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<th>H ( \infty ) and positive-real control for linear neutral delay systems</th>
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IV. Conclusion

In this note, an algorithm is provided for testing the diagnosability of discrete-event systems. Compared to the existing testing method in [4], our algorithm does not require the construction of a diagnoser for the system. The complexity of our algorithm is of fourth order in the number of states of the system and linear in the number of failure types of the system, whereas the complexity of the testing method in [4] is exponential in the number of states of the system and doubly exponential in the number of failure types of the system.

REFERENCES

H∞ and Positive-Real Control for Linear Neutral Delay Systems

Shengyuan Xu, James Lam, and Chengwu Yang

Abstract—This note is concerned with the H∞ and positive-real control problems for linear neutral delay systems. The purpose of H∞ control is the design of a memoryless state feedback controller which stabilizes the neutral delay system and reduces the H∞ norm of the closed-loop transfer function from the disturbance to the controlled output to a prescribed level, while the purpose of positive-real control is to design a memoryless state feedback controller such that the resulting closed-loop system is stable and the closed-loop transfer function is extended strictly positive real. Sufficient conditions for the existence of the desired controllers are given in terms of a linear matrix inequality (LMI). When this LMI is feasible, the expected memoryless state feedback controllers can be easily constructed via convex optimization.

Index Terms—H∞ control, linear matrix inequality, memoryless state feedback, neutral delay systems, positive-real control.

I. INTRODUCTION

Since the late 1980s, the H∞ control problem has attracted much attention due to its both practical and theoretical importance. Various approaches have been developed and a great number of results for continuous systems as well as discrete systems have been reported in the literature; see, for instance, [4], [18]. Very recently, interest has been focused on H∞ control problem for delay systems. Lee et al. [7] generalized the H∞ results for continuous systems to systems with state delay, which was further extended to systems with both state and input delays in [3] and [9], respectively. In the context of discrete systems with state delay, similar results can be found in [12] and references therein.

On the other hand, since the introduction of the notion of positive realness, many researchers have considered the positive-real control problem for linear time-invariant systems [1], [15]. The objective is to design controllers such that the resulting closed-loop system is stable and the closed-loop transfer function is positive real. It has been shown in [13] that a solution to this problem involves solving a pair of Riccati inequalities. These results have been extended to uncertain linear systems with time-invariant uncertainty in [11] and [16], respectively. It is worth noting that some positive realness results have also been generalized to time-delay systems [8].

Recently, much attention has been focused on the study of the theory of neutral delay systems and some issues, such as stability and stabilization, related to such systems have been studied [5], [10], [14]. To date, however, very little attention has been drawn to the problem of H∞ control, as well as positive-real control, for linear neutral delay systems, these are more complex and still open.

In this note, we deal with the H∞ control and positive-real control problems for linear neutral delay systems. The size of the delays appearing in the state and derivative of the state may not be identical. The H∞ control problem we address is to design a memoryless state feedback controller such that the resulting closed-loop system is asymptotically stable while the closed-loop transfer function from the disturbance to the controlled output meets a prescribed H∞-norm bound constraint. In terms of a linear matrix inequality, a sufficient condition for the existence of H∞ state feedback controllers is presented. Then, based on the relationship between bounded realness and positive realness and the results on H∞ control, we obtain a sufficient condition for extended strictly positive realness (ESPR) for neutral delay systems. The condition for the solvability of positive-real control problem is also given in terms of a linear matrix inequality.

Notation: Throughout this note, for symmetric matrices X and Y, the notation X ≥ Y (respectively, X > Y) means that the matrix X − Y is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimension. The superscript “T” and “*” represent the transpose and the complex conjugate transpose, ||x|| is the Euclidean norm of the vector x. For a given stable transfer function matrix G(s), its H∞ norm is given by ||G(s)||∞ = supω∈R,|ω|=1 |G(jω)|, where σmax represents the maximum singular value of a matrix. ρ(A) denotes spectral radius of a matrix A. L2[0, ∞) stands for the space of square integrable functions on [0, ∞). Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. MAIN RESULTS

