LMI Synthesis of H_2 and Mixed H_2/H_∞ Controllers for Singular Systems

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***Abstract*—This paper considers the H_2 control problems for continuous-time singular systems with and without an H_∞ constraint. Without the constraint, we derive necessary and sufficient conditions for the existence of H_2 output feedback controllers using the linear matrix inequality (LMI) approach. With the H_∞ constraint, sufficient LMI conditions on the existence of the H_2 controller are obtained. In both cases, the desired H_2 controller can be constructed through the feasible solutions of the LMIs. The proposed synthesis method is illustrated through numerical examples.**

***Index Terms*—H_2 controller, linear matrix inequality (LMI), mixed H_2/H_∞ controller, singular systems.**

I. Introduction

THE SINGULAR system model is a natural representation of dynamic systems and describes a larger class of systems than the normal linear system model. The usual state-space description of linear systems cannot represent the algebraic constraints between state variables. For example, in chemical processes, such algebraic constraints are often obtained from thermodynamic equilibrium relations, empirical corrections, pseudo-steady-state assumptions, and closure conditions [9]. Moreover, it is illustrated by Kumar and Daoutidis [9] that the algebraic equations are implicit and singular in nature in a wide variety of chemical process applications with simultaneous fast and slow phenomena such as a reactor with fast heat transfer through a heating jacket, a reactor with fast and slow reactions, a two-phase reactor with fast mass transfer, a cascade of reactors with high-pressure gaseous flow, and so on. Impulsive and hysteretic phenomena, which exhibit in circuit theory, also cannot be treated properly using linear state-space models [10], [15]. Fortunately, singular system models offer an effective way to describe these behaviors. Such models are also widely encountered in large-scale systems, economics, networks, power, and neural systems [2], [10].

The H_2 norm of the system is one of the most important control performance measures in control system design. In H_2 optimal control, the quadratic performance of a system can be equivalently characterized by the H_2 norm of an associated system. In a white noise attenuation design, the variance of the output of the system error caused by white noise can also be represented in the form of the H_2 norm [6]. Hence, the H_2 control problem, which aims at finding a controller such that the closed-loop system is internally stable and the H_2 norm of the closed-loop system is as small as possible, has drawn considerable attention over the last few decades. Under the assumptions of $D_{12}^T D_{12} > 0$ and $D_{21} D_{21}^T > 0$ (see [16]), an optimal H_2 controller was given by solving two Riccati equations in [3] for linear state-space systems and, for singular systems, similar results were given in [14]. A mathematically elegant solution has been found when $D_{12}^T D_{12} > 0$ and $D_{21} D_{21}^T > 0$ are satisfied. In the case when such conditions are not satisfied, Sato and Liu [12] gave the linear matrix inequality (LMI) solution to the general H_2 problems for linear state-space systems. For linear singular systems, the state feedback H_2 control was considered in [7] based on strict LMIs and, in [13], a state feedback H_2 control problem for uncertain time-varying singular systems was studied involving nonstrict LMIs (of the form $E^T X = X^T E \geq 0$) such that an upper bound of the H_2 norm of the closed-loop system is less than a scalar $\gamma > 0$. However, the dynamic output feedback H_2 control problem in the case when $D_{12}^T D_{12} > 0$ and $D_{21} D_{21}^T > 0$ are not satisfied is still open for singular systems.

The H_∞ norm of a system is another popularly used control performance measure for control system synthesis. For linear state-space systems, the existence of H_∞ controllers was tested by solving two Riccati equations in terms of the state-space matrices in [3], and all suitable controllers were parameterized via a linear fractional transformation. Recently, the H_∞ control problem has been investigated based on an LMI formulation [5], [8]. For linear singular systems, the H_∞ control problem has been also considered in [11], [15], and [16] using generalized Riccati equations and matrix inequalities. Due to the importance of the H_2 and H_∞ norms, the mixed H_2/H_∞ control problem has received much attention recently. For linear state-space systems, such works can be seen in [1], [4], [17], [18], and so on, and for linear singular systems, to our knowledge, it remains an open problem.

In this paper, we study the dynamic output feedback control problems of singular systems when $D_{12} = 0$ and $D_{21} = 0$. It is shown that under a certain condition, the H_2 norm of a singular system is bounded by a given $\gamma > 0$ if and only if a set of LMIs are feasible. Based on this result, we establish necessary and sufficient conditions in strict LMIs for the existence of a dynamic output feedback controller such that the closed-loop system is admissible and the H_2 norm of the closed-loop system

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is less than a prescribed $\gamma > 0$. Moreover, explicit expressions of the state-space matrices of the controller are given in terms of the solutions of the derived LMIs. A mixed H_2/H_∞ control problem for singular systems is also considered in this paper. Sufficient conditions for such a controller are obtained in LMIs. Numerical examples are used to illustrate the main results.

II. Preliminaries

A continuous-time singular system, which is denoted as (E, A, B, C), takes the form of

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^l$ is the output vector, E, A, B, C are constant real matrices with appropriate dimensions, and $\mathrm{rank}(E) = r(r \le n)$. The pair (E, A) is *regular* if $\det(sE - A)$ is not identically zero. The zeros of $\det(sE - A)$ are called the *finite poles* of (E, A). (E, A) is said to be *stable* if all the finite poles of (E, A) lie in $\mathrm{Re}(s) < 0$. (E, A) is called *impulse free* if $\deg\det(sE - A) = r$. (E, A) is *admissible*, if it is regular, stable, and impulse free.

Since $\mathrm{rank}(E) = r$, there exist two orthogonal matrices U and V such that E has the decomposition as

$$E = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T \tag{1}$$

where $\Sigma_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ with $\sigma_i > 0$ for $i = 1, \ldots, r$. Partition

$$U = [\, U_1 \quad U_2 \,] \qquad V = [\, V_1 \quad V_2 \,] \tag{2}$$

conformably with (1) and let $\mathcal{V} = [V_1\Sigma_r \;\; V_2]$. We then have

$$E = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}^T. \tag{3}$$

From (1), it can be seen that V_2 spans the right null space of E and U_2^T spans the left null space of E, i.e., $EV_2 = 0$ and $U_2^T E = 0$.

Lemma 1: The following items are true.

i) All P satisfying

$$E^T P = P^T E \ge 0 \tag{4}$$

can be parameterized as

$$P = U_1 W U_1^T E + U_2 S$$

where $W \ge 0 \in \mathbb{R}^{r\times r}$ and $S \in \mathbb{R}^{(n-r)\times n}$ are parameter matrices; furthermore, when P is nonsingular, $W > 0$.

ii) All X satisfying

$$XE^T = EX^T \ge 0 \tag{5}$$

can be parameterized as

$$X = EV_1 \tilde{W} V_1^T + \tilde{S} V_2^T$$

where $\tilde{W} \ge 0 \in \mathbb{R}^{r\times r}$ and $\tilde{S} \in \mathbb{R}^{n\times(n-r)}$ are parameter matrices; furthermore, when X is nonsingular, $\tilde{W} > 0$.

iii) If $U_1 W U_1^T E + U_2 S$ is nonsingular with $W > 0$, then there exist $\tilde{W}$ and $\tilde{S}$ such that

$$\left(U_1 W U_1^T E + U_2 S\right)^{-T} = EV_1 \tilde{W} V_1^T + \tilde{S} V_2^T$$

with $\tilde{W} = \Sigma_r^{-1} W^{-1} \Sigma_r^{-1}$.

Proof: i): Let

$$\overline{P} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = U^T P \mathcal{V}^{-T} \quad \overline{E} = U^T E \mathcal{V}^{-T} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Equation (4) then gives $\overline{E}^T\overline{P} = \overline{P}^T\overline{E} \ge 0$, which implies $P_1 = P_1^T \ge 0$ and $P_2 = 0$. With $\overline{E}\mathcal{V}^T = U^T E$, we have

$$\begin{aligned} P &= U\overline{P}\mathcal{V}^T \\ &= [\, U_1 \;\; U_2 \,] \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \overline{E}\mathcal{V}^T + [\, U_1 \;\; U_2 \,] \begin{bmatrix} 0 & 0 \\ P_3 & P_4 \end{bmatrix} \mathcal{V}^T \\ &= [\, U_1 P_1 \quad 0 \,] U^T E + U_2 [\, P_3 \quad P_4 \,] \mathcal{V}^T \\ &= U_1 P_1 U_1^T E + U_2 [\, P_3 \quad P_4 \,] \mathcal{V}^T. \end{aligned}$$

Let

$$W = P_1 \quad S = [\, P_3 \quad P_4 \,] \mathcal{V}^T$$

P is then parameterized by W and S. If P is nonsingular, $W = P_1 > 0$.

ii): Let

$$\overline{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = U^T X \mathcal{V}.$$

