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\[ \dot{\gamma}(q, \dot{q}, t) = \frac{\partial W}{\partial t} \text{ q.m. energy} \]

\[ \overline{H}(q, \dot{q}, t) = \text{as in (11) q.m. energy operator, obtained} \]

\[ L(x, v, t) = pv - H \text{ classical Lagrangian} \]

\[ \dot{L}(q, \dot{v}, t) = \dot{p}v - \overline{H} \text{ q.m. Lagrangian} \]

\[ \begin{align*}
p &= -\frac{\partial W}{\partial x} \text{ classical momentum} \\
\dot{p} &= -\frac{\partial W}{\partial q} \text{ q.m. momentum} \\
\dot{v} &= -i\hbar \frac{\partial}{\partial q} \text{ q.m. momentum operator} \\
\end{align*} \]

\[ V(x, t) \text{ classical potential energy} \]

\[ \begin{align*}
\text{obtained by substituting } g \text{ for } x \text{ in } V(x, t) \\
W &= \text{as in (1) classical action integral with} \\
\text{reversed sign} \\
\dot{W} &= \text{as in (6) q.m. action integral} \\
p &= \text{density of physical particles on the} \\
\text{real axis} \\
\psi &= \exp(\frac{\overline{W}}{i\hbar}) \text{ wave function satisfying} \\
\text{Schrödinger’s equation} \\
\text{stat} &= \text{stationary value obtained by} \\
\text{varying the functions } v, p \\
\text{stat} &= \text{stationary value obtained by} \\
\text{varying the functions } \dot{v}, \dot{p} \\
\delta &= \text{variation of the succeeding} \\
\text{expression with respect to } v, p \\
\dot{\delta} &= \text{variation of the succeeding} \\
\text{expression with respect to } \dot{v}, \dot{p} \\
\end{align*} \]

**REFERENCES**


**Robust $H_\infty$ Control of Uncertain Markovian Jump Systems with Time-Delay**

Yong-Yan Cao and James Lam

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**Abstract**—This correspondence is concerned with the robust stochastic stabilizability and robust $H_\infty$ disturbance attenuation for a class of uncertain linear systems with time delay and randomly jumping parameters. The transition of the jumping parameters is governed by a finite-state Markov process. Sufficient conditions on the existence of a robust stochastic stabilizing and $\gamma$-suboptimal $H_\infty$ state-feedback controller are presented using the Lyapunov functional approach. It is shown that a robust stochastically stabilizing $H_\infty$ state-feedback controller can be constructed through the numerical solution of a set of coupled linear matrix inequalities.

**Index Terms**—Jumping parameters, linear matrix inequality (LMI), linear uncertain systems, robust control, time-delay systems.

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**I. INTRODUCTION**

A great deal of attention has recently been devoted to the Markovian jump linear systems. This family of systems is modeled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. It was introduced by Krasovskii and Lidskii in 1961 [13] and may represent a large variety of processes, including those in production systems and economic problems. Developments in control engineering regarding applications, stability conditions, and optimal control problems for jump linear systems are reported in [1], [3], [8]–[10], [12], and [17]. On the other hand, time-delay systems have been studied extensively on the subject of stability and control over the years; see [4], [5], and [16] for instance. The problem of robust $H_\infty$ control of linear uncertain systems with time delay has gathered much attention, and some sufficient conditions have been presented [6], [7], [11], [15]. In this correspondence, we study the robust stochastic stabilizability and robust $H_\infty$ disturbance attenuation for a class of uncertain linear systems.
time-delay systems with Markovian jumping parameters using the stochastic Lyapunov functional approach first developed by Kushner [14]. Marion [17] used the stochastic Lyapunov function approach to obtain a sufficient condition for the mean square stability and the almost sure stability of the linear jump systems without delay. In [10] and [12], the stochastic stability, stabilizability, and controllability are studied based on the same methodology, and in [2], the authors discussed stochastic stability for a class of uncertain jump linear systems with time delay. In general, these results are in the form of a set of coupled algebraic Riccati equations that is difficult to solve. To facilitate the solution process, the linear matrix inequality (LMI) approach will be employed in the present development.

II. PROBLEM STATEMENT

In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. The notation $M \geq (\geq, <, \leq, \leq 0)$ is used to denote a symmetric positive-definite (positive-semidefinite, negative, negative-semidefinite) matrix. $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix, respectively. $\| \cdot \|$ denotes the Euclidean norm for vectors or the spectral norm of matrices. $E[\cdot]$ stands for the mathematical expectation.

Consider the class of stochastic, uncertain linear state-delay systems with Markovian jumping parameters

\[
\dot{x}(t) = A_1(t, r(t))x(t) + A_2(t, r(t))x(t - \tau) + B_1(t, r(t))u(t) + B_2(t, r(t))w(t) \tag{1}
\]

\[
z(t) = \hat{C}_1(t, r(t))x(t) + \hat{C}_2(t, r(t))x(t - \tau) + \hat{D}_1(t, r(t))u(t) + \hat{D}_2(t, r(t))w(t) \tag{2}
\]

\[
x(t) = \psi(t), \quad t \in [-\tau, 0], \quad r(0) = r_0 \tag{3}
\]

where

$x(t) \in \mathbb{R}^n$ system state;

$w(t) \in \mathbb{R}^m$ exogenous disturbance input;

$u(t) \in \mathbb{R}^n$ control input;

$z(t) \in \mathbb{R}^m$ output to be controlled.