Consider the following linear neutral delay system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_dx(t - h) + A_{d}x(t - d) \\
+ Bu(t) + E \omega(t) \\
z(t) &= Cx(t) + D \omega(t) \\
x(t_0 + \theta) &= \phi(\theta) \quad \forall \theta \in [-l, 0]
\end{align*}
\]

(1) (2) (3)
From (12), it is easy to show that

\[
\mathcal{D}(\theta) = \mathcal{A}_N G_{\omega}(\mathcal{A}_N)^{-1} \mathcal{A}_N^T + \mathcal{A}_N S^{-1} \mathcal{A}_N^T \mathcal{P} \mathcal{D}(x_t)
\]

(13)

Thus, the operator \( \mathcal{D} \) is stable.

Now, we introduce the following Lyapunov functional candidate for the system (9) and (10):

\[
V(x_t) = (x(t) - A_\omega x(t - d))^T P(x(t) - A_\omega x(t - d)) + \int_{-d}^0 (x(\tau))^T Q(x(\tau)) d\tau + \int_{-h}^0 x(\tau)^T S x(\tau) d\tau
\]

where \( x_t = x(t + \theta), \theta \in [-I, 0] \). It can be shown that there exist scalars \( c_1 > 0 \) and \( c_2 > 0 \) such that the following holds:

\[
c_1 \|\mathcal{D}(\theta)\|_2^2 \leq V(\theta) \leq c_2 \sup_{\theta \in [-I, 0]} \|\dot{\theta}(\theta)\|^2.
\]

(15)

Differentiating \( V(x_t) \) along the solution of (9) and (10) results in

\[
\dot{V}(x_t) = 2(x(t) - A_\omega x(t - d))^T P(Ax(t) + A_\omega x(t - h)) + x(t)^T (Q + S)x(t) - x(t - d)^T Qx(t - d)
\]

\[
- x(t - h)^T S x(t - h) = (x(t) - A_\omega x(t - d))^T (PA + A_\omega x(t - d))^T (Q + S + PA) x(t - d)
\]

\[
+ 2(x(t) - A_\omega x(t - d))^T (Q + S + PA) x(t - d) + 2(x(t) - A_\omega x(t - d))^T PA_\omega x(t - h)
\]

\[
- x(t - h)^T S x(t - h).
\]

(16)

Noting the definition of the operator \( \mathcal{D} \), this equality can be rewritten as

\[
\dot{V}(x_t) = \mathcal{D}(x_t)^T (PA + A_\omega x(t - d))^T (Q + S + PA) x(t - d)
\]

\[
- x(t - d)^T W x(t - d) + 2 \mathcal{D}(x_t)^T (Q + S + PA) A_\omega x(t - d) + 2 \mathcal{D}(x_t)^T P A_\omega x(t - h)
\]

\[
- x(t - h)^T S x(t - h).
\]

(17)

By considering (12), it follows that:

\[
V(x_t) = \mathcal{D}(x_t)^T (PA + A_\omega x(t - d))^T (Q + S + PA) A_\omega x(t - d)
\]

\[
+ \mathcal{A}_N^T (Q + S + PA) A_\omega x(t - d)
\]

\[
+ \mathcal{A}_N S^{-1} \mathcal{A}_N^T \mathcal{P} (x_t)
\]

\[
- x(t - d)^T W (Q + S + PA) A_\omega x(t - d)
\]

\[
- x(t - h)^T S x(t - h) = \mathcal{D}(x_t)^T (PA + A_\omega x(t - d))^T (Q + S + PA) x(t - d)
\]

(18)

This equality, together with (11), implies that there exists a scalar \( c > 0 \) such that

\[
\dot{V}(x_t) \leq -c \mathcal{D}(x_t)^T \mathcal{D}(x_t)
\]

Finally, noting the stability of the operator \( \mathcal{D} \) and the above inequality and (15), the desired result follows immediately from [6, Th. 7.1].

Remark 1: Lemma 1 provides a delay-independent stability condition for the neutral delay system (9) and (10), it is worth noting that, for any \( d > 0 \) and \( h > 0 \), Lemma 1 is always applicable. However, [5, Th. 1, p. 93] is only applicable to the case when \( d = h \). In this sense, our stability result extends that of [5] and is more general.