Equation (5) then gives $\overline{X}\overline{E}^T = \overline{E}\overline{X}^T \ge 0$, which implies $X_1 = X_1^T \ge 0$ and $X_3 = 0$. Therefore, noticing $EV_2 = 0$, all the solutions of (5) are parameterized by

$$\begin{aligned} X &= U\overline{X}\mathcal{V}^{-1} \\ &= U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}^{-1} + U \begin{bmatrix} 0 & X_2 \\ 0 & X_4 \end{bmatrix} \mathcal{V}^{-1} \\ &= U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}^T \mathcal{V}^{-T} \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}^{-1} + U \begin{bmatrix} 0 & X_2 \\ 0 & X_4 \end{bmatrix} \mathcal{V}^{-1} \\ &= E\mathcal{V}^{-T} \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{V}^{-1} + U \begin{bmatrix} 0 & X_2 \\ 0 & X_4 \end{bmatrix} \begin{bmatrix} \Sigma_r^{-1} V_1^T \\ V_2^T \end{bmatrix} \\ &= EV_1 \Sigma_r^{-1} X_1 \Sigma_r^{-1} V_1^T + U \begin{bmatrix} X_2 \\ X_4 \end{bmatrix} V_2^T. \end{aligned}$$

Let $\tilde{W} = \Sigma_r^{-1} X_1 \Sigma_r^{-1}$, $\tilde{S} = U\left[\begin{smallmatrix} X_2 \\ X_4 \end{smallmatrix}\right]$, X is then parameterized by $\tilde{W}$ and $\tilde{S}$. If X is nonsingular, then $\tilde{W} = X_1 > 0$.

iii): It can be seen that $P = U_1 W U_1^T E + U_2 S$ is a solution of (4), and P^{-T} is a solution of a solution of (5). From the proof of i), we may see that $W = P_1$ and P^{-T} can be expressed as

$$P^{-T} = U\overline{P}^{-T}\mathcal{V}^{-1}$$

where

$$\overline{P}^{-T} = \begin{bmatrix} P_1^{-1} & -P_1^{-1} P_3^T P_4^{-T} \\ 0 & P_4^{-T} \end{bmatrix}.$$

On the other hand, from ii) and its proof, P^{-T} can also be expressed as

$$P^{-T} = EV_1 \tilde{W} V_1^T + \tilde{S} V_2^T$$

and

$$\tilde{W} = \Sigma_r^{-1} P_1^{-1} \Sigma_r^{-1} = \Sigma_r^{-1} W^{-1} \Sigma_r^{-1}.$$

■

Lemma 2: (E, A) is admissible if and only if either of the following items is true.

i) There exist $W > 0$ and S such that

$$A^T \left(U_1 W U_1^T E + U_2 S\right) + \left(U_1 W U_1^T E + U_2 S\right)^T A < 0$$

holds.

ii) There exist $\tilde{W} > 0$ and $\tilde{S}$ such that

$$\left(EV_1 \tilde{W} V_1^T + \tilde{S} V_2^T\right) A^T + A \left(EV_1 \tilde{W} V_1^T + \tilde{S} V_2^T\right)^T < 0$$

holds.

Proof: The proof can be obtained by Lemma 1 and the fact that (E, A) is admissible if and only if there exists P (or X) such that [11]

$$E^T P = P^T E \geq 0 \quad A^T P + P^T A < 0$$

(or $XE^T = EX^T \geq 0$, $XA^T + AX^T < 0$) hold. ■

III. Computation of the H_2 Norm

Let $G(s)$ denote the transfer function of singular system (E, A, B, C), i.e.,

$$G(s) = C(sE - A)^{-1} B.$$

The H_2 norm of an admissible singular system (E, A, B, C) can then be defined as

$$\|G(s)\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left(G^T(-j\omega) G(j\omega)\right)\right)^{1/2}. \quad (6)$$

It can be seen that $\|G(s)\|_2$ is finite if and only if $\lim_{s \to \infty} G(s) = 0$. To ensure $\|G(s)\|_2$ to be finite, we assume that [7], [13]

$$\ker C \supseteq \ker E \quad (7)$$

where $\ker(\cdot)$ denotes the kernel of a matrix.

In the following theorem, we can see that if the H_2 norm of (E, A, B, C) is finite, the H_2 norm of (E, A, B, C) can be expressed explicitly.

Theorem 1: Assume that (E, A) is admissible and (7) holds. The H_2 norm of (E, A, B, C) can be expressed as

$$\|G(s)\|_2 = \sqrt{\text{trace}\left(CV_1 W V_1^T C^T\right)}$$

where $W \geq 0 \in \mathbb{R}^{r \times r}$ and $S \in \mathbb{R}^{n \times (n-r)}$ are the solutions of

$$\left(EV_1 W V_1^T + SV_2^T\right) A^T + A \left(EV_1 W V_1^T + SV_2^T\right)^T + BB^T = 0. \quad (8)$$

Proof: Since (E, A) is admissible, there exist two nonsingular matrices T and Q such that (see [2])

$$TEQ = \overline{E} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (9)$$
$$TAQ = \overline{A} = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}$$
$$TB = \overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$$
$$CQ = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix}$$

with $\overline{A}_1$ Hurwitz. Let $Q = [Q_1 \;\; Q_2]$. From (9), it can be seen that $EQ_2 = 0$. Hence, we have $CQ = [\overline{C}_1 \;\; 0]$ from (7). Notice that

$$G(s) = C(sE - A)^{-1} B = \overline{C}_1 \left(sI - \overline{A}_1\right)^{-1} \overline{B}_1$$

and, hence,

$$\|G(s)\|_2 = \sqrt{\text{trace}\left(\overline{C}_1 \overline{P}_1 \overline{C}_1^T\right)}$$

where $\overline{P}_1 \geq 0$ satisfies

$$\overline{A}_1 \overline{P}_1 + \overline{P}_1 \overline{A}_1^T + \overline{B}_1 \overline{B}_1^T = 0. \quad (10)$$

From (1) and (9), we have

$$U^T T^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} V = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

which implies $Q^{-1}V$ has the form of

$$Q^{-1} V = [\, Q^{-1} V_1 \quad Q^{-1} V_2 \,] = \begin{bmatrix} Q_{v1} & 0 \\ Q_{v2} & Q_{v3} \end{bmatrix}$$

with Q_{v1} and Q_{v3} nonsingular. By premultiplying and postmultiplying both sides of (8) using T and T^T, respectively, we have

$$TEQQ^{-1} V_1 W V_1^T Q^{-T} Q^T A^T T^T + TSV_2^T Q^{-T} Q^T A^T T^T + TAQQ^{-1} V_1 W V_1^T Q^{-T} Q^T E^T T^T + TAQQ^{-1} V_2 S^T T^T + TBB^T T^T = 0. \quad (11)$$

Let $TS = \left[\begin{smallmatrix} S_{t1} \\ S_{t2} \end{smallmatrix}\right]$. Equation (11) can then be written as shown in the equation at the bottom of the page. From the $(1,1)$ block of this equation, $Q_{v1} W Q_{v1}^T \geq 0$ is the solution of (10), and the H_2 norm $\|G(s)\|_2$ can be computed by

$$\|G(s)\|_2 = \sqrt{\text{trace}\left(\overline{C}_1 Q_{v1} W Q_{v1}^T \overline{C}_1^T\right)}.$$

Moreover, since $Q_{v1} = [I_r \;\; 0] Q^{-1} V_1$ and $\overline{C}_1 [I_r \;\; 0] Q^{-1} = C$, then

$$\|G(s)\|_2 = \sqrt{\text{trace}\left(\overline{C}_1 [\, I_r \;\; 0\,] Q^{-1} V_1 W V_1^T Q^{-T} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \overline{C}_1^T\right)} = \sqrt{\text{trace}\left(CV_1 W V_1^T C^T\right)}.$$

Hence, the proof follows. ■

$$\begin{bmatrix} Q_{v1} W Q_{v1}^T \overline{A}_1^T + \overline{A}_1 Q_{v1} W Q_{v1}^T + \overline{B}_1 \overline{B}_1^T & Q_{v1} W Q_{v2}^T + S_{t1} Q_{v3}^T + \overline{B}_1 \overline{B}_2^T \\ \left(Q_{v1} W Q_{v2}^T + S_{t1} Q_{v3}^T + \overline{B}_1 \overline{B}_2^T\right)^T & S_{t2} Q_{v3}^T + Q_{v3} S_{t2}^T + \overline{B}_2 \overline{B}_2^T \end{bmatrix} = 0$$

Remark 1: When $E = I$, then (E, A, B, C) reduces to a state-space model (A, B, C). In this case, $V_1 = I$, $V_2 = 0$. Hence, the H_2 norm given in Theorem 1 becomes

$$\|G(s)\|_2 = \sqrt{\text{trace}\left(CWC^T\right)}$$

and (8) reduces to a usual Lyapunov equation for linear systems described by

$$WA^T + AW + BB^T = 0.$$

It can be seen that W is the controllability gramian of the state-space model (A, B, C).