$A_1(t, r(t)), A_2(t, r(t)), B_1(t, r(t)), B_2(t, r(t)), \hat{C}_1(t, r(t)), \hat{C}_2(t, r(t)), \hat{D}_1(t, r(t)), \hat{D}_2(t, r(t))$ are matrix functions of the random jumping process $\{r(t)\}$. $r(t)$ is a finite-state Markov jump process representing the system mode; that is, $r(t)$ takes discrete values in a given finite set $\mathcal{S} = \{1, 2, \ldots, s\}$. Let $\Pi = [\pi_{ij}]$, where $i, j = 1, 2, \ldots, s$, denote the transition probability matrix with

\[
\Pr\{r(t + \Delta) = j| r(t) = i\} = \begin{cases} 
\pi_{ij}\Delta + o(\Delta), & i \neq j \\
1 + \pi_{ii}\Delta + o(\Delta), & i = j
\end{cases} \tag{4}
\]

where $\Delta > 0$, $\pi_{ij} \geq 0$ for $i \neq j$, $\pi_{ii} \equiv \Delta - \pi_{ii} \leq 0$ with $\sum_{i=1, i \neq j}^{s} \pi_{ij} \equiv -\pi_{ii}$ for each mode $i$, $i = 1, 2, \ldots, s$, and $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. $\tau$ is the constant delay time of the state in the system, $\psi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau, 0]$, and $r_0 \in \mathcal{S}$ are the initial conditions of the continuous state and the mode, respectively. To simplify the notation, $M(t, r)$ will be denoted by $M(t)$, and $\dot{A}_1(t, r(t))$ is denoted by $\dot{A}_1$, and so on.

Time-varying uncertainties may appear in these matrices; that is

\[
\dot{A}_1(t) = A_{1r}, \Delta A_1(t), \dot{A}_2(t) = A_{2r} + \Delta A_2(t) \tag{5}
\]

\[
\dot{B}_1(t) = B_{1r} + \Delta B_1(t), \dot{C}_1(t) = C_{1r} + \Delta C_1(t) \tag{6}
\]

\[
\dot{C}_2(t) = C_{2r} + \Delta C_2(t), \dot{D}_1(t) = D_{1r} + \Delta D_1(t) \tag{7}
\]

where $A_{1r}, A_{2r}, B_{1r}, C_{1r}, C_{2r},$ and $D_{1r}$ are governed only by the Markovian jump process and $\Delta A_1(t), \Delta A_2(t), \Delta B_1(t), \Delta C_1(t), \Delta C_2(t),$ and $\Delta D_1(t)$ are real-valued functions representing time-varying parameter uncertainties. We assume that the uncertainties are norm-bounded and can be described as

\[
\begin{bmatrix} 
\Delta A_1(t) \\
\Delta A_2(t) \\
\Delta B_1(t) \\
\Delta C_1(t) \\
\Delta C_2(t) \\
\Delta D_1(t)
\end{bmatrix} = 
\begin{bmatrix} 
E_{1i} \\
E_{2i}
\end{bmatrix}
F_i(t)[H_{1i}, H_{2i}, H_{3i}], \quad \text{when } r(t) = i \tag{8}
\]

where $E_{1i} \in \mathbb{R}^{p \times r_f}, E_{2i} \in \mathbb{R}^{p \times r_f}$, and $H_{1i}, H_{2i}, H_{3i} \in \mathbb{R}^{r_f \times r_f}$ are known constant matrices for each index $i \in \mathcal{S}$ and $F_i(t) \in \mathbb{R}^{r_f \times r_f}$ are unknown matrix functions satisfying

\[
F_i^T(t)F_i(t) \leq I, \quad \forall i \in \mathcal{S}.
\]

It is assumed the elements of $F_i(t)$ are Lebesgue measurable. When $F_i(t) \equiv 0$, then system \((1)-(4)\) is referred to as a nominal jump linear system. It is said to be a free system if $u(t) \equiv 0$.

In this correspondence, we will assume that for all $\delta \in [-\tau, 0]$, a scalar $\varepsilon > 0$ exists such that

\[
||x(t + \delta)|| \leq \varepsilon ||x(t)|| \tag{9}
\]

As was indicated by [16], this assumption does not represent a restriction because $\varepsilon$ can be chosen arbitrarily.

This correspondence is concerned with the design of a robust state-feedback controller

\[
u(t) = -K_i x(t), \quad \text{when } r(t) = i \tag{10}
\]

where $K_i$ is constant for each value $i \in \mathcal{S}$ such that the closed-loop system

\[
\dot{x}(t) = (\hat{A}_{i1} + E_{1i}F_i\hat{H}_{i1})x(t) + (A_{2i} + E_{2i}F_iH_{2i})x(t - \tau) + B_{2i}w(t) \tag{11}
\]

\[
z(t) = (\hat{C}_{i1} + E_{1i}F_i\hat{H}_{i1}x(t) + (C_{2i} + E_{2i}F_iH_{2i})x(t - \tau) + D_{2i}w(t) \tag{12}
\]

where $A_{i1}$, $B_{i1}$, $\hat{H}_{i1}$, $\hat{L}_{i1} - H_{3i}K_i$, $\hat{C}_{i1}$, $C_{2i}$ and $D_{2i}$ are stochastic stable with $\gamma$-disturbance attenuation. It is assumed that the controller has complete access to the state variables $\{x(t)\}$ and the jumping process $\{r(t)\}$. In the following, we will denote by $x(t, \psi, r_0, u)$ the solution of system \((1)-(4)\) at time $t$ under the initial conditions $x(t_0)$ and $r_0$, and the control input $u(t)$, and $x_0$ represents $x(t, \psi, r_0, u)$ at $t = 0$.