The following result will play an important role in solving the \( H_\infty \) control problem in this section.

Theorem 1: Consider the neutral delay system (1)-(3) with \( u(t) \equiv 0 \), that is

\[
\dot{x}(t) = Ax(t) + A_\omega x(t - h) + A_\omega \dot{x}(t - d) + E \omega(t)
\]

\[
z(t) = Cx(t) + D \omega(t)
\]

\[
x(t_0 + \theta) = \phi(\theta) \quad \forall \theta \in [-I, 0].
\]

(19)

(20)

If there exist matrices \( P > 0 \), \( Q > 0 \), and \( S > 0 \) such that

\[
PA + A_\omega x(t - d))^T (Q + S + PA) x(t - d)
\]

\[
- x(t - d)^T W (Q + S + PA) A_\omega x(t - d)
\]

\[
- x(t - h)^T S x(t - h) = \mathcal{D}(x_t)^T (PA + A_\omega x(t - d))^T (Q + S + PA) x(t - d)
\]

(21)

\[
W = Q - A_N^T (Q + S + PA) A_N > 0
\]

\[
W = Q - A_N^T (Q + S + PA) A_N > 0
\]

(22)
then the system (16)–(18) is asymptotically stable, and
\[
\left\| C [ s ( I - A_d e^{-ds}) - ( A + A_h e^{-dh} ) ]^{-1} E + D \right\|_\infty < \gamma.
\]
For the proof of Theorem 1, the following two lemmas will be used.

**Lemma 2:** If there exist matrices \( P > 0, Q > 0 \), and \( S > 0 \) such that
\[
P A + A^T P + C^T C + Q + S + \gamma^{-2} P E E^T P + P A_h S^{-1} A_h^T P
\]
\[
+ (C^T C + Q + S + P A_h) A_d W^{-1} A_d^T (C^T C + Q + S + P A)^T < 0
\]
(23)
\[
W = Q - A_d (C^T C + Q + S) A_d > 0
\]
(24)
then system (9) is asymptotically stable, and
\[
\left\| C [ s ( I - A_d e^{-ds}) - ( A + A_h e^{-dh} ) ]^{-1} E \right\|_\infty < \gamma.
\]

**Proof:** Let
\[
S_1 = S + C^T C.
\]
From (23), it can be deduced that
\[
P A + A^T P + Q + S_1 + (Q + S_1 + P A)
\]
\[
\cdot A_d W^{-1} A_d^T (Q + S_1 + P A)^T + P A_h S^{-1} A_h^T P < 0.
\]
By Lemma 1, this inequality, together with (24), implies the asymptotic stability of the neutral delay system (9). Next, we will show that the \( H_\infty \) norm bound constraint is satisfied. To this end, we set
\[
\Psi (j \omega) = I - A_d e^{-j \omega d}.
\]
From (24), we have
\[
A_d^T Q A_d - Q < 0.
\]
Therefore, \( \rho (A_d) < 1 \). This implies that for all \( \omega \in \mathbb{R} \), \( \Psi (j \omega) \) is invertible.