The following corollary can be easily obtained by considering the dual system of (E, A, B, C):

$$\begin{cases} E^T\tilde{x}(t) = A^T\tilde{x}(t) + C^T\tilde{u}(t) \\ \tilde{y}(t) = B^T\tilde{x}(t) \end{cases} \tag{12}$$

when

$$\text{img}B \subseteq \text{img}E \tag{13}$$

where $\text{img}(\cdot)$ denotes the range of a matrix.

Corollary 1: Assume that (E, A) is admissible and (13) holds. The H_2 norm of (E, A, B, C) can be expressed as

$$\|G(s)\|_2 = \sqrt{\text{trace}\left(B^T U_1 W U_1^T B\right)}$$

where $W \geq 0 \in \mathbb{R}^{r\times r}$ and $S \in \mathbb{R}^{(n-r)\times n}$ are the solutions of

$$A^T\left(U_1WU_1^TE + U_2S\right) + \left(U_1WU_1^TE + U_2S\right)^T A + C^TC = 0.$$

Theorem 2: Assume that (7) holds. The following statements are then equivalent.

i) Given a scalar $\gamma_2 > 0$, $\|G(s)\|_2 < \gamma_2$ and (E, A) is admissible.

ii) There exist $W > 0$ and S such that

$$\left(EV_1WV_1^T + SV_2^T\right)A^T + A\left(EV_1WV_1^T + SV_2^T\right)^T + BB^T < 0 \tag{14}$$

$$\text{trace}(CV_1WV_1^TC^T) < \gamma_2^2 \tag{15}$$

hold.

Proof: i) ⇒ ii): If $\|G(s)\|_2 < \gamma_2$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\left\|C(sE - A)^{-1}\left[B \quad \varepsilon^{1/2}I\right]\right\|_2 < \gamma_2$$

holds. From Theorem 1, we have (15), where $W \geq 0$ and S satisfy

$$\left(EV_1WV_1^T + SV_2^T\right)A^T + A\left(EV_1WV_1^T + SV_2^T\right)^T + BB^T = -\varepsilon I < 0.$$

Furthermore, $W > 0$. Otherwise, since $W \geq 0$, a perturbation on W exists such that $W > 0$ without violating (14) and (15).

ii) ⇒ i): By (14) and Lemma 2, (E, A) is admissible. Equation (14) implies that there exists $\hat{B} \neq 0$ satisfying

$$\left(EV_1WV_1^T + SV_2^T\right)A^T + A\left(EV_1WV_1^T + SV_2^T\right)^T + BB^T + \hat{B}\hat{B}^T = 0$$

which means

$$\left\|C(sE - A)^{-1}\left[B \quad \hat{B}\right]\right\|_2 < \gamma_2.$$

Hence, $\|G(s)\|_2 < \gamma_2$. ∎

Corollary 2: Assume that (13) holds. The following statements are then equivalent.

i) Given a scalar $\gamma_2 > 0$, $\|G(s)\|_2 < \gamma_2$ and (E, A) is admissible.

ii) There exist $W > 0$ and S such that

$$A^T\left(U_1WU_1^TE + U_2S\right) + \left(U_1WU_1^TE + U_2S\right)^T A + C^TC < 0$$

$$\text{trace}(B^TU_1WU_1^TB) < \gamma_2^2$$

hold.

IV. Strict LMI Conditions for H_2 Controller

Now we consider the H_2 control problem for the following system:

$$\begin{aligned} E\dot{x} &= Ax + B_w w + B_u u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \tag{16}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^q$ is the disturbance, $u \in \mathbb{R}^m$ is the control input, $z \in \mathbb{R}^p$ is the controlled output, and $y \in \mathbb{R}^l$ is the measured output. Here, we assume that $D_{ij} = 0$, $i = 1, 2$, $j = 1, 2$. Suppose that a dynamic output feedback controller of (16) described by

$$\hat{E}\dot{\xi} = \hat{A}\xi + \hat{B}y \quad u = \hat{C}\xi, \tag{17}$$

is used, where $\xi \in \mathbb{R}^n$ is the state vector, and $\hat{E} \in \mathbb{R}^{n\times n}$, $\hat{A} \in \mathbb{R}^{n\times n}$, $\hat{B} \in \mathbb{R}^{n\times l}$, and $\hat{C} \in \mathbb{R}^{m\times n}$ are unknown matrices. The resulting closed-loop system is

$$E_c\dot{x}_c = A_cx_c + B_cw \quad z = C_cx_c \tag{18}$$

where

$$\begin{aligned} x_c &= \begin{bmatrix} x \\ \xi \end{bmatrix} \\ E_c &= \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} \\ A_c &= \begin{bmatrix} A & B_u\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix} \\ B_c &= \begin{bmatrix} B_w \\ 0 \end{bmatrix} \\ C_c &= \left[C_1 \quad 0\right]. \end{aligned} \tag{19}$$

It can be seen that when $\ker C_1 \supseteq \ker E$ or $\text{img}B_w \subseteq \text{img}E$ holds, $\ker C_c \supseteq \ker E_c$ or $\text{img}B_c \subseteq \text{img}E_c$ holds. In this case, the H_2 norm of the closed-loop system (18) is finite so that we may consider the H_2 control problem for (16). In this section, we assume that $\ker C_1 \supseteq \ker E$, and the formulations for the case when $\text{img}B_w \subseteq \text{img}E$ can also be given by considering the dual system of (16).

The H_2 control problem is to find a dynamic controller (17) such that the closed-loop system (18) is admissible and

$$\|G_{zw}\|_2 = \left\|C_c(sE_c - A_c)^{-1}B_c\right\|_2 < \gamma_2 \tag{20}$$

where $\gamma_2 > 0$ is a prescribed scalar. Denote $\hat{r} = \text{rank}(\hat{E})$. Similar to E, let $\hat{E}$ be decomposed as

$$\hat{E} = \hat{U} \begin{bmatrix} \hat{\Sigma}_{\hat{r}} & 0 \\ 0 & 0 \end{bmatrix} \hat{V}^T \tag{21}$$

where $\hat{\Sigma}_{\hat{r}} = \text{diag}(\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{\hat{r}})$ with $\hat{\sigma}_i > 0$, and $\hat{U}$, $\hat{V}$ are orthogonal matrices. Partition $\hat{U}$ and $\hat{V}$ conformably with (21) as

$$\hat{U} = [\,\hat{U}_1 \quad \hat{U}_2\,] \quad \hat{V} = [\,\hat{V}_1 \quad \hat{V}_2\,].$$

It can be observed that

$$E_c = \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & \hat{U} \end{bmatrix} \Phi \begin{bmatrix} \Sigma_c & 0 \\ 0 & 0 \end{bmatrix} \Phi^T \begin{bmatrix} V^T & 0 \\ 0 & \hat{V}^T \end{bmatrix}$$

where

$$\Phi = \begin{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ I_{n-r} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & I_{\hat{r}} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & I_{n-\hat{r}} \end{bmatrix} \end{bmatrix} \quad \Sigma_c = \begin{bmatrix} \Sigma_r & 0 \\ 0 & \hat{\Sigma}_{\hat{r}} \end{bmatrix}.$$

Let

$$U_c = [\,U_{c1} \quad U_{c2}\,] = \begin{bmatrix} U & 0 \\ 0 & \hat{U} \end{bmatrix} \Phi$$
$$V_c = [\,V_{c1} \quad V_{c2}\,] = \begin{bmatrix} V & 0 \\ 0 & \hat{V} \end{bmatrix} \Phi.$$

It can then be shown that U_c and V_c are orthogonal matrices, and

$$U_{c1} = \begin{bmatrix} U_1 & 0 \\ 0 & \hat{U}_1 \end{bmatrix} \quad U_{c2} = \begin{bmatrix} U_2 & 0 \\ 0 & \hat{U}_2 \end{bmatrix} \quad V_{c1} = \begin{bmatrix} V_1 & 0 \\ 0 & \hat{V}_1 \end{bmatrix} \quad V_{c2} = \begin{bmatrix} V_2 & 0 \\ 0 & \hat{V}_2 \end{bmatrix}. \tag{22}$$

By Theorem 1, the H_2 norm of the closed-loop system is given by

$$\left\| C_c(sE_c - A_c)^{-1} B_c \right\|_2^2 = \text{trace}\left(C_c V_{c1} W_c V_{c1}^T C_c^T\right) = \text{trace}\left(C_1 V_1 W_c^{11} V_1^T C_1^T\right) \tag{23}$$

where

$$W_c = \begin{bmatrix} W_c^{11} & W_c^{12} \\ (W_c^{12})^T & W_c^{22} \end{bmatrix} \quad S_c = \begin{bmatrix} S_c^{11} & S_c^{12} \\ S_c^{21} & S_c^{22} \end{bmatrix} \tag{24}$$

are the solutions of

$$A_c\left(E_c V_{c1} W_c V_{c1}^T + S_c V_{c2}^T\right)^T + \left(E_c V_{c1} W_c V_{c1}^T + S_c V_{c2}^T\right) A_c^T + B_c^T B_c = 0.$$

Hence, by Theorem 2, our problem is to find $\hat{E}, \hat{A}, \hat{B}, \hat{C}, W_c > 0$, and S_c such that

$$A_c\left(E_c V_{c1} W_c V_{c1}^T + S_c V_{c2}^T\right)^T + \left(E_c V_{c1} W_c V_{c1}^T + S_c V_{c2}^T\right) A_c^T + B_c^T B_c < 0 \tag{25}$$
$$\text{trace}(C_1 V_1 W_c^{11} V_1^T C_1^T) < \gamma_2^2 \tag{26}$$

hold.