**Definition 1:** The free nominal jump linear system is said to be *stochastically stabilizable* if, when $w(t) \equiv 0$, for all finite $v(t) \in \mathbb{R}^n$ defined on $[-\tau, 0]$ and initial mode $r_0 \in \mathcal{S}$, a linear feedback control law \((10)\) exists satisfying

\[
\lim_{t \to -\infty} \mathbb{E}\{\int_0^t \beta(x(t, \psi, r_0, u) - x(t)) dt\psi, r_0\} \leq \beta x_0^T M x_0
\]

for some $\beta > 0$. The uncertain system \((1)-(4)\) is said to be *robust stochastically stabilizable* if it is stochastically stabilizable for all possible uncertainty $||F_i(t)|| \leq 1$.

This definition is similar to that of stochastic stabilizability of jump linear systems without time delay [10], [12]. Under the above definition, stochastic stabilizability of a system means that a linear state-feedback control law exists that drives the $x$ state from any given initial condition $(\psi, t_0)$ asymptotically to the origin, in the mean square sense, which implies the asymptotic stability of the closed-loop system.
Definition 2: For a given control law (10) and a real number $\gamma > 0$, the nominal jump linear system is said to be stochastically stabilizable with $\gamma$-disturbance attenuation if for every $T > 0$ and for every piecewise continuous function $w: [0, \infty) \to \mathbb{R}^r$, the closed-loop system (1)-(4), (11) is asymptotically stable and the response $z: [0, \infty) \to \mathbb{R}^r$ satisfies

$$
\int_0^T z^T(t)z(t)dt < \gamma^2 \int_0^T w^T(t)w(t)dt. \quad (13)
$$

The uncertain system (1)-(4) is said to be robust stochastically stabilizable with $\gamma$-disturbance attenuation if it is stochastically stabilizable with $\gamma$-disturbance attenuation for all possible uncertainty $\| F_i(u) \| \leq 1$.

Let

$$
\| u \|_2 = \mathbb{E} \left\{ \int_0^T w^T(t)w(t)dt \right\}^{1/2}
$$

and

$$
\| z \|_2 = \mathbb{E} \left\{ \int_0^T z^T(t)z(t)dt \right\}^{1/2}
$$

and let $T_{\omega\epsilon}$ denote the system from the exogenous input $w(t)$ to the controlled output $z(t)$; then, the $H_{\infty}$-norm of $T_{\omega\epsilon}$ is

$$
\| T_{\omega\epsilon} \|_{\infty} = \sup_{w(t) \in \mathbb{Z} \forall \epsilon \in (0, \infty)} \frac{\| z \|_2}{\| w \|_2},
$$

Hence, (13) implies $\| T_{\omega\epsilon} \|_{\infty} < \gamma$. In other words, $\gamma$-disturbance attenuation implies $\gamma$-suboptimal $H_{\infty}$ control. The following matrix inequalities are essential for the proofs in sections [20].

Lemma 1: For any vectors $x, y \in \mathbb{R}^r$, matrices $A, P \in \mathbb{R}^{m \times m}$, $D \in \mathbb{R}^{m \times s}$, $E \in \mathbb{R}^{s \times s}$, and $F \in \mathbb{R}^{s \times s}$ with $P > 0$, $\| F \| \leq 1$, and scalar $\varepsilon > 0$, we have:

i) $2x^Ty \leq x^TP^{-1}x + y^TPy$;

ii) $DFE + EF^TE^T \leq \varepsilon^{-1}D^TD + \varepsilon EE^T$;

iii) if $\varepsilon I - EPE^T > 0$,

$$(A + DFE)P(A + DFE)^T \leq APAT + APE^T(\varepsilon I - EPE^T)^{-1}EPA^T + \varepsilon DD^T;
$$

iv) if $P - \varepsilon DD^T > 0$,

$$(A + DFE)^T P^{-1} (A + DFE) \leq A^T(P - \varepsilon DD^T)^{-1}A + \epsilon EE^T.
$$

III. ROBUST STOCHASTIC STABILIZABILITY

In this section, we present a sufficient condition for the robust stochastic stabilizability of the jump linear system (1)-(4) with $w(t) \equiv 0$.

Theorem 1: The nominal jump linear system is stochastically stabilizable if $Q > 0$, $P_i > 0$, and $K_{i,j} = 1, \ldots, s$, exist, satisfying the coupled matrix inequalities

$$
\begin{bmatrix}
\tilde{A}_{i,1}^T P_i + P_i \tilde{A}_{i,1} + \sum_{j=1}^s \pi_{i,j} P_j + Q & P_i A_{2i} \\
A_{2i}^T P_i & -Q
\end{bmatrix} < 0,
$$

$$
M_i \triangleq \begin{bmatrix}
\tilde{A}_{i,1}^T P_i + P_i \tilde{A}_{i,1} + \sum_{j=1}^s \pi_{i,j} P_j + Q & P_i A_{2i} \\
A_{2i}^T P_i & -Q
\end{bmatrix} < 0,
$$

with $x(t) = \psi(t), t \in [-\tau, 0], r(0) = r_0$. In the following, we will simply use $x(t)$ to denote the solution $x(t, \psi, r_0, u)$ under the initial condition $\psi(t)$ and $r_0$ with control $u(t)$.