Now, through some routine algebraic manipulations, we obtain
\[
C^T C + Q
\]
\[
= \Psi (j \omega)^{-1} [ \Psi (j \omega)^* (C^T C + Q + S) \Psi (j \omega) ] \Psi (j \omega)^{-1}
\]
\[
= \Psi (j \omega)^{-1} (S + C^T C + W) \Psi (j \omega)^{-1} + X (j \omega) + X (j \omega)^*
\]
(25)
for all \( \omega \in \mathbb{R} \), where
\[
X (j \omega) = - e^{-j \omega d} (C^T C + Q + S) A_d \Psi (j \omega)^{-1}.
\]
Then, (23) can be rewritten as
\[
P A + A^T P + \gamma^{-2} P E E^T P + P A_h S^{-1} A_h^T P + X (j \omega)
\]
\[
+ X (j \omega)^* + \Psi (j \omega)^{-1} (S + C^T C + W) \Psi (j \omega)^{-1}
\]
\[
+ (C^T C + Q + S + P A_h) A_d W^{-1}
\]
\[
\cdot A_d^T (C^T C + Q + S + P A)^T < 0.
\]
(26)
Define
\[
Y (j \omega) = e^{-j \omega d} (C^T C + Q + S + P A_h) A_d \Psi (j \omega)^{-1},
\]
\[
Z (j \omega) = e^{-j \omega d} P A_h \Psi (j \omega)^{-1}.
\]
Recalling that for any matrices \( K_1, K_2 \) and \( K_3 \) of appropriate dimensions with \( K_2 > 0 \)
\[
K_1^T K_2 K_3 \leq K_1^T K_2 K_1 + K_3^T K_2^{-1} K_3.
\]
Therefore
\[
Y (j \omega) + Y (j \omega)^* \leq (C^T C + Q + S + P A) A_d W^{-1} A_d^T (C^T C + Q + S + P A)^T + \Psi (j \omega)^{-1} W \Psi (j \omega)^{-1}
\]
(27)
and
\[
Z (j \omega) + Z (j \omega)^* \leq P A_h S^{-1} A_h^T P + \Psi (j \omega)^{-1} S \Psi (j \omega)^{-1}.
\]
(28)
From (26)–(28), it follows that for all \( \omega \in \mathbb{R} \):
\[
P A + A^T P + \gamma^{-2} P E E^T P + X (j \omega) + X (j \omega)^* + Y (j \omega)
\]
\[
+ Y (j \omega)^* + Z (j \omega) + Z (j \omega)^* + \Psi (j \omega)^{-1} C^T C \Psi (j \omega)^{-1} < 0.
\]
(29)

Observing that
\[
P (A + A_h e^{-j \omega d}) \Psi (j \omega)^{-1}
\]
\[
= P A + e^{-j \omega d} P A A_d + e^{-j \omega d} A_h \Psi (j \omega)^{-1}
\]
\[
= P A + X (j \omega) + Y (j \omega) + Z (j \omega).
\]
Substituting this equality into (29) yields
\[
P (A + A_h e^{-j \omega d}) \Psi (j \omega)^{-1} + \Psi (j \omega)^{-1} (A^T + A_h^T e^{j \omega d}) P
\]
\[
+ \gamma^{-2} P E E^T P + \Psi (j \omega)^{-1} C^T C \Psi (j \omega)^{-1} < 0.
\]
That is
\[
P [ j \omega I - (A + A_h e^{-j \omega d}) ] \Psi (j \omega)^{-1}
\]
\[
+ [ j \omega I - (A + A_h e^{-j \omega d}) ] P
\]
\[
\leq \gamma^{-2} P E E^T P + \Psi (j \omega)^{-1} C^T C \Psi (j \omega)^{-1} > 0.
\]
(30)
Let \( \Gamma (j \omega) = j \omega I - (A + A_h e^{-j \omega d}) \Psi (j \omega)^{-1} \), then for all \( \omega \in \mathbb{R} \), \( \Gamma (j \omega) \) is invertible since the neutral delay system (9) is stable.

Premultiplying (30) by \( E^T \Gamma (j \omega)^{-1} \) and postmultiplying (30) by \( \Gamma (j \omega)^{-1} E \) give
\[
E^T \Gamma (j \omega)^{-1} P E + E^T P \Gamma (j \omega)^{-1} E
\]
\[
- \gamma^{-2} E^T \Gamma (j \omega)^{-1} P E E^T P \Gamma (j \omega)^{-1} E
\]
\[
= E^T \Gamma (j \omega)^{-1} \Psi (j \omega)^{-1} C^T C \Psi (j \omega)^{-1} \Gamma (j \omega)^{-1} E > 0.
\]
In completing the squares in this inequality, it follows that, for all \( \omega \in \mathbb{R} \)
\[
\gamma^2 I - \gamma^{-2} [ E^T \Gamma (j \omega)^{-1} P E - \gamma^2 I ] [ E^T \Gamma (j \omega)^{-1} E - \gamma^2 I ]
\]
\[
= E^T \Gamma (j \omega)^{-1} \Psi (j \omega)^{-1} C^T C \Psi (j \omega)^{-1} \Gamma (j \omega)^{-1} E < \gamma^2 I.
\]
(31)

Finally, by noting that
\[
\Psi (j \omega)^{-1} \Gamma (j \omega)^{-1} \equiv [ j \omega (I - A_d e^{-j \omega d}) - (A + A_h e^{-j \omega d} ) ]^{-1}
\]
and using (31), we conclude that \( \| C [ s ( I - A_d e^{-ds}) - (A + A_h e^{-dh} ) ]^{-1} E \|_\infty < \gamma \) holds.