Theorem 3: For a given scalar $\gamma_2 > 0$, there exists a controller given by (17), $W_c > 0$, and S_c such that (25) and (26) are feasible if and only if there exist $W_1 > 0$, $W_2 > 0$, S_1, S_2, F, and G such that the following matrix inequalities:

$$(A + GC_2)\left(EV_1W_1V_1^T + S_1V_2^T\right)^T + \left(EV_1W_1V_1^T + S_1V_2^T\right)(A + GC_2)^T + B_wB_w^T < 0 \tag{27}$$
$$(A + B_uF)\left(EV_1W_2V_1^T + S_2V_2^T\right)^T + \left(EV_1W_2V_1^T + S_2V_2^T\right)(A + B_uF)^T + B_wB_w^T < 0 \tag{28}$$
$$W_2 - W_1 > 0 \tag{29}$$
$$\text{trace}\left(C_1V_1W_2V_1^TC_1^T\right) < \gamma_2^2 \tag{30}$$

hold. Moreover, such a controller is given by

$$\hat{A} = \left(X^{-1} - P^T\right)^{-1}\left(PA^T + (A + B_uF + GC_2)X^{-1} + B_wB_w^T\right) \cdot \left(X^{-1} - P^T\right)^{-1} \tag{31}$$
$$\hat{E} = E$$
$$\hat{B} = G$$
$$\hat{C} = -FX^{-1}\left(X^{-1} - P^T\right)^{-1} \tag{32}$$

with

$$X^{-1} = \left(EV_1W_2V_1^T + S_2V_2^T\right)^T$$
$$P = EV_1W_1V_1^T + S_1V_2^T. \tag{33}$$

Proof: Sufficiency: It can be seen that $EV_1W_1V_1^T + S_1V_2^T$ and $EV_1W_2V_1^T + S_2V_2^T$ are nonsingular by (27) and (28). Let X and P be defined as in (33). Equations (27) and (28) imply

$$(A + GC_2)P^T + P(A + GC_2)^T + B_wB_w^T < 0 \tag{34}$$
$$(A + B_uF)X^{-1} + X^{-T}(A + B_uF)^T + B_wB_w^T < 0 \tag{35}$$

from which we can see that X and P are nonsingular. Furthermore, we can assume that $X^{-1} - P^T$ is nonsingular from (29). Otherwise, small perturbations on W_1, W_2, S_1, S_2 can ensure the nonsingularity of $X^{-1} - P^T$. With $\hat{E}$, $\hat{A}$, $\hat{B}$, $\hat{C}$ given by (31) and (32), we have the first equation shown at the bottom of the next page. Let

$$X_c = \begin{bmatrix} X^{-1} & -X^{-1} + P^T \\ -X^{-1} + P^T & X^{-1} - P^T \end{bmatrix}.$$

We have the second equation shown at the bottom of the page. By (34), (35), and the Schur complement, it can be shown that

$$A_cX_c + X_c^TA_c^T + B_cB_c^T < 0. \tag{36}$$

Notice that with $\hat{E} = E$, V_{c1} and V_{c2} in (22) become

$$V_{c1} = \begin{bmatrix} V_1 & 0 \\ 0 & V_1 \end{bmatrix} \quad U_{c2} = \begin{bmatrix} V_2 & 0 \\ 0 & V_2 \end{bmatrix} \tag{37}$$

and (38), shown at the bottom of the page, where

$$W_c = \begin{bmatrix} W_2 & -W_2 + W_1 \\ -W_2 + W_1 & W_2 - W_1 \end{bmatrix} > 0 \tag{39}$$
$$S_c = \begin{bmatrix} S_2 & -S_2 + S_1 \\ -S_2 + S_1 & S_2 - S_1 \end{bmatrix}.$$

Noticing (38) into (36), (25) holds, and by substituting (30) and (39), (26) holds.

Necessity: By assumption, we know that there exists a controller given by (17) such that (E_c, A_c) is admissible and (20) holds, i.e., there exist $\hat{E}$, $\hat{A}$, $\hat{B}$, $\hat{C}$, $W_c > 0$, and S_c such that (25) and (26) hold. With the partition in (24), if we let $W_2 = W_c^{11}$ and $S_2 = S_c^{11}$, (30) can be obtained from (26). Denote the fourth equation shown at the bottom of the page, and, thus, (25) gives

$$A_cX_c + X_c^TA_c^T + B_c^TB_c = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} < 0$$

where

$$\begin{aligned} Y_{11} &= AX_c^{11} + (X_c^{11})^TA^T + B_u\hat{C}X_c^{21} + (X_c^{21})^T\hat{C}^TB_u^T \\ &\quad + B_wB_w^T \\ Y_{12} &= AX_c^{12} + (X_c^{21})^T\hat{A}^T + B_u\hat{C}X_c^{22} + (X_c^{11})^TC_2^T\hat{B}^T \\ Y_{22} &= \hat{A}X_c^{22} + (X_c^{22})^T\hat{A}^T + \hat{B}C_2X_c^{12} + (X_c^{12})^TC_2^T\hat{B}^T. \end{aligned}$$

By letting $F = \hat{C}X_c^{21}(X_c^{11})^{-1}$, (28) is followed from $Y_{11} < 0$. Furthermore, define

$$\begin{aligned} \tilde{P} &= \begin{bmatrix} I & 0 \\ -(X_c^{22})^{-T}(X_c^{12})^T & I \end{bmatrix} \\ \tilde{Q} &= \begin{bmatrix} I & 0 \\ -(X_c^{22})^{-1}X_c^{21} & I \end{bmatrix} \\ \tilde{E}_c &= \tilde{Q}^TE_c\tilde{P}^{-T} \\ \tilde{A}_c &= \tilde{Q}^TA_c\tilde{P}^{-T} \\ \tilde{B}_c &= \tilde{Q}^TB_c \\ \tilde{C}_c &= C_c\tilde{P}^{-T}. \end{aligned}$$

Then

$$\begin{aligned} \tilde{X}_c &= \tilde{P}^TX_c\tilde{Q} = \begin{bmatrix} X_c^{11} - X_c^{12}(X_c^{22})^{-1}X_c^{21} & 0 \\ 0 & X_c^{22} \end{bmatrix} \\ &\tilde{A}_c\tilde{X}_c + \tilde{X}_c^T\tilde{A}_c^T + \tilde{B}_c\tilde{B}_c^T \\ &\quad = \tilde{Q}^T\left(A_cX_c + X_cA_c^T + B_cB_c^T\right)\tilde{Q} < 0. \end{aligned} \tag{40}$$

$$E_c = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \quad A_c = \begin{bmatrix} A & -B_uFX^{-1}\left(X^{-1} - P^T\right)^{-1} \\ GC_2 & \left(PA^T + (A + B_uF + GC_2)X^{-1} + B_wB_w^T\right)\left(X^{-1} - P^T\right)^{-1} \end{bmatrix}$$

$$\begin{aligned} &A_cX_c + X_c^TA_c^T + B_cB_c^T \\ &= \begin{bmatrix} (A + B_uF)X^{-1} + X^{-T}(A + B_uF)^T + B_wB_w^T & -(A + B_uF)X^{-1} - X^{-T}(A + B_uF)^T - B_wB_w^T \\ -(A + B_uF)X^{-1} - X^{-T}(A + B_uF)^T - B_wB_w^T & \begin{matrix}(A + GC_2)P^T + P(A + GC_2)^T \\ +(A + B_uF)X^{-1} + X^{-T}(A + B_uF)^T + 2B_wB_w^T\end{matrix} \end{bmatrix} \end{aligned}$$

$$X_c = \begin{bmatrix} V_1W_2V_1^TE^T + V_2S_2^T & V_1(W_1 - W_2)V_1^TE^T + V_2\left(S_1^T - S_2^T\right) \\ V_1(W_1 - W_2)V_1^TE^T + V_2\left(S_1^T - S_2^T\right) & V_1(W_2 - W_1)V_1^TE^T + V_2\left(S_2^T - S_1^T\right) \end{bmatrix} = V_{c1}W_cV_{c1}^TE_c^T + V_{c2}S_c^T \tag{38}$$

$$X_c = V_{c1}W_cV_{c1}^TE_c^T + V_{c2}S_c^T = \begin{bmatrix} V_1W_2V_1^TE^T + V_2S_2^T & V_1W_c^{12}\hat{V}_1^T\hat{E}^T + V_2(S_c^{21})^T \\ \hat{V}_1(W_c^{12})^TV_1^TE^T + \hat{V}_2\left(S_c^{12}\right)^T & \hat{V}_1W_c^{22}\hat{V}_1^T\hat{E}^T + \hat{V}_2\left(S_c^{22}\right)^T \end{bmatrix} \equiv \begin{bmatrix} X_c^{11} & X_c^{12} \\ X_c^{21} & X_c^{22} \end{bmatrix}$$