Take the stochastic Lyapunov functional $V(\cdot): \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \rightarrow \mathbb{R}_+$ to be

$$
V(x(t), r(t)) = i \quad \equiv V(x, i)
$$

$$
\Delta x^T(t)P_i x(t) + \int_{-\tau}^0 x^T(t + \theta)Qx(t + \theta) d\theta \quad (16)
$$

where $Q$ constant $P_i$ constant for each $i$.

The weak infinitesimal operator $A$ [12], [14] of the stochastic process \{r(t), x(t)\}, $t \geq 0$, is given by

$$
\mathcal{A} V(x(t), r(t)) = \lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{E} \{ V(x(t + \Delta), r(t + \Delta))|x(t), r(t) = i \}
$$

$$
- V(x(t), r(t) = i)
$$

$$
x^T(t) \mathbb{E} \left\{ \tilde{A}_{i,1}^T P_i + P_i \tilde{A}_{i,1} + \sum_{j=1}^s \pi_{i,j} P_j + Q \right\} x(t)
$$

$$
+ x^T(t) \mathbb{E} \left\{ P_i A_{2i} \right\} x(t)
$$

$$
+ x^T(t) \mathbb{E} \left\{ Qx(t) \right\}
$$

$$
= x^T(t) M_i x(t)
$$

where $x_i(t) \triangleq [x^T(t) x^T(t - \tau)]^T$. With the assumption in (9), we have $\| x(t) \| \leq \| x_i(t) \|$. Note that $M_i < 0$ and $P_i > 0$; hence, we have for $x \neq 0$

$$
\mathcal{A} V(x, i) \leq \frac{x^T(t) M_i x_i(t)}{x^T(t) P_i x(t) + \int_{-\tau}^0 x^T(t + \theta)Qx(t + \theta) d\theta}
$$

$$
= \frac{x^T(t) P_i x(t) + \int_{-\tau}^0 x^T(t + \theta)Qx(t + \theta) d\theta}{x^T(t) P_i x(t) + \int_{-\tau}^0 x^T(t + \theta)Qx(t + \theta) d\theta}
$$

$$
\leq \min_{0 < \alpha \leq \alpha_0} \left\{ \lambda_{\min}(-M_i) \lambda_{\max}(P_i) + \tau \alpha^2 \lambda_{\max}(Q) \right\}.
$$

Define

$$
\alpha \triangleq \min_{0 < \alpha \leq \alpha_0} \left\{ \lambda_{\min}(-M_i) \lambda_{\max}(P_i) + \tau \alpha^2 \lambda_{\max}(Q) \right\}.
$$

We have $0 > \alpha$ and $\mathcal{A} V(x, i) \leq -\alpha V(x, i)$. Then, by Dynkin’s formula (see [14])

$$
\mathbb{E} \{ V(x(t), r(t)) \} - V(x_0, r_0)
$$

$$
= \mathbb{E} \left( \int_0^T \mathcal{A} V(x(s), r(s)) ds \right)
$$

$$
\leq -\alpha \mathbb{E} \left( \int_0^T x^T(t + s)Qx(t + s) ds \right) > 0.
$$

The Gronwall–Bellman lemma gives $\mathbb{E} \{ V(x(t), r(t)) \} \leq \exp(-\alpha t) V(x_0, r_0)$. Because $Q > 0$, then

$$
\mathbb{E} \left\{ \int_{-\tau}^0 x^T(t + s)Qx(t + s) ds \right\} > 0.
$$

Thus, for all $r_0 \in \mathcal{S}$

$$
\mathbb{E} \left\{ x^T(t) P_i x(t) \psi, r_0 \right\}
$$

$$
= \mathbb{E} \{ V(x, i) \psi, r_0 \}.
$$
Because \( \frac{\partial V}{\partial t} \leq \alpha \left[ \exp(-\alpha T) - 1 \right] V(x_0, i) \), taking the limit as \( T \to \infty \), we have
\[
\lim_{T \to \infty} \mathbb{E} \left\{ \int_0^T x^T(t) x(t) \, dt \right\} \leq \frac{1}{\alpha} \left[ \exp(-\alpha T) - 1 \right] V(x_0, i).
\]

Taking the limit as \( T \to \infty \), we have
\[
\lim_{T \to \infty} \mathbb{E} \left\{ \int_0^T x^T(t) P_i x(t) \, dt \right\} \leq \frac{1}{\alpha} \mathbb{E} \left\{ \lambda_{\max}(P_i) + \varepsilon^2 \tau \lambda_{\max}(Q) \right\} I x_0^T P_i x_0
\]

where
\[
\hat{P} = \max_{i \in S} \left\{ \frac{\lambda_{\max}(P_i)}{\alpha \lambda_{\min}(P_i)} + \varepsilon^2 \tau \lambda_{\max}(Q) \right\} I
\]

implies that the closed-loop system under control law (10) is stochastically stable.