**Lemma 3:** Let \( H > 0, \Psi (s) = I - A_d e^{-ds} \)
\[
\Phi (s) = \left[ E^T (-s \Psi (s)^{-1} - (A + A_h e^{-dh} )^{-1} I) \right]
\]
\[
\cdot \left[ \begin{array}{c} P \\ S \\ H \end{array} \right] \left[ \begin{array}{c} (s \Psi (s) - (A + A_h e^{-dh} )^{-1} E) \end{array} \right]^T I
\]

Applying Lemma 2 to this inequality yields

\[
\begin{bmatrix}
\hat{A} + A - E H^{-1} S^T, \\
\hat{E} = E H^{-1/2}, \\
\hat{P} = P - S H^{-1} S^T
\end{bmatrix}
\]

Then, \( \hat{\Phi}(\omega) > 0 \) if and only if \( \hat{\Phi}(\omega) > 0 \).

\textbf{Proof:} The proof follows the same idea as in [17, Lemma 13.18] and is thus omitted. \( \square \)

\textbf{Proof of Theorem 1:} The asymptotic stability of the neutral delay system (9) can be inferred from (19) and Lemma 1. To show the result.

Applying Lemma 2 to this inequality yields

\[
\begin{bmatrix}
\hat{A} + A - E H^{-1} D^T C, \\
\hat{C} = (I + D V^{-1} D^T)^{1/2} C, \\
\hat{E} = E V^{-1/2} D^T C
\end{bmatrix}
\]

Then, the memoryless state feedback controller

\[
u(t) = Y X^{-1} x(t),
\]

stabilizes system (1)–(3) and guarantees that the \( H_\infty \) norm bound of the closed-loop transfer function constraint has a prescribed level \( \gamma > 0 \).

\textbf{Proof:} Applying the controller (38) to the neutral delay system (1)–(3), we obtain the resulting closed-loop system in the form of (5) and (6) with

\[
A_c = A + B Y X^{-1}.
\]

By Schur complement, (37) implies

\[
X A_c^T + A_c X + E E^T + Q + S + (X C^T + E D^T) V^{-1}(X C^T + E D^T)^T + X A_c^T S^{-1} A_c X + M_1 A_c^T W_1^{-1} A_c M_1^T < 0
\]

where

\[
M_1 = E (I + D^T V^{-1} D) E^T + Q + S + X (A_c^T + C^T D) V^{-1}(PE + C^T D)^T
\]

\[
W_1 = Q - A_d [E (I + D^T V^{-1} D) E^T + Q + S] A_d^T > 0.
\]

Using Theorem 1, we have that the following neutral delay system:

\[
th(t) = A x(t) + A_h x(t - h) + E \omega(t)
\]

\[
z(t) = C x(t) + D \omega(t)
\]

\[
x(t_0 + \theta) = \phi(\theta) \quad \forall \theta \in [-h, 0].
\]

If there exist matrices \( P > 0 \) and \( S > 0 \) such that

\[
P A + A^T P + C^T C + (P E + C^T D) V^{-1}(P E + C^T D)^T + P A_h S^{-1} A_h^T P + S < 0
\]

with

\[
V = \gamma^2 I - D^T D > 0
\]

then the system (33)–(35) is asymptotically stable when \( \omega(t) \equiv 0 \), and the \( H_\infty \) norm of the transfer function satisfies

\[
\left\| C [s I - A - A_h e^{-s h}]^{-1} E + D \right\|_{\infty} < \gamma.
\]

\textbf{Remark 2:} It is easy to see that Corollary 1 is the same as [7, Th. 1] when \( D = 0 \), thus Theorem 1 here can be viewed as an extension of the existing results on \( H_\infty \) disturbance attenuation for delay systems to neutral delay systems.