From the $(1,1)$ block of (40), we have

$$\left(A-(X_c^{21})^T(X_c^{22})^{-T}\hat{B}C_2\right)\left(X_c^{11}-X_c^{12}(X_c^{22})^{-1}X_c^{21}\right) + \left(X_c^{11}-X_c^{12}(X_c^{22})^{-1}X_c^{21}\right)^T \cdot\left(A-(X_c^{21})^T(X_c^{22})^{-T}\hat{B}C_2\right)^T + B_wB_w^T<0. \quad (41)$$

Letting $G=-(X_c^{21})^T(X_c^{22})^{-T}\hat{B}$ and noticing iii) of Lemma 1 and $\hat{U}_1^T\hat{E}\hat{V}_1=\hat{\Sigma}_{\hat{r}}$, we have

$$\begin{aligned}
X_c^{11}&-X_c^{12}(X_c^{22})^{-1}X_c^{21}\\
&=V_1W_2V_1^TE^T+V_2S_2^T\\
&\quad-\left(V_1W_c^{12}\hat{V}_1^T\hat{E}^T+V_2(S_c^{21})^T\right)\\
&\quad\cdot\left(\hat{U}_1\hat{\Sigma}_{\hat{r}}^{-1}(W_c^{22})^{-1}\hat{\Sigma}_{\hat{r}}^{-1}\hat{U}_1^T\hat{E}+\hat{U}_2\tilde{S}\right)\\
&\quad\cdot\left(\hat{V}_1(W_c^{12})^TV_1^TE^T+\hat{V}_2\left(S_c^{12}\right)^T\right)\\
&=V_1\left(W_c^{11}-W_c^{12}(W_c^{22})^{-1}(W_c^{12})^T\right)V_1^TE^T\\
&\quad+V_2\Big(S_2^T-(S_c^{21})^T\hat{U}_2\tilde{S}\hat{V}_2(S_c^{12})^T\\
&\quad-(S_c^{21})^T\hat{U}_1\hat{\Sigma}_{\hat{r}}^{-1}(W_c^{22})^{-1}(W_c^{12})^TV_1^TE^T\\
&\quad-(S_c^{21})^T\hat{U}_2\tilde{S}\hat{V}_1(W_c^{12})^TV_1^TE^T\Big).
\end{aligned}$$

Let

$$\begin{aligned}
W_1&=W_c^{11}-W_c^{12}(W_c^{22})^{-1}(W_c^{12})^T\\
S_1&=\Big(S_2^T-(S_c^{21})^T\hat{U}_2\tilde{S}\hat{V}_2(S_c^{12})^T\\
&\quad-(S_c^{21})^T\hat{U}_1\hat{\Sigma}_{\hat{r}}^{-1}(W_c^{22})^{-1}(W_c^{12})^TV_1^TE^T\\
&\quad-(S_c^{21})^T\hat{U}_2\tilde{S}\hat{V}_1(W_c^{12})^TV_1^TE^T\Big)^T.
\end{aligned}$$

Notice $W_1>0$ as $W_c>0$. Equation (27) then follows from (41). Without loss of generality, W_c^{12} is assumed to be invertible. Otherwise, a small perturbation of W_c^{12} exists such that W_c^{12} is invertible without violating (25). Equation (29) is then obvious from

$$\begin{aligned}
W_2-W_1&=W_c^{11}-\left(W_c^{11}-W_c^{12}(W_c^{22})^{-1}(W_c^{12})^T\right)\\
&=W_c^{12}(W_c^{22})^{-1}(W_c^{12})^T>0.
\end{aligned}$$

■

Notice that the matrix inequalities (27) and (28) are not linear. The following theorem gives equivalent LMI conditions to (27)–(30).

Theorem 4: For a given scalar $\gamma_2>0$, there exist a controller given by (17), $W_c>0$, and S_c such that (25) and (26) are feasible if and only if there exist $\tilde{W}>0$, $W_2>0$, $\tilde{S}$, S_2, M, and N such that the following LMIs hold, as shown in (42)–(45) at the bottom of the page. Moreover, such a controller is given by

$$\hat{A}=\left(PA^T+(A+B_uF+GC_2)X^{-1}+B_wB_w^T\right)\cdot\left(X^{-1}-P^T\right)^{-1} \quad (46)$$

$$\begin{aligned}
\hat{E}&=E\\
\hat{B}&=G\\
\hat{C}&=-FX^{-1}\left(X^{-1}-P^T\right)^{-1}
\end{aligned} \quad (47)$$

with

$$\begin{aligned}
G&=PM\\
F&=NX\\
X^{-1}&=\left(EV_1W_2V_1^T+S_2V_2^T\right)^T\\
P^{-T}&=U_1\tilde{W}U_1^TE+U_2\tilde{S}.
\end{aligned}$$

Proof: Notice that $EV_1W_1V_1^T+S_1V_2^T$ in (27) is nonsingular. Premultiplying and postmultiplying both sides of (27) by $(EV_1W_1V_1^T+S_1V_2^T)^{-1}$ and its transpose, respectively, then from iii) of Lemma 1, we have

$$\left(U_1\tilde{W}U_1^TE+U_2\tilde{S}\right)^T(A+GC_2) + (A+GC_2)^T\left(U_1\tilde{W}U_1^TE+U_2\tilde{S}\right) + \left(U_1\tilde{W}U_1^TE+U_2\tilde{S}\right)^TB_wB_w^T\left(U_1\tilde{W}U_1^TE+U_2\tilde{S}\right)<0$$

where

$$\left(EV_1W_1V_1^T+S_1V_2^T\right)^{-T}=U_1\tilde{W}U_1^TE+U_2\tilde{S}$$

with $\tilde{W}=\Sigma_r^{-1}W_1^{-1}\Sigma_r^{-1}$. By letting $(U_1\tilde{W}U_1^TE+U_2\tilde{S})^TG=M$, we may then see that (42) is equivalent to (27). Moreover, the equivalence between (43) and (28) can be seen by letting $N=F(EV_1W_2V_1^T+S_2V_2^T)^T$. Finally, (44) is equivalent to (29) by noticing $W_1=\Sigma_r^{-1}\tilde{W}^{-1}\Sigma_r^{-1}$ and the Schur complement. ■

V. LMI Conditions for Mixed H_2/H_∞ Controller

In the above section, we obtain the LMI conditions for H_2 controllers. In this section, we will consider the mixed H_2/H_∞ control problem for singular systems. Consider the following singular system:

$$\begin{aligned}
E\dot{x}&=Ax+B_ww+B_uu\\
z&=C_1x\\
z_\infty&=C_\infty x\\
y&=C_2x
\end{aligned} \quad (48)$$

$$\begin{bmatrix}(U_1\tilde{W}U_1^TE+U_2\tilde{S})^TA+A^T(U_1\tilde{W}U_1^TE+U_2\tilde{S})+MC_2+C_2^TM^T & (U_1\tilde{W}U_1^TE+U_2\tilde{S})^TB_w\\ B_w^T(U_1\tilde{W}U_1^TE+U_2\tilde{S}) & -I\end{bmatrix}<0 \quad (42)$$

$$A\left(EV_1W_2V_1^T+S_2V_2^T\right)^T+\left(EV_1W_2V_1^T+S_2V_2^T\right)A^T+B_uN+N^TB_u^T+B_wB_w^T<0 \quad (43)$$

$$\begin{bmatrix}-W_2 & I\\ I & -\Sigma_r\tilde{W}\Sigma_r\end{bmatrix}<0 \quad (44)$$

$$\mathrm{trace}(C_1V_1W_2V_1^TC_1^T)<\gamma_2^2 \quad (45)$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^q$ is the disturbance, $u \in \mathbb{R}^m$ is the control input, $z \in \mathbb{R}^{p_1}$ and $z^\infty \in \mathbb{R}^{p_2}$ are the controlled outputs, and $y \in \mathbb{R}^l$ is the measured output. The design objective is to find a controller (17) such that when w is a Gaussian white noise, the H_2 norm of the transfer function from w to z is less than a given γ_2, and when w is a deterministic signal of unity power, the H_∞ norm of the transfer function from w to z_∞ is less than a given γ_∞. A similar control problem for normal linear systems was considered in [1].