From Schur complement, \( M_i < 0 \) if and only if
\[
(A_{i1} - B_{i1} K_i)^T P_i + P_i (A_{i1} - B_{i1} K_i) + \sum_{j=1}^s \pi_{ij} P_j + Q + P_i A_{2i} Q_i^{-1} A_{2i}^T P_i < 0.
\]

Define \( X_i \triangleq P_i^{-1} Y_i \triangleq K_i X_i \). Pre- and postmultiplying the last matrix inequality by \( X_i^{-1} \), it is easy to find that \( M_i < 0 \), \( i = 1, \ldots, s \) if and only if for \( i = 1, \ldots, s \)
\[
X_i, A_{i1} X_i - Y_i^T B_{i1}^T - B_{i1} Y_i + \sum_{j=1}^s \pi_{ij} X_j X_j^{-1} X_i + X_i Q X_i + A_{2i} Q_i^{-1} A_{2i}^T < 0
\]

which are equivalent to the following LMI’s
\[
\begin{bmatrix}
\Delta_i + A_{2i} R A_{2i}^T & \Xi_i \\
\Xi_i^T & -Y_i
\end{bmatrix} < 0, \quad i = 1, \ldots, s
\]

where
\[
\Delta_i \triangleq X_i A_{i1} + A_{i1} X_i - Y_i^T B_{i1}^T - B_{i1} Y_i + \pi_{i} X_i, \quad \Xi_i \triangleq \begin{bmatrix} \sqrt{\pi_{i1}} X_i & \cdots & \sqrt{\pi_{is}} X_i \\ \sqrt{\pi_{i1}} X_i^T & \cdots & \sqrt{\pi_{is}} X_i \\ \end{bmatrix}, \quad Y_i \triangleq \text{diag} (X_1, \ldots, X_i-1, X_{i+1}, \ldots, X_s, R)
\]

The foregoing analysis leads to a stochastic stabilizability result in terms of LMI’s.

**Theorem 2**: The nominal jump linear system is stochastically stabilizable if \( R > 0, X_i > 0, \) and \( Y_i \), \( i = 1, \ldots, s \) exist, satisfying the coupled LMI’s (19). A stabilizing controller is constructed as \( K_i = Y_i X_i^{-1} \).

**Remark 1**: The proof of Theorem 1 is similar to the corresponding result in [12, Theorem 1]. If the nominal jump linear system has zero time delay, that is, \( A_{2i} = 0 \) for all \( i \in S \), condition (14) on stochastic stabilizability becomes
\[
(A_{i1} - B_{i1} K_i)^T P_i + P_i (A_{i1} - B_{i1} K_i) + \sum_{j=1}^s \pi_{ij} P_j < 0, \quad i = 1, \cdots, s
\]

which are equivalent to the following LMI’s
\[
\begin{bmatrix}
\Delta_i & -\Xi_i \\
\Xi_i^T & -Y_i
\end{bmatrix} < 0, \quad i = 1, \ldots, s
\]

given in [19]. We can show that it is a necessary and sufficient condition for the stochastic stabilizability of jump linear systems without time delay [12, Theorem 1]. Obviously, if \( P_i > 0 \) and \( K_i \) exist, satisfying the coupled matrix inequality (24), then some \( N_i > 0 \) exists such that the following Lyapunov equations are solvable:
\[
(A_{i1} - B_{i1} K_i)^T P_i + P_i (A_{i1} - B_{i1} K_i) + \sum_{j=1}^s \pi_{ij} P_j = -N_i, \quad i = 1, \cdots, s
\]

which is the coupled equations for the stochastic stabilizability in [12]. It is very difficult, however, to solve the above-coupled matrix equations [1], [18]. Fortunately, the LMI’s (19) and (25) can be numerically solved efficiently.

**Remark 2**: If we consider free jump linear systems, then the feasibility of LMI’s (19) with \( B_{i1} = 0 \) is sufficient for stochastic stability of this class of time-delay systems. It can also be extended to test the stochastic stability of free jump linear systems without time delay [10].

**Theorem 3**: The uncertain jump linear system (1)–(4) is robust stochastically stabilizable if scalars \( \alpha_i > 0, \beta_i > 0 \) and matrices \( R > 0, X_i > 0, \) and \( Y_i, i = 1, \cdots, s \) exist, satisfying the coupled matrix inequalities
\[
\begin{bmatrix}
\Delta_i & A_{2i} R A_{2i}^T \\
H_{2i} R A_{2i}^T & -\beta_i \hat{I} + H_{2i} R H_{2i}^T & 0 & 0 \\
H_{1i} & 0 & -\alpha_i \hat{I} & 0 \\
\Xi_i^T & 0 & 0 & -Y_i
\end{bmatrix} < 0,
\]

where
\[
\Delta_i \triangleq \Delta_i + A_{2i} R A_{2i}^T + \alpha_i E_i E_i^T + \beta_i E_i E_i^T
\]

**Proof**: From Theorem 1, the uncertain closed-loop system is stochastically stable, if for each mode \( i \in S \) and \( \|F_i\| \leq 1 \), the following coupled matrix inequalities hold for \( i = 1, \ldots, s \):

\[
X_i \left( A_{1i} + E_i F_i H_{1i} \right)^T + \left( A_{1i} + E_i F_i H_{1i} \right) X_i + X_i Q X_i + A_{2i} R A_{2i}^T < 0
\]