Now we are in a position to give a solution to the \( H_\infty \) control problem specified above.

\textbf{Theorem 2:} Suppose that there exist matrices \( X > 0 \), \( Y \), \( Q \), \( S > 0 \) satisfying the LMI, as shown in (37) at the bottom of the page, where

\[
L = [E (I + D^T V^{-1} D) E^T + Q + S + XC^T V^{-1} D E^T] + X A^T + Y D^T B^T A_d^T
\]

\[
H = [X A_d^T X C^T + E D^T]
\]

\[
J = \text{diag}(S, V)
\]

\[
V = \gamma^2 I - D^T D > 0
\]

then the memoryless state feedback controller

\[
u(t) = Y X^{-1} x(t),
\]

stabilizes system (1)–(3) and guarantees that the \( H_\infty \) norm bound of the closed-loop transfer function constraint has a prescribed level \( \gamma > 0 \).

\textbf{Proof:} Applying the controller (38) to the neutral delay system (1)–(3), we obtain the resulting closed-loop system in the form of (5) and (6) with

\[
A_c = A + B Y X^{-1}.
\]

By Schur complement, (37) implies

\[
X A_c^T + A_c X + E E^T + Q + S + (X C^T + E D^T) V^{-1}(X C^T + E D^T)^T + X A_c^T S^{-1} A_c X + M_1 A_c^T W_1^{-1} A_c M_1^T < 0
\]

where

\[
M_1 = E (I + D^T V^{-1} D) E^T + Q + S + X (A_c^T + C^T D) V^{-1}(PE + C^T D)^T
\]

\[
W_1 = Q - A_d [E (I + D^T V^{-1} D) E^T + Q + S] A_d^T > 0.
\]

Using Theorem 1, we have that the following neutral delay system:

\[
x(t) = A x(t) + A_h x(t - h) + E \omega(t)
\]

\[
z(t) = C x(t) + D \omega(t)
\]

\[
x(t_0 + \theta) = \phi(\theta) \quad \forall \theta \in [-h, 0].
\]

If there exist matrices \( P > 0 \) and \( S > 0 \) such that

\[
P A + A^T P + C^T C + (P E + C^T D) V^{-1}(P E + C^T D)^T + P A_h S^{-1} A_h^T P + S < 0
\]

with

\[
V = \gamma^2 I - D^T D > 0
\]

then the system (33)–(35) is asymptotically stable when \( \omega(t) \equiv 0 \), and the \( H_\infty \) norm of the transfer function satisfies

\[
\left\| C [s I - A - A_h e^{-s h}]^{-1} E + D \right\|_{\infty} < \gamma.
\]
and satisfies

\[ \begin{bmatrix} \left[ \mathcal{C} \left( I - A_d e^{-s\tau} \right) - (A_c + A_h e^{-s\tau}) \right]^{-1} E + D \end{bmatrix}_{\ell_\infty} < \gamma \]

holds. This completes the proof. \hfill \Box

By considering the connections between bounded realness and positive realness, next we shall consider the positive-real control problem for neutral delay systems. Theorem 3 provides sufficient conditions for solvability of the problems of \( H_\infty \) and positive-real control for neutral delay systems, respectively. It is worth pointing out that the LMI (37) in Theorem 2 and the LMI (46) in Theorem 4 can be solved efficiently, and no tuning of parameters is required [2].

### References


A Note on Uniform Observability

Bernard Delyon

Abstract—We prove in this note that the classical inequality $P_t \leq Q_t^{-1} + C_t$, relating the variance of the Kalman filter estimate, the observability matrix, and the controllability matrix is not true. This inequality is the cornerstone of the asymptotic stability theory of the Kalman filter for time-varying systems. We provide another inequality of the same type.

Index Terms—Kalman filter, time-varying, uniform observability.