Notice that the resulting closed-loop system from (48) and (17) is

$$E_c\dot{x}_c = A_c x_c + B_c w \quad z = C_c x_c \quad z_\infty = C_{c\infty} x_c$$

where

$$\begin{aligned} x_c &= \begin{bmatrix} x \\ \xi \end{bmatrix} \\ E_c &= \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} \\ A_c &= \begin{bmatrix} A & B_u\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix} \\ B_c &= \begin{bmatrix} B_w \\ 0 \end{bmatrix} \\ C_c &= [C_1 \quad 0] \\ C_{c\infty} &= [C_\infty \quad 0]. \end{aligned}$$

The transfer function from w to z is

$$G_{zw}(s) = C_c(sE_c - A_c)^{-1}B_c$$

and the transfer function from w to z_∞ is

$$G_{z_\infty w}(s) = C_{c\infty}(sE_c - A_c)^{-1}B_c.$$

In order that the H_2 norm of $G_c(s)$ is finite, we assume that $\ker C_1 \supseteq \ker E$. Thus, our control problem can be stated as follows: for a singular system given by (48), find a controller with the form of (17) such that (E_c, A_c) is admissible and

$$\|G_{zw}(s)\|_2 < \gamma_2 \tag{49}$$

$$\|G_{z_\infty w}(s)\|_\infty < \gamma_\infty \tag{50}$$

where $\gamma_2 > 0$ and $\gamma_\infty > 0$ are prescribed scalars.

The following lemma gives a necessary and sufficient condition of the existence of a controller by (17) such that (E_c, A_c) is admissible and (50) holds.

Lemma 3: For a given scalar $\gamma_\infty > 0$, there exists a controller given by (17) such that (E_c, A_c) is admissible and (50) holds if and only if there exist $\hat{W}_1 > 0$, $\hat{W}_2 > 0$, $\hat{S}_1$, $\hat{S}_2$, $\hat{M}$, and $\hat{N}$ such that the following LMIs hold, as shown in (51)–(53) at the bottom of the page. Moreover, such a controller is given by

$$\begin{aligned} \hat{A} &= \left(\hat{P}A^T + \gamma_\infty^2(A + B_u\hat{F} + \hat{G}C_2)\hat{X}^{-1} \right. \\ &\qquad \left. + B_wB_w^T + \hat{P}C_\infty^T C_\infty \hat{X}^{-1}\right)\left(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T\right)^{-1} \\ \hat{E} &= E \\ \hat{B} &= G \\ \hat{C} &= -\gamma_\infty^2 \hat{F}\hat{X}^{-1}(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T)^{-1} \end{aligned}$$

with

$$\begin{aligned} \hat{G} &= \hat{P}\hat{M} \\ \hat{F} &= \hat{N}\hat{X} \\ \hat{X}^{-1} &= \left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)^T \\ \hat{P}^{-T} &= U_1\hat{W}U_1^TE + U_2\hat{S}. \end{aligned}$$

The proof of this lemma needs the following lemma.

Lemma 4 [11]: For a given scalar $\gamma_\infty > 0$, there exists a controller given by (17) such that (E_c, A_c) is admissible and (50) holds if and only if there exist $\hat{X}$, $\hat{P}$, $\hat{F}$, and $\hat{G}$ satisfying

$$E\hat{P}^T = \hat{P}E^T \geq 0 \tag{54}$$

$$\begin{aligned} &\left(A + \hat{G}C_2\right)\hat{P}^T + \hat{P}\left(A + \hat{G}C_2\right)^T + B_wB_w^T \\ &\quad + \frac{1}{\gamma_\infty^2}\hat{P}C_\infty^TC_\infty\hat{P}^T < 0 \end{aligned} \tag{55}$$

$$E^T\hat{X} = \hat{X}^TE \geq 0 \tag{56}$$

$$\begin{aligned} &\hat{X}^T\left(A + B_u\hat{F}\right) + \left(A + B_u\hat{F}\right)^T\hat{X} + C_\infty^TC_\infty \\ &\quad + \frac{1}{\gamma_\infty^2}\hat{X}^TB_wB_w^T\hat{X} < 0 \end{aligned} \tag{57}$$

$$E\left(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T\right) \geq 0. \tag{58}$$

$$\begin{bmatrix} \left(U_1\hat{W}U_1^TE + U_2\hat{S}\right)^T A + A^T\left(U_1\hat{W}U_1^TE + U_2\hat{S}\right) + \frac{1}{\gamma_\infty^2}C_\infty^TC_\infty + \hat{M}C_2 + C_2^T\hat{M}^T & \left(U_1\hat{W}U_1^TE + U_2\hat{S}\right)^T B_w \\ B_w^T\left(U_1\hat{W}U_1^TE + U_2\hat{S}\right) & -I \end{bmatrix} < 0 \tag{51}$$

$$\begin{bmatrix} A\left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)^T + \left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)A^T + B_u\hat{N} + \hat{N}^TB_u^T + \frac{1}{\gamma_\infty^2}B_wB_w^T & \left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)C_\infty^T \\ C_\infty\left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)^T & -I \end{bmatrix} < 0 \tag{52}$$

$$\begin{bmatrix} -\gamma_\infty^2\hat{W}_2 & I \\ I & -\Sigma_r\hat{W}\Sigma_r \end{bmatrix} < 0 \tag{53}$$

Such a controller is then given by

$$\hat{A} = \left(\hat{P}A^T + \gamma_\infty^2\left(A + B_u\hat{F} + \hat{G}C_2\right)\hat{X}^{-1} + B_wB_w^T + \hat{P}C_\infty^TC_\infty\hat{X}^{-1}\right)\left(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T\right)^{-1}$$
$$\hat{E} = E$$
$$\hat{B} = G$$
$$\hat{C} = -\gamma_\infty^2\hat{F}\hat{X}^{-1}\left(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T\right)^{-1}.$$

Proof of Lemma 3: Multiplying both sides of (55) by $\hat{P}^{-1}$ and $\hat{P}^{-T}$, respectively, we obtain

$$\hat{P}^{-1}\left(A + \hat{G}C_2\right) + \left(A + \hat{G}C_2\right)^T\hat{P}^{-T} + \hat{P}^{-1}B_wB_w^T\hat{P}^{-T} + \frac{1}{\gamma_\infty^2}C_\infty^TC_\infty < 0.$$

By ii) and iii) of Lemma 1, $\hat{P}$ and $\hat{P}^{-T}$ can be expressed as $\hat{P} = EV_1\hat{W}_1V_1^T + \hat{S}_1V_2^T$ and $\hat{P}^{-T} = U_1\hat{W}U_1^TE + U_2\hat{S}$ with $\hat{W} = \Sigma_r^{-1}\hat{W}_1^{-1}\Sigma_r^{-1}$. Thus, by letting $\hat{P}^{-1}\hat{G} = \hat{M}$, we can see that (54) and (55) are equivalent to (51). On the other hand, multiplying both sides of (56) and (57) by $\hat{X}^{-T}$ and $\hat{X}^{-1}$, respectively, we obtain

$$\hat{X}^{-T}E^T = E\hat{X}^{-1}$$
$$\left(A + B_u\hat{F}\right)\hat{X}^{-1} + \hat{X}^{-T}\left(A + B_u\hat{F}\right)^T + \hat{X}^{-T}C_\infty^TC_\infty\hat{X}^{-1} + \frac{1}{\gamma_\infty^2}B_wB_w^T < 0.$$

Again, by ii) of Lemma 1, $\hat{X}^{-T}$ can be expressed as $\hat{X}^{-T} = EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T$. It can be seen that (56) and (57) are equivalent to (52) by letting $\hat{F}\hat{X}^{-1} = \hat{N}$. Notice that $EV_2 = 0$ and

$$\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T = V_1\left(\gamma_\infty^2\hat{W}_2 - \hat{W}_1\right)V_1^TE^T + V_2\left(\gamma_\infty^2\hat{S}_2^T - \hat{S}_1^T\right).$$

Therefore, (53) implies (58) since $\hat{W} = \Sigma_r^{-1}\hat{W}_1^{-1}\Sigma_r^{-1}$. Conversely, from (58)

$$EV_1\left(\gamma_\infty^2\hat{W}_2 - \hat{W}_1\right)V_1^TE^T \geq 0.$$

From (3) and noticing $V^TV_1 = \begin{bmatrix} V_1^TV_1 \\ V_2^TV_1 \end{bmatrix} = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$, we then have

$$\begin{aligned} &U^TEVV^TV_1\left(\gamma_\infty^2\hat{W}_2 - \hat{W}_1\right)V_1^TVV^TE^TU \\ &= \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_r \\ 0 \end{bmatrix}\left(\gamma_\infty^2\hat{W}_2 - \hat{W}_1\right)\begin{bmatrix} I_r & 0 \end{bmatrix}\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_r\left(\gamma_\infty^2\hat{W}_2 - \hat{W}_1\right)\Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \\ &\geq 0 \end{aligned}$$

which implies $\gamma_\infty^2\hat{W}_2 - \hat{W}_1 > 0$ or $\gamma_\infty^2\hat{W}_2 - \Sigma_r^{-1}\hat{W}^{-1}\Sigma_r^{-1} > 0$ without loss of generality. Otherwise, a small perturbation of $\hat{W}_2$ exists such that $\gamma_\infty^2\hat{W}_2 - \hat{W}_1 > 0$ without violating LMI (52). ■

Based on Lemma 3 and Theorem 3, the mixed H_2 and H_∞ control problem can be solved.