On the other hand, from Lemma 1, LMI’s (28) hold if scalars \( \alpha_i > 0, \beta_i > 0 \) exist such that
\[
\begin{bmatrix}
\Delta_i & \alpha_i E_i E_i^T + \alpha_i H_{1i}^T H_{1i} + A_{2i} R H_{2i} \\
\beta_i \hat{I} - H_{2i} R H_{2i}^T & 0 & 0 & 0 \\
\end{bmatrix} < 0,
\]

where
\[
\Delta_i = \Delta_i + \sum_{j=1, j \neq i}^s \pi_{ij} X_j X_j^{-1} X_i + X_i Q X_i + A_{2i} R A_{2i}^T
\]

which are equivalent to (27) from the Schur complement.
IV. ROBUST $H_{\infty}$ DISTURBANCE ATTENUATION

In this section, we consider robust $H_{\infty}$ disturbance attenuation for the uncertain jump linear state-delay systems.

**Theorem 4.** For the nominal jump linear system, a state feedback control (10) exists such that the closed-loop system possesses the $\gamma$-disturbance attenuation property; that is, $\|z\|_2 < \gamma \|w\|_2$ for all $w \in L_2[0, \infty]$, $w \not= 0$, if $Q > 0$ and $P_i > 0$, $i = 1, \cdots, s$ exist, satisfying the following coupled matrix inequalities:

$$\Theta_i \triangleq \begin{bmatrix} \hat{A}_i & P_i A_{2i} + \hat{C}_{1i}^T C_{2i} + C_{2i}^T \hat{C}_{1i} & P_i B_{2i} + \hat{C}_{1i}^T D_{2i} & \hat{C}_{1i}^T D_{2i} \\ P_i A_{2i} + \hat{C}_{1i}^T C_{2i} + C_{2i}^T \hat{C}_{1i} & -Q + C_{2i}^T C_{2i} & C_{2i}^T D_{2i} & -\gamma^2 I - D_{2i}^T D_{2i} \end{bmatrix} < 0$$

(29)

as shown in (29) for $i = 1, \cdots, s$, where $\hat{A}_i \triangleq \hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + \sum_{j=1}^s \pi_{ij} \hat{P}_j + Q + \hat{C}_{1i}^T \hat{C}_{1i}$.

**Proof:** Let the mode at time $t$ be $i$; that is, $r(t) = i \in \mathcal{S}$. Consider feedback control $u(t) = -K_i x(t)$ for $t \geq 0$. Then, the nominal closed-loop system becomes

$$\dot{x}(t) = \hat{A}_i x(t) + A_2 x(t - \tau) + B_2 w(t)$$

(30)

$$\dot{z}(t) = \hat{C}_i x(t) + C_2 x(t - \tau) + D_2 w(t).$$

(31)

Choose the stochastic Lyapunov functional $V(x(t); R^n \times \mathbb{R}_+ \times \mathcal{S} \rightarrow \mathbb{R}_+)$ as in (16). Then,

$$\dot{V}(x(t), r(t)) = x^T(t) M_i x(t) + w^T(t) B_2^T P_i x(t) + x^T(t) P_i B_2^T w(t).$$

Notice that, when the disturbance input is zero, that is, $w(t) = 0$, based on Theorem 1 and (29), we have $\dot{V}(x(t), r(t)) < 0$, which ensures the asymptotic stability of the closed-loop system.

In the following, we assume zero initial condition, that is, $x(t) = 0$ for $t \in [-\tau, 0]$, and define

$$J_T \triangleq \mathbb{E} \left\{ \int_0^T [z^T(t) z(t) - \gamma^2 w^T(t) w(t)] dt \right\}.$$

From Dynkin’s formula [14] and the fact that $x_0 = x(0, 0, r_0, \omega) = 0$, we have

$$\mathbb{E} \{V(x(T), r(T)) \} = \mathbb{E} \left\{ \int_0^T \dot{V}(x(s), r(s)) ds \right\}$$

since, $V(x_0, r_0) = 0$. Then, for any nonzero $w(t) \in L_2[0, \infty]$,

$$J_T = \mathbb{E} \left\{ \int_0^T [z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(x(t), r(t))] dt \right\}
- \mathbb{E} \{V(x(T), r(T))\}.$$

So

$$J_T \leq \mathbb{E} \left\{ \int_0^T [z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(x(t), r(t))] dt \right\}
= \mathbb{E} \left\{ \int_0^T \sigma^T(t) \Theta_i \sigma(t) dt \right\} < 0$$

where $\sigma(t) \triangleq [z^T(t) x^T(t - \tau) w^T(t)]^T$. Therefore, the dissipativity inequality (13) holds for all $T \geq 0$. In other words, we have that $z \in L_2[0, \infty]$, for any nonzero $w \in L_2[0, \infty]$, and $\|z\|_2 < \gamma \|w\|_2$. □

Theorem 4 can be rewritten in the form of LMI’s. From the Schur complement, it is easy to find that the coupled matrix inequalities in (29) are equivalent to the following matrix inequalities:

$$\begin{bmatrix} \hat{A}_{1i}^T P_i & P_i A_{2i} + \hat{C}_{1i}^T C_{2i} & P_i B_{2i} & \hat{C}_{1i}^T D_{2i} \\ P_i A_{2i} + \hat{C}_{1i}^T C_{2i} & -Q + C_{2i}^T C_{2i} & C_{2i}^T D_{2i} & -\gamma^2 I - D_{2i}^T D_{2i} \end{bmatrix} < 0$$