I. INTRODUCTION

We consider the following system:

\[
\begin{align*}
\dot{x}_t &= A_t x_t + v_t \\
y_t &= C_t x_t + w_t \\
E(v,v') &= Qt^{-1}R_t^{-1}S_t(1) = \delta(t-s).
\end{align*}
\]

with an initial value with Gaussian distribution $x_0 \sim N(\bar{x}_0, P_0)$. The corresponding Kalman filter is

\[
\begin{align*}
\hat{x}_t &= A_t x_t + (P_t C_t + R_t) S_t^{-1}(y_t - C_t \hat{x}_t) \\
\hat{P}_t &= A_t P_t + P_t A_t^T + Q_t \\
&\quad - (P_t C_t + R_t) S_t^{-1}(C_t P_t + R_t^T).
\end{align*}
\]

The matrix $P_t$ is the variance of the estimation error $\hat{x}_t - x_t$. Bounding $P_t$ is, for obvious reasons, an important issue. In [2, p. 359], R. E. Kalman considers the case where $R_t = 0$ and states the following lemma (we set $W_t = C_t^T S_t^{-1} C_t$).

Lemma 1 (Case $R_t = 0$): Let $P_t$, $C_t$, $C_t$ be the solutions to

\[
\begin{align*}
\dot{P}_t &= A_t P_t + P_t A_t^T + Q_t - P_t W_t P_t \\
\dot{C}_t &= -C_t A_t - A_t^T C_t + W_t \\
\dot{C}_t &= A_t C_t + C_t A_t^T + Q_t .
\end{align*}
\]

as soon as $O_t^{-1}$ exists.

This lemma is more explicitly stated and “proved” in [1, p. 234, 243]. The flaw in the proof is apparent in the last correlation inequality at the end of [1, p. 234].

We show here in Section II that this lemma is untrue and prove in Section III the following modified version of it.

**Lemma 2 (Case $R_t = 0$):** Let $A_t$, $Q_t$. $W_t$ be arbitrary square matrices with same dimensions, piecewise continuous w.r.t. $t$, such that $Q_t$ and $W_t$ are symmetric and $W_t$ is nonnegative for all $t$. Let $P_t$, $C_t$, $D_t$ be the solutions to

\[
\begin{align*}
\dot{P}_t &= A_t P_t + P_t A_t^T + Q_t - P_t W_t P_t \\
\dot{C}_t &= -C_t A_t - A_t^T C_t + W_t \\
\dot{C}_t &= A_t C_t + C_t A_t^T + Q_t \\
\dot{O}_t &= 0
\end{align*}
\]

then

\[
P_t \leq O_t^{-1} + C_t
\]

as soon as $O_t^{-1}$ exists. Furthermore, one has

\[
O_t \leq \int_0^t e^{2 \alpha(t-s)} \|W_s\| ds
\]

Some classical comments, which are shared by both lemmas, are in order.

2) The important point here is that the bound is independent of $P_0$. This allows indeed to get bounds for $P_t$, $t \geq 0$, by considering the system on a finite-time interval $(t - \sigma, t)$:

\[
P_t \leq P_0^{-1} A_t - A_t^T P_t^{-1} + W_t + P_t^{-1} \alpha \leq 0 \leq \alpha \leq \sup_{0 \leq s \leq \alpha} \|A_s\|
\]

3) A bound on $D_t$ is easily obtained assuming boundedness of $A_t$ and integrability of $\|W_t\| + \|Q_t\|$ over finite intervals. The main condition is the invertibility of $O_t$.

4) The case $R_t \neq 0$ is actually covered via simple changes in $A_t$ and $Q_t$.

5) Since $P_t^{-1}$ satisfies

\[
P_t^{-1} = -P_t^{-1} A_t - A_t^T P_t^{-1} + W_t - P_t^{-1} Q_t P_t^{-1}
\]

another application of this theorem leads to a lower bound on $P_t$ based on the invertibility of the controllability matrix $C_t$.

6) The solution to the equation for $P_t^{-1}$ with initial value $P_0^{-1} = 0$ is smaller than $O_t$; this implies that in the limit $P_0 \to \infty$, one has $P_t \geq O_t^{-1}$. This is why the term $O_t^{-1}$ cannot be avoided.

II. COUNTEREXAMPLE

This example is made with $A_t = 0$. Consider

\[
P_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_t = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

then we have

\[
P_t W_t = \begin{pmatrix} 1 & 1 \\ 1 & (t+1)^2 \end{pmatrix} = Q_t - P_t.
\]