Theorem 5: For given scalars $\gamma_2 > 0$ and $\gamma_\infty > 0$, there exists a controller given by (17) such that (E_c, A_c) is admissible and (49) holds under the constraint of (50) if there exist $\hat{W}_1 > 0$, $\hat{W}_2 > 0$, $\hat{S}_1$, $\hat{S}_2$, $\hat{F}$, and $\hat{G}$ such that the following LMIs hold, as shown in (59)–(62) at the bottom of the page. Moreover, such a controller is given by

$$\hat{A} = \left(\hat{P}A^T + \gamma_\infty^2\left(A + B_u\hat{F} + \hat{G}C_2\right)\hat{X}^{-1} + B_wB_w^T + \hat{P}C_\infty^TC_\infty\hat{X}^{-1}\right)\left(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T\right)^{-1} \tag{63}$$

$$\hat{E} = E$$
$$\hat{B} = \hat{G}$$
$$\hat{C} = -\gamma_\infty^2\hat{F}\hat{X}^{-1}\left(\gamma_\infty^2\hat{X}^{-1} - \hat{P}^T\right)^{-1} \tag{64}$$

with

$$\begin{aligned} \hat{G} &= \hat{P}\hat{M} \\ \hat{F} &= \hat{N}\hat{X}, \\ \hat{X}^{-1} &= \left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)^T \\ \hat{P}^{-T} &= U_1\hat{W}U_1^TE + U_2\hat{S}. \end{aligned} \tag{65}$$

$$\begin{bmatrix} \left(U_1\hat{W}U_1^TE + U_2\hat{S}\right)^TA + A^T\left(U_1\hat{W}U_1^TE + U_2\hat{S}\right) + \frac{1}{\gamma_\infty^2}C_\infty^TC_\infty + \hat{M}C_2 + C_2^T\hat{M}^T & \left(U_1\hat{W}U_1^TE + U_2\hat{S}\right)^TB_w \\ B_w^T\left(U_1\hat{W}U_1^TE + U_2\hat{S}\right) & -I \end{bmatrix} < 0 \tag{59}$$

$$\begin{bmatrix} A\left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)^T + \left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)A^T + B_u\hat{N} + \hat{N}^TB_u^T + \frac{1}{\gamma_\infty^2}B_wB_w^T & \left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)C_\infty^T \\ C_\infty\left(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T\right)^T & -I \end{bmatrix} < 0 \tag{60}$$

$$\begin{bmatrix} -\gamma_\infty^2\hat{W}_2 & I \\ I & -\Sigma_r\hat{W}\Sigma_r \end{bmatrix} < 0 \tag{61}$$

$$\operatorname{trace}\left(C_1V_1\hat{W}_2V_1^TC_1^T\right) < \frac{\gamma_2^2}{\gamma_\infty^2} \tag{62}$$

Proof: It can be seen from Lemma 3 that if (59)–(61) hold, then the controller of (17) with $\hat{E}$, $\hat{A}$, $\hat{B}$, and $\hat{C}$ by (63) and (64) is such that (E_c, A_c) is admissible and (50) holds. We will show that the controller (17) with $\hat{E}$, $\hat{A}$, $\hat{B}$, and $\hat{C}$ by (63) and (64) is also such that (49) holds. With $\hat{E}$, $\hat{A}$, $\hat{B}$, and $\hat{C}$ given by (63) and (64), we then have the first equation shown at the bottom of the page. It can be seen that $U_1\hat{W}U_1^TE + U_2\hat{S}$ and $(EV_1\hat{W}_2V_1^T + \hat{S}_2V_2^T)$ are nonsingular from (59) and (60). Let $\hat{X}$ and $\hat{P}$ be defined as in (65). Equations (59) and (60) imply

$$\left(A+\hat{G}C_2\right)\hat{P}^T+\hat{P}\left(A+\hat{G}C_2\right)^T+B_wB_w^T + \frac{1}{\gamma_\infty^2}\hat{P}C_\infty^TC_\infty\hat{P}^T<0 \tag{66}$$

$$\left(A+B_u\hat{F}\right)\hat{X}^{-1}+\hat{X}^{-T}\left(A+B_u\hat{F}\right)^T + \frac{1}{\gamma_\infty^2}B_wB_w^T+\hat{X}^{-T}C_\infty^TC_\infty\hat{X}^{-1}<0. \tag{67}$$

Furthermore, we can assume that $\gamma_\infty^2\hat{X}^{-1}-\hat{P}^T$ is nonsingular from (61). Otherwise, small perturbations on $\hat{W}_1$, $\hat{W}_2$, $\hat{S}_1$, and $\hat{S}_2$ can ensure the nonsingularity of $\gamma_\infty^2\hat{X}^{-1}-\hat{P}^T$. Let

$$\hat{X}_c=\begin{bmatrix}\gamma_\infty^2\hat{X}^{-1} & -\gamma_\infty^2\hat{X}^{-1}+\hat{P}^T\\ -\gamma_\infty^2\hat{X}^{-1}+\hat{P}^T & \gamma_\infty^2\hat{X}^{-1}-\hat{P}^T\end{bmatrix}.$$

We then have

$$A_c\hat{X}_c+\hat{X}_c^TA_c^T+B_cB_c=\begin{bmatrix}Z_1 & Z_2\\ Z_2^T & Z_3\end{bmatrix}$$

where

$$\begin{aligned}Z_1 &= \gamma_\infty^2\left(A+B_u\hat{F}\right)\hat{X}^{-1}+\gamma_\infty^2\hat{X}^{-T}(A+B_u\hat{F})^T\\ &\quad + B_wB_w^T\\ Z_2 &= -Z_1-\hat{X}^{-T}C_\infty^TC_\infty\hat{P}^T\\ Z_3 &= \left(A+\hat{G}C_2\right)\hat{P}^T+\hat{P}\left(A+\hat{G}C_2\right)^T+B_wB_w^T\\ &\quad + Z_1+\hat{X}^{-T}C_\infty^TC_\infty\hat{P}^T+\hat{P}C_\infty^TC_\infty\hat{X}^{-1}\end{aligned}$$

and as shown in the second equation at the bottom of the page by (66) and (67). Hence,

$$A_c\hat{X}_c+\hat{X}_c^TA_c^T+B_cB_c^T<0.$$

Since $\hat{P}^{-T}=U_1\hat{W}U_1^TE+U_2\hat{S}$, by iii) of Lemma 1, there exist $\hat{W}_1$ and $\hat{S}_1$ such that $\hat{P}=EV_1\hat{W}_1V_1^T+\hat{S}_1V_2^T$. Notice the equation shown at the bottom of the next page, where U_{c1} and U_{c2} are given in (37) since $\hat{E}=E$ and

$$\hat{W}_c=\begin{bmatrix}\gamma_\infty^2\hat{W}_2 & -\gamma_\infty^2\hat{W}_2+\hat{W}_1\\ -\gamma_\infty^2\hat{W}_2+\hat{W}_1 & \gamma_\infty^2\hat{W}_2-\hat{W}_1\end{bmatrix}>0$$

$$\hat{S}_c=\begin{bmatrix}\gamma_\infty^2\hat{S}_2 & -\gamma_\infty^2\hat{S}_2+\hat{S}_1\\ -\gamma_\infty^2\hat{S}_2+\hat{S}_1 & \gamma_\infty^2\hat{S}_2-\hat{S}_1\end{bmatrix}$$

which satisfies

$$A_c\left(V_{c1}\hat{W}_cV_{c1}^TE_c^T+V_{c2}\hat{S}_c^T\right) + \left(V_{c1}\hat{W}_cV_{c1}^TE_c^T+V_{c2}\hat{S}_c^T\right)^TA_c^T+B_c^TB_c<0.$$