(32)

for $i = 1, \cdots, s$. Obviously, the matrix $M_i$ is a principal submatrix of $\Lambda_i$. So, if $\Lambda_i \prec 0$, then $M_i \prec 0$, which proves that Theorem 3 ensures Theorem 1 again. Let $X_i \triangleq P_i^{-1}$ and $Y_i \triangleq K_i X_i$, and define $T_i = \text{diag}(X_i, I, I, I)$. Pre- and postmultiplying (32) by $T_i$, we find

$$\begin{bmatrix} \Omega_i & A_2 Q^{-1} A_{2i}^T \\ C_{1i} X_i - D_{1i} Y_i + C_{2i} Q^{-1} A_{2i}^T \\ B_{2i}^T \\ -I + C_{2i} Q^{-1} C_{2i}^T \\ C_{2i}^T D_{2i} \\ D_{2i}^T \\ -\gamma^2 I \end{bmatrix} \Lambda_i \begin{bmatrix} A_{2i}^T P_i \\ -Q \\ 0 \\ C_{2i}^T \\ B_{2i}^T P_i \\ 0 \end{bmatrix} < 0$$

(33)

$$\begin{bmatrix} \Delta_i + A_2 R A_{2i}^T \\ C_{1i} X_i - D_{1i} Y_i + C_{2i} R A_{2i}^T \\ B_{2i}^T \\ -I + C_{2i} R C_{2i}^T \\ D_{2i}^T \\ -\gamma^2 I \end{bmatrix} \Xi_i \begin{bmatrix} A_{2i}^T P_i \\ -Q \\ 0 \\ C_{2i}^T \\ B_{2i}^T P_i \\ 0 \end{bmatrix} < 0$$

(34)

$$\begin{bmatrix} \Delta_i + \alpha_i E_i E_i^T + A_2 R A_{2i}^T \\ C_{1i} X_i - D_{1i} Y_i + C_{2i} R A_{2i}^T \\ Z_i \\ H_i X_i - H_{2i} Y_i + H_{2i} R A_{2i}^T \\ H_i Y_i + H_{2i} R A_{2i}^T \\ B_{2i}^T \\ -\gamma^2 I \end{bmatrix} \Xi_i \begin{bmatrix} A_{2i}^T P_i \\ -Q \\ 0 \\ C_{2i}^T \\ B_{2i}^T P_i \\ 0 \end{bmatrix} < 0.$$
that the coupled matrix inequalities (32) are equivalent to the following matrix inequalities:

\[
\begin{bmatrix}
\Omega_i & A_{2i} & B_{2i} & (C_{1i}X_i - D_{1i}Y_i)^T \\
A_{2i}^T & -Q & 0 & C_{2i}^T \\
B_{2i}^T & 0 & -\gamma^2 I & D_{2i}^T \\
C_{1i}X_i - D_{1i}Y_i & C_{2i} & D_{2i} & -I
\end{bmatrix} < 0,
\]

where

\[
\Omega_i = X_iA_{1i}^T + A_{1i}X_i - Y_j^TB_{1i}^T - B_{1i}Y_j + \pi_{ij}X_j^T X_j + X_jQX_j.
\]

From the Schur complement, we find the above matrix inequalities are equivalent to (33), given at the bottom of the previous page, for \(i = 1, \ldots, s\), which are in turn equivalent to the following LMI’s as shown in (34), given at the bottom of the previous page, for \(i = 1, \ldots, s\), where \(\Delta_i, \Xi, \Upsilon_i\), and \(R\) are defined in (20)–(23), respectively.

From the above derivations, the \(\gamma\)-disturbance attenuation result is summarized in the following theorem involving LMI’s.

**Theorem 5:** For the nominal jump linear system, a state feedback control law (10) exists such that the closed-loop system is stochastically stable with \(\gamma\)-disturbance attenuation; that is, \(\|\tilde{z}\| < \gamma\|w\|\) for all \(w \in L^2[0, \infty), \|w\| \neq 0\), if \(r > 0, X_i > 0, i = 1, \ldots, s\), exist, satisfying the coupled LMI’s shown in (34). A stabilizing controller to provide \(\gamma\)-disturbance attenuation can be constructed as \(K_i \equiv Y_jX_j^{-1}\).

**Proof:** Let the mode at time \(t = i\) be \(i\); that is, \(r(t) = i \in \mathcal{S}\). Consider the closed-loop system (11) and (12). From Theorem 5, it is stochastically stable with \(\gamma\)-disturbance attenuation if for all possible uncertainties \(\|F_i(t)\| \leq 1, Q > 0, X_i > 0, i = 1, \ldots, s\) exist, satisfying the following coupled matrix inequalities:

\[
\begin{bmatrix}
\hat{\Omega}_i & A_{2i} + E_{2i}F_iH_{2i} & B_{2i} \\
(A_{2i} + E_{2i}F_iH_{2i})^T & -Q & 0 \\
B_{2i}^T & 0 & -\gamma^2 I
\end{bmatrix} + \tilde{\Gamma}_i^T \tilde{\Gamma}_i < 0
\]

for \(i = 1, \ldots, s\), where

\[
\hat{\Omega}_i = X_i(A_{1i} + E_{1i}F_iH_{1i})^T + (A_{1i} + E_{1i}F_iH_{1i})X_i + \sum_{j=1}^{s} \pi_{ij}X_j^T X_j + X_jQX_j
\]

