ sites in state one among the $2q$ neighboring sites of α. Thus, the majority rule implies that the state of site α in the next time step is equal to its present state. Consequently, this system configuration is a fixed point of the modified majority rule.

Lemma 4. Suppose the local car density at every site is greater than or equal to ρ_c and also that the total density of the system is strictly greater than ρ_c. After applying the modified traffic rule $\lceil N/2(q-1) \rceil$ times, the state of every site in the system will become one. Similarly, if the local car density at every site is less than or equal to ρ_c and also if the total density of the system is strictly less than ρ_c, then after applying the modified traffic rule $\lceil N/2(q-1) \rceil$ times, the state of every site in the system will become zero.

Proof. For simplicity, we only consider the case in which $\rho > \rho_c$. The proof for the case in which $\rho < \rho_c$ is similar. From Lemma 3, we know that the local car density at every site must be greater than or equal to ρ_c after repeated applications of the modified majority rule. Moreover, if the local car density of a site α exceeds ρ_c, then the states of α and its $q-1$ left- and right-neighboring sites must be one in the next (and hence all subsequent) time step. In other words, the propagation speed for state one is $q-1$ sites per time step both leftward and rightward. Therefore, after $\lceil N/2(q-1) \rceil$ time steps, all sites in the system will be in state one. Hence, our assertion is proved.

Combining Theorem 1, Lemma 3, and Lemma 4 with the fact that the size of the rule tables for both the modified traffic and the modified majority rules are independent of N,

we obtain the following CA density classification theorem:

Theorem 2 (Density Classification By CA). Let $\rho_c = p/q$ be a rational number between zero and one, with p and q being relatively prime positive integers. Then the density classification problem can be solved using the following two CA rules: apply the modified traffic rules $\lceil N(\max(q,2p) -1)\max(q-p,p)/pq \rceil + q - 2$ times and then the modified majority rule $\lceil N/2(q-1) \rceil$ times.

IV. DISCUSSION

In summary, we report a way to classify the density of ones using two one-dimensional binary CAs provided that the density threshold ρ_c is a rational number. Our result, therefore, generalizes that of Fukś [5]. Besides, our construction also takes into account the case where the density ρ is equal to its critical value ρ_c—something Fukś does not consider seriously. Because of the local nature of CA rules, we believe that the time complexity of a general density classification problem using any combination of CA rules is at least O(N). Thus, apart from a constant speedup, our set of density classification CA rules is probably the least time-consuming.

As we have discussed in Sec. I, if the number of sites in the system N is finite, it suffices to restrict ourselves to considering rational density classification. Any irrational density can be well approximated by a corresponding rational density. Actually, by approximating an irrational density by better and better rational numbers, the size of the CA rule table grows. Thus, we believe that combinations of CA rules are unlikely to be powerful enough to classify irrational density in an infinite one-dimensional system.

[1] See, for example, S. Wolfram, *Cellular Automata and Complexity* (Addison-Wesley, Reading, MA, 1994).

[2] See, for example, T. Toffoli, in *Encyclopedia of Physics*, edited by R. G. Lerner and G. L. Trigg, 2nd ed. (VCH Publishers, New York, 1990), p. 126 for a definition of cellular automaton.

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