From (62) and Theorem 2, the proof is obtained. ■

$$A_c=\begin{bmatrix}A & -\gamma_\infty^2B_u\hat{F}\hat{X}^{-1}\left(\gamma_\infty^2\hat{X}^{-1}-\hat{P}^T\right)^{-1}\\ \hat{G}C_2 & \left(\hat{P}A^T+\gamma_\infty^2(A+B_u\hat{F}+\hat{G}C_2)\hat{X}^{-1}+B_wB_w^T+\hat{P}C_\infty^TC_\infty\hat{X}^{-1}\right)\left(\gamma_\infty^2\hat{X}^{-1}-\hat{P}^T\right)^{-1}\end{bmatrix}$$

$$\begin{aligned}&\begin{bmatrix}I & 0\\ I & I\end{bmatrix}\begin{bmatrix}Z_1 & Z_2\\ Z_2^T & Z_3\end{bmatrix}\begin{bmatrix}I & I\\ 0 & I\end{bmatrix}\\ &=\begin{bmatrix}Z_1 & Z_1+Z_2\\ Z_1+Z_2^T & Z_1+Z_3+Z_2+Z_2^T\end{bmatrix}\\ &=\begin{bmatrix}\gamma_\infty^2\left(A+B_u\hat{F}\right)\hat{X}^{-1}+\gamma_\infty^2\hat{X}^{-T}\left(A+B_u\hat{F}\right)^T+B_wB_w^T & -\hat{X}^{-T}C_\infty^TC_\infty\hat{P}^T\\ -\hat{P}C_\infty^TC_\infty\hat{X}^{-1} & \left(A+\hat{G}C_2\right)\hat{P}^T+\hat{P}\left(A+\hat{G}C_2\right)^T+B_wB_w^T\end{bmatrix}\\ &\le\begin{bmatrix}\begin{matrix}\gamma_\infty^2\left(A+B_u\hat{F}\right)\hat{X}^{-1}+\gamma_\infty^2\hat{X}^{-T}\left(A+B_u\hat{F}\right)^T+B_wB_w^T\\ +\hat{X}^{-T}C_\infty^TC_\infty\hat{X}^{-1}\end{matrix} & 0\\ 0 & \begin{matrix}\left(A+\hat{G}C_2\right)\hat{P}^T+\hat{P}\left(A+\hat{G}C_2\right)^T+B_wB_w^T\\ +\frac{1}{\gamma_\infty^2}\hat{P}C_\infty^TC_\infty\hat{P}^T\end{matrix}\end{bmatrix}\\ &<0\end{aligned}$$

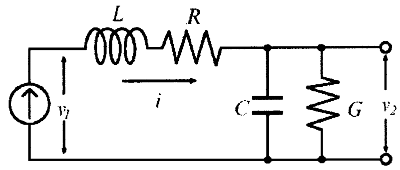

Fig. 1. Electrical circuit.

VI. Illustrative Example

Example 1: Consider the electrical circuit given in [13], which is depicted by Fig. 1. In the circuit, L and C are the inductance and capacitance, respectively; $v_k(t)$ $(k = 1, 2)$ and $i(t)$ denote the voltages and current flow, respectively. Here, it is assumed that

$$i(t) = u(t) + w(t)$$

where $u(t)$ is the control input and $w(t)$ is the white noise disturbance with zero mean and unit intensity. The controlled output is defined as

$$z(t) = v_2(t)$$

and the measured output is defined as

$$y(t) = v_1(t) + v_2(t).$$

This dynamic system is then a singular system, which has the form of

$$\begin{bmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -R & -1 & 1 \\ 0 & -1/G & 0 \\ 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u(t)$$
$$z(t) = [0 \quad 1 \quad 0]\, x(t)$$
$$y(t) = [0 \quad 1 \quad 1]\, x(t)$$

where $x(t) := [i(t) \;\; v_2(t) \;\; v_1(t)]^T$. Our objective is to find a dynamic controller with the form of (17) such that the resulting closed-loop system is admissible and its H_2 norm $\|G_{zw}\|_2$ is less than a given $\gamma_2 > 0$ for the given dynamic system with

$$L = 3 \quad G = 1 \quad C = 2 \quad R = 2.$$

It is noticed that this system is not impulse free since it has one finite pole at $s = -(1/2)$, but rank$E = 2$. Here, it can be also seen that $\ker C_1 \supseteq \ker E$ is satisfied.

U_1, U_2, V_1, V_2, and Σ_r can be obtained by the singular value decomposition of E as

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$U_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\Sigma_r = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

By solving LMIs (42)–(45) in Theorem 4, we obtain

$$\tilde{W} = \begin{bmatrix} 0.2076 & 0.0318 \\ 0.0318 & 0.7361 \end{bmatrix} > 0$$
$$W_2 = \begin{bmatrix} 2.3029 & -0.1956 \\ -0.1956 & 0.4273 \end{bmatrix} > 0$$
$$\tilde{S} = \begin{bmatrix} 0.2523 \\ 1.4683 \\ 0.0039 \end{bmatrix}^T$$
$$S_2 = \begin{bmatrix} 12.3717 \\ -1.9324 \\ -5.4907 \end{bmatrix}$$
$$M = \begin{bmatrix} -0.6249 \\ 0.7066 \\ -0.8292 \end{bmatrix}$$
$$N = \begin{bmatrix} 0.2836 \\ -0.0772 \\ 0.2482 \end{bmatrix}^T$$
$$\gamma_2 = 0.7.$$

Thus, a stabilizing controller $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ achieving $\|G_{zw}\|_2 < 0.7$ can be calculated by (47) and (46), which is

$$\hat{E} = E$$
$$\hat{A} = 10^3 \times \begin{bmatrix} -0.1121 & -2.1924 & 0.0011 \\ -0.4257 & -8.5487 & 0.0044 \\ 0.4309 & 8.6340 & -0.0045 \end{bmatrix}$$
$$\hat{B} = \begin{bmatrix} 52.5414 \\ 208.6767 \\ -211.0282 \end{bmatrix}$$
$$\hat{C} = \begin{bmatrix} -0.1847 \\ -1.2868 \\ 0.0010 \end{bmatrix}^T.$$

$$\hat{X}_c = \begin{bmatrix} \gamma_\infty^2 V_1 \hat{W}_2 V_1^T E^T + \gamma_\infty^2 V_2 \hat{S}_2^T & V_1\left(\hat{W}_1 - \gamma_\infty^2 \hat{W}_2\right)V_1^T E^T + V_2\left(\hat{S}_1^T - \gamma_\infty^2 \hat{S}_2^T\right) \\ V_1\left(\hat{W}_1 - \gamma_\infty^2 \hat{W}_2\right)V_1^T E^T + V_2\left(\hat{S}_1^T - \gamma_\infty^2 \hat{S}_2^T\right) & V_1\left(\gamma_\infty^2 \hat{W}_2 - \hat{W}_1\right)V_1^T E^T + V_2\left(\gamma_\infty^2 \hat{S}_2^T - \hat{S}_1^T\right) \end{bmatrix}$$
$$= V_{c1} \hat{W}_c V_{c1}^T E_c^T + V_{c2} \hat{S}_c^T$$

Example 2: Still consider Example 1, but here we assume that there is another controlled output z_∞, which is defined as

$$z_\infty(t) = v_1(t) = [0 \quad 0 \quad 1]x(t).$$

The objective is to find a dynamic controller with the form of (17) such that the resulting closed-loop system is admissible, $\|G_{zw}\|_2 < \gamma_2$ and $\|G_{z_\infty w}\|_\infty < \gamma_\infty$. By solving LMIs (59)–(62) in Theorem 5, we obtain

$$\begin{aligned}
\hat{W} &= \begin{bmatrix} 1.2999 & 0.4838 \\ 0.4838 & 4.6011 \end{bmatrix} > 0 \\
\hat{W}_2 &= \begin{bmatrix} 2.6075 & 0.0133 \\ 0.0133 & 2.6782 \end{bmatrix} > 0 \\
\hat{S} &= \begin{bmatrix} 1.4215 \\ 9.2248 \\ -0.0226 \end{bmatrix}^T \\
\hat{S}_2 &= \begin{bmatrix} 0.8071 \\ -0.0484 \\ -0.0484 \end{bmatrix} \\
\hat{M} &= \begin{bmatrix} -2.9078 \\ 3.9808 \\ -8.1890 \end{bmatrix} \\
\hat{N} &= \begin{bmatrix} 1.3272 \\ -0.3761 \\ 1.4474 \end{bmatrix}^T \\
\gamma_2 &= 1 \\
\gamma_\infty &= 0.5.
\end{aligned}$$

Thus, a stabilizing controller $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ achieving $\|G_{zw}\|_2 < 1$ and $\|G_{z_\infty w}\|_\infty < 0.5$ can be calculated by (47) and (46), which is

$$\begin{aligned}
\hat{E} &= E \\
\hat{A} &= \begin{bmatrix} -2.6832 & 1.6932 & 0.0056 \\ 14.2809 & -394.0588 & 0.0357 \\ -13.0275 & 392.0607 & -0.0363 \end{bmatrix} \\
\hat{B} &= \begin{bmatrix} 2.2901 \\ -362.8606 \\ 362.1625 \end{bmatrix} \\
\hat{C} &= \begin{bmatrix} -0.1870 \\ -0.2143 \\ -0.0082 \end{bmatrix}^T.
\end{aligned}$$

VII. Conclusions

The main conclusions of this paper are as follows.

i) An expression of the H_2 norm for the admissible singular systems in terms of system matrices has been derived under a certain condition.

ii) The H_2 control problem of finding a dynamic output feedback controller such that the closed-loop system is admissible and its H_2 norm is bounded by a given $\gamma > 0$ has been considered. The desired controller has been obtained by solving a set of strict LMIs.

iii) A mixed H_2/H_∞ control problem has also been considered. It has been shown that such a controller exists when a set of LMIs are feasible. Explicit expressions of a desirable controller are given through the solutions of the derived LMIs.

iv) Two numerical examples have been used to illustrate the effectiveness of the proposed methods.

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