\[
\tilde{\Gamma}_i = \begin{bmatrix} \hat{C}_{1i}X_i + E_{2i}F_iH_{1i}X_i & C_{2i} + E_{2i}F_iH_{2i} & D_{2i} \end{bmatrix}
\]

which can be rewritten as

\[
G_0 + G_1F_iL_{1i} + (G_{1i}F_iL_{1i})^T + (G_{2i} + E_{2i}F_iL_{1i})^T (G_{2i} + E_{2i}F_iL_{1i}) < 0
\]

for \(i = 1, \ldots, s\), where

\[
G_0 = \begin{bmatrix} A_{2i} & -Q & 0 \\
B_{2i}^T & 0 & -\gamma^2 I
\end{bmatrix},
\]

\[
G_{1i} = [E_{2i}F_iL_{1i}^T, 0]^T,
\]

\[
G_{2i} = [\hat{C}_{1i}X_i, C_{2i}, D_{2i}].
\]

From Lemma 1, the matrix inequalities in (37) hold if scalars \(\alpha_i > 0\) and \(\delta_i > 0\) exist such that, for \(i = 1, \ldots, s\)

\[
I - \delta_i E_{2i}F_i^T > 0,
\]

\[
G_0 + \alpha_i G_{1i}G_{1i}^T + \alpha_i^2 L_{1i}^T L_{1i} + G_{2i}^T + (I - \delta_i E_{2i}F_i^T)^{-1} G_{2i} + \delta_i^2 L_{1i}^T L_{1i},
\]

which are equivalent to the LMI’s in (35).

**Remark 3:** Theorems 5 and 6 can be easily adapted to \(\gamma\)-suboptimal \(H_{\infty}\) control of jump linear systems without delay. In [9], the \(\gamma\)-suboptimal \(H_{\infty}\) control was addressed based on a set of coupled algebraic Riccati equations for a special class of jump linear systems without delay. No solution method for these coupled equations is presented, however.

V. CONCLUSION

In this correspondence, we have studied the robust stochastic stabilizability and \(H_{\infty}\) disturbance attenuation for a class of uncertain jump linear systems with time delay. Sufficient conditions on robust stochastic stabilizability and robust \(\gamma\)-disturbance attenuation are presented based on coupled LMI’s. All of these results established are independent of size of the delay time and applicable to situations in which an a priori knowledge of delay time is available. The results can also be extended to the jump linear systems with multiple time delays using the method of [5]. A possible direction for future work is to obtain delay-dependent conditions that are expected to be less conservative.

REFERENCES

Exponential Stability of Constrained Receding Horizon Control with Terminal Ellipsoid Constraints

Jae-Won Lee

Abstract—In this correspondence, state- and output-feedback receding horizon controllers are proposed for linear discrete time systems with input and state constraints. The proposed receding horizon controllers are obtained from the finite horizon optimization problem with the finite terminal weighting matrix and the artificial invariant ellipsoid constraint, which is less restrictive than the conventional terminal equality constraint. Both hard constraints and mixed constraints are considered in the state-feedback case, and mixed constraints are considered in the output-feedback case. It is shown that all proposed state- and output-feedback receding horizon controllers guarantee the exponential stability of closed-loop systems for all feasible initial sets using the Lyapunov approach.

Index Terms—Discrete linear system with input and state constraints, exponential stability, output-feedback control, receding horizon control.

I. INTRODUCTION

The receding horizon control has emerged as a powerful strategy for constrained systems with limitations on inputs, states, and outputs [4]–[6], [8]–[11]. Especially, the stability issue of the receding horizon control for constrained systems has been focused on in recent literatures [4], [6], [8], [11].

For the state-feedback case, the terminal equality constraint has been utilized to guarantee the closed-loop stability of the receding horizon controller for unconstrained systems [2], [3] and for constrained systems [8], [11]. This artificial constraint is satisfied by driving the state (or unstable mode) to the origin at the finite terminal time. This terminal equality constraint, however, is rather restrictive, because it is generally more difficult to drive a state to a specified point than into a specified set such as an ellipsoid or a ball. Moreover, this approach may make the optimization problem infeasible under the hard state constraint. Hence, the horizon size may have to be made longer so as to make the problem feasible. Even though the mixed constraint has been introduced to relax the hard state constraint [11], issues still need to be covered regarding feasibility and stability, because a somewhat restrictive terminal equality constraint should still be satisfied under the input constraint, and only “attractivity” rather than “asymptotic” or “exponential” stability has been shown in existing results [8], [11]. Recently, efforts have been made to overcome restriction of the terminal equality constraint [4], [7]. In [4], the invariant ellipsoid constraint has been introduced. This artificial constraint is satisfied by putting the state (or unstable mode) into an invariant ellipsoid. In this result, however, exponential stability is shown only for initial states inside the invariant ellipsoid defined by the terminal weighting matrix. In [7], it is shown that the receding horizon control with a sufficiently long horizon size can make the problem feasible. Even though the mixed constraint has been introduced, issues still need to be covered regarding feasibility and stability, because a somewhat restrictive terminal equality constraint should still be satisfied under the input constraint, and only “attractivity” rather than “asymptotic” or “exponential” stability has been shown in existing results [8], [11